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# Crack arrest by residual bonding in resistor and spring networks

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A propagating crack in a resistor (spring) network may stop because of residual resistance (residual or ligamentary bonding) behind the crack tip. In breakdown networks containing random flaws, this leads to failure by microcracking rather than by the growth of a dominant crack. Numerical simulations of two-dimensional random resistor networks indicate that the transition between these two regimes occurs near $R \sim 35$ for $p=0.75$, where $R$ is the ratio of the residual resistance to the original resistance.

Once crack propagation has begun in brittle materials, it often proceeds to the complete fracture of the material. This is an undesirable feature, as although they are often very strong, this type of material is susceptible to sudden and rapid fracture—they are not very durable. Fiber reinforcement, transformation toughening, and mixtures of tough and strong materials are all ways of reducing brittleness.[1,2]

Random flaws play an important role in electrical and mechanical strength, and it has been difficult traditionally to analyze theoretically the strength of materials with randomness. This situation has been recently improved by the introduction of simple network models,[3–10] and methods which are extensions of those used to study transport and elastic properties in random media. This class of model also provides a unified framework for understanding both electrical and mechanical breakdown,[3,4] a novel and fruitful development. In this paper, we use these new ideas to study the onset and suppression of brittle fracture in materials with randomness. In particular, using random spring and resistor networks[1] we introduce a simple mechanism that leads to the arrest of even *a single propagating crack* in both electrical and mechanical networks. In the presence of random flaws, we show that in some cases, materials which exhibit this type of crack arrest locally are toughened by nonlocal microcracking in which well-separated regions scattered throughout the sample fail before macroscopic fracture occurs. We introduce an "order parameter" that distinguishes between the brittle and toughened regimes.

Fracture of solids is often analyzed in terms of the stress enhancement at the tip of a growing crack.[2] If there is no bonding behind the crack tip or any other toughening mechanism operating in the material, this stress grows as

$$\sigma_{\text{tip}} \sim c^{1/2}, \tag{1a}$$

where $c$ is the crack length. Similarly, in electrical networks, the current enhancement at the tip of an isolated void grows as

$$I_{\text{tip}} \sim c^{1/2}. \tag{1b}$$

However, if there is residual bonding (residual resistance after fuse failure) behind the crack tip the stress enhancement is reduced, and in some cases crack arrest may occur. Although this has been realized before in the context of the failure of ceramics (where it is called ligamentary bonding[11,12]), it is difficult to include the effect of disorder and nonlocal effects into a continuum analysis. We have thus introduced numerically tractible models of residual bonding in spring and resistor networks. They are defined as follows. We take random networks with a percolation distribution of random voids and with $p > p_c$ so that the networks are geometrically connected. In the resistor case, present bonds are circuit elements that change from resistance 1 to resistance $R$ when a current of 1 A passes through them. If $R \rightarrow \infty$, the fuse network[3–5] is recovered, and the network typically fails in a brittle manner.[4] In the mechanical problem, present bonds are springs that change from an elastic constant of 1 to $1/R$ when they are extended beyond a threshold strain. The bonds of lower elastic constant (or higher resistance) are the ligamentary bonds that provide the residual bonding. In physical situations, there is a second strain (current) level at which the residual bonds fail, and here we study the case where a great deal of microcracking has occurred before this second threshold is reached.

To demonstrate the effect of residual bonding on the propagation of an isolated crack in two-dimensional network models, consider Fig. 1(a). In this figure, there is an intrinsic flaw (crack) of size 3 a.u. in the center of a lattice whose present bonds are central-force springs with unit spring constant. An external tensile stress is applied in the $y$ direction. We solve the force equilibrium equations for the network, and calculate the stress in the bond at the crack tip. We then fail the bond at the crack tip, resolve the network equations and find the stress in the bond at the tip of the extended crack. In the zero-ligamentary-bonding case, a failed bond has zero elastic constant, and carried no load. In this case, the stress at the crack tip increases with the law given in Eq. (1). In the residual bonding case, the failed bond does not separate completely, but rather has a reduced elastic constant in comparison to that in the bulk. The ratio of the spring constant for the bulk as compared to that for the ligamentary bonding is held constant and is denoted by $R$. As $R$ goes to infinity, Eq. (1) is recovered. (In polymer composites this model may be directly applicable. There,

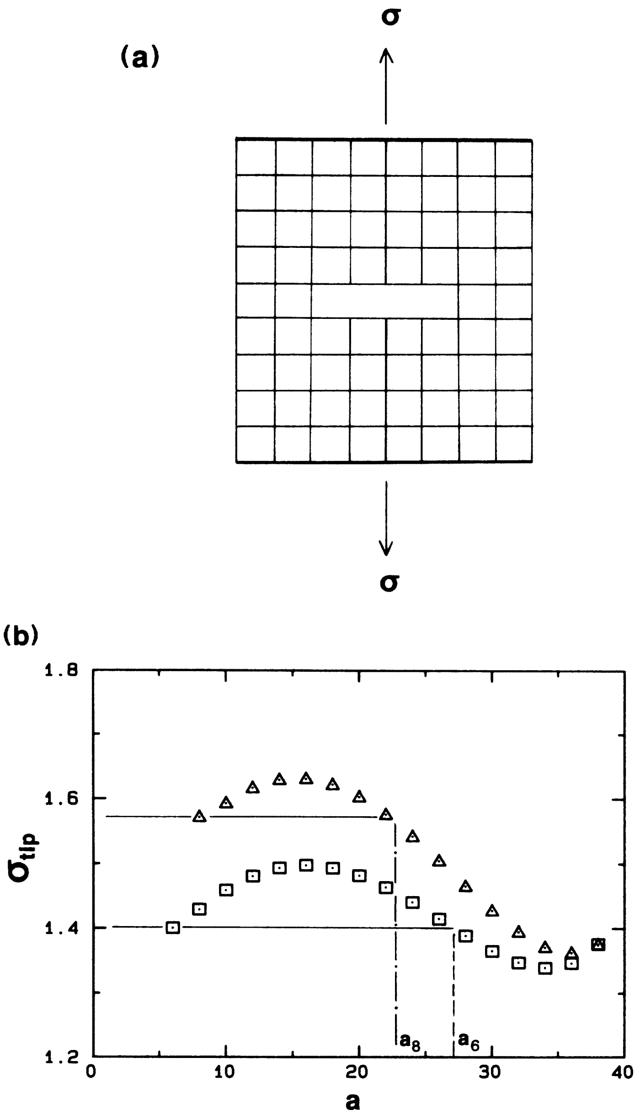

FIG. 1. (a) A square network with three bonds missing in the center of the lattice is shown before relaxation. A fixed external stress is applied in the $y$ direction, and the two edges at which the stress is applied are held rigid. (b) The stress at the crack tip $\sigma_{\mathrm{tip}}$ as a function of the size of the crack, calculated on a $40\times40$ network and with $R=20.0$, for initial crack sizes: 6 bonds (□); and 8 bonds (△). When the external stress is just large enough to initiate crack growth, crack arrest occurs at lengths $a_6$ and $a_8$, respectively.

the crosslinks between polymers are broken at lower loads, after which the polymer chains across a crack uncoil with a low elastic constant until they finally form extended chains or fibrils across the crack. This phenomenon, called crazing,[13] occurs at much lower loads than the failure of the macroscopic sample.)

In Fig. 1(b), the stress at the crack tip of a network with residual bonding $R$ is plotted as a function of crack size. The important feature of this figure is that after an initial rise in stress enhancement with increasing crack length, the enhancement decreases and goes *below the stress required to initiate the crack propagation.* At a constant external stress, this crack then exhibits *crack arrest*, at the length indicated in the figure. (The increase in stress enhancement for the largest crack sizes shown in the figure is due to the finite size of the network used in the simulation. In an infinite system $\sigma_{\mathrm{tip}}$ is expected to decrease monotonically from the crack arrest point onwards.) It is easy to understand qualitatively why the nonmonotonic behavior of Fig. 1(b) occurs for finite $R$, by considering the behavior for $R=\infty$ given in Eq. (1), and the $R=1$ limit where the stress decays for large $c$ as

$$\sigma_{\mathrm{tip}}\sim(c-c_{\mathrm{initial}})^{-D}, \qquad (2)$$

where $D$ is the space dimension. (This may be deduced from the long-distance behavior of the stress near an elliptical void in an isotropic elastic background.[14])

For intermediate $R$, the tip enhancement thus crosses over from a monotonically decreasing behavior for $R=1$ [see Eq. (2)] to a monotonically increasing behavior for $R\rightarrow\infty$ [see Eq. (1)], and so the intermediate behavior found in Fig. 1(b) is not surprising. In electrical networks, a precisely analogous crossover occurs for the current at the tip of a region of electrically failed bonds. There $R$ measures the ratio of the resistance of a bond after failure to that before failure. In dielectric problems, an analogous effect occurs in two dimensions where the metal inclusion that induces failure is in the direction of the applied field, and the residual resistance is less than one. (The $R\rightarrow0$ limit recovers the dielectric breakdown limit usually studied.[7,8]) In three-dimensional dielectrical problems, the behavior is expected to be somewhat different than for the fuse or mechanical problems, as there, the critical defects are finger-shaped rather than penny-shaped.[3,4,8] This makes it more difficult to induce crack arrest in three-dimensional dielectric networks than for three-dimensional fuse or mechanical networks.

Once crack arrest occurs locally, it is possible that many local regions may exhibit some degree of cracking before the entire network fails.[6,9] This is in contrast to the $R\rightarrow\infty$ limit of the electrical problem where most of the failed bonds lie on the eventual-failure path, or brittle fracture where a single dominant crack often propagates to induce macroscopic failure. As $R$ is reduced, we expect that more local failure occurs, and that eventually, the system fails by microcracking rather than by single-crack propagation. In this case, we say that the material is tough as the failure of many bonds implies that a large amount of energy must be added to the system to produce the global fracture. To demonstrate the crossover from brittle to tough behavior, we study the fracture of random electrical networks in two dimensions. For a fixed initial configuration of the random resistor network (with fraction $p$ of present bonds) and different ratios $R$, we solve Kirchhoff's equations and find the hottest bond and the external voltage needed to switch it. The resistance of the hottest bond is changed from 1 to $R$, and the solutions to Kirchhoff's equation is found using the new network configuration. This procedure is iterated until the network has failed (i.e., a connected path of resisters of size greater than or equal to $R$ exists across the network).[4] A clear indication of the difference in the failure mechanism

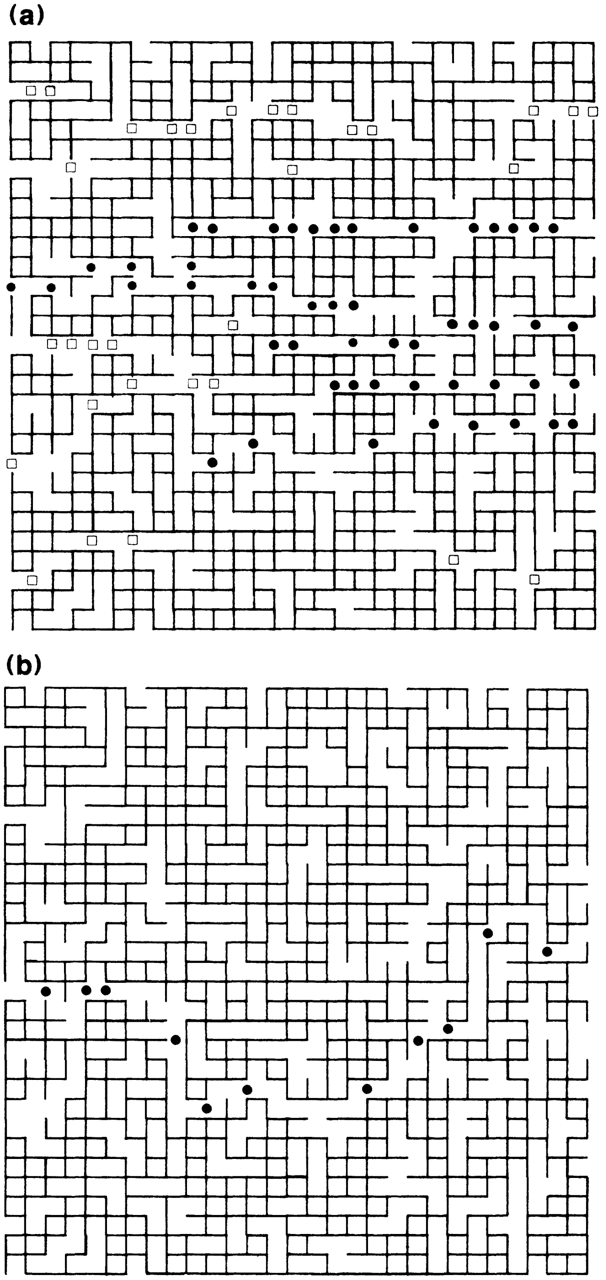

FIG. 2. The failure configuration for a 31×30 square lattice with the same initial configurations (with $p=0.75$), but different values of $R$ used in the crack propagation. Dots are bonds which fail, and which lie on the final fracture path. Squares are bonds which fail, but do not lie on the final failure path. (a) $R=2.0$; (b) $R=1000.0$.

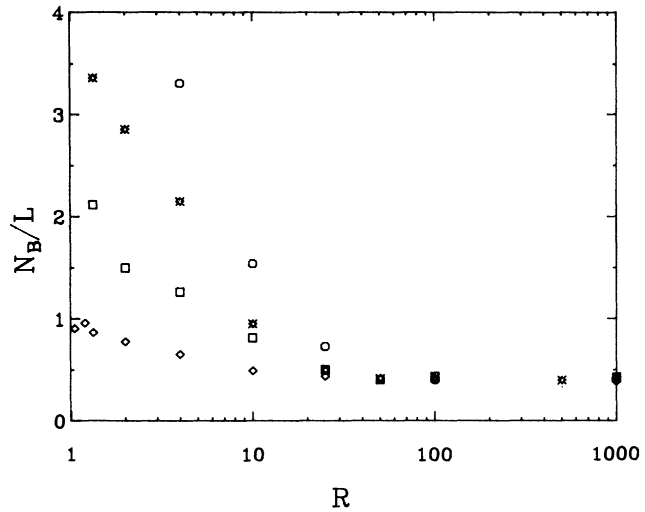

FIG. 3. A plot of $N_B/L$ as a function of $R$ for $p=0.75$ and $L=10$ (◇); $L=20$ (□); $L=30$ (*); and $L=40$ (○). Each point represents an average over 20 realizations of the initial configuration.

that occurs in the large as opposed to small $R$ regimes is given in Fig. 2, where we show the final (failure) configurations for $R=2$ [Fig. 2(a)] and $R=1000$ [Fig. 2(b)] for the *same initial configuration*, on a 31×30 array at $p=0.75$. For $R=1000$ [Fig. 2(b)], we needed to switch only eleven bonds to induce failure and all of the broken bonds contribute to the final breakdown path. But for $R=2$ [Fig. 2(a)], we have to switch 83 bonds that are throughout the network to induce failure. In the large $R$ regime the system fails as soon as a few critical bonds are broken. In the small $R$ regime, however, the failure of a bond rarely leads to the failure of neighboring bonds, and so a deterministic crack propagation does not occur.

To distinguish quantitatively the different regimes, we introduce the variable $N_B$ which is equal to the number of bonds broken in the global failure process. In the brittle fracture (dominant crack) case, $N_B \sim L$, the linear dimension of the network. In the case where a great deal of residual resistivity (ligamentary bonding) is present however, $N_B \sim L^x$ (where $x > D-1$). To illustrate the crossover between these two limiting behaviors, we have calculated $N_B$ for a series of square lattices and for various values of $R$. The results are shown in Fig. 3 for $p=0.75$. From this figure, we deduce that for $R > 35$ the networks fail by brittle fracture, while for $R < 35$ they are toughened by a residual bonding. The numerical simulations show that the transition is sharper with increasing system size, and that $x \sim 2$ for $R$ sufficiently small. At present, it is beyond our numerical accuracy to determine whether the transition becomes a single singular point in the thermodynamic limit.

In conclusion, we have shown that residual bonding leads to crack arrest in both electrical and mechanical networks, and that in random electrical networks this leads to toughening if the residual bonding is sufficiently strong. It is important to note that this toughening mechanism is nonlocal in that all parts of the network participate in increasing the work of fracture. In this sense, it is essential to discuss the process using statistical methods, rather than trying to treat the effect by an expanded zone near the crack tip. This nonlocality makes analytical and numerical analysis difficult, and makes the use of novel ideas from statistical mechanics and statistics necessary. Toughening (or ductility) induced by suitable distributions of bond fracture strengths,[6,9,15] is also nonlocal and, hence, is similarly difficult to analyze in detail.

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