Linear dielectric-breakdown electrostatics

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(Received 19 May 1988)

The dielectric breakdown of solids is a problem of great practical and theoretical interest. It is the electrical analog of the fracture of solids under applied loads. In the case of fracture, the reigning theory for linear elastic materials is linear elastic fracture mechanics. This paper develops the analogous theory, linear dielectric-breakdown electrostatics, based on a Griffith-like energy-balance calculation applied to a single conducting crack in an isotropic dielectric medium. Results include the development of the critical field-intensity factor, K_c^e, and the introduction of a contour-independent line integral, J^e, which is analogous to the J integral of linear and nonlinear elastic fracture mechanics. Some discussion of the relation between these results and recent lattice models of dielectric breakdown is given.

I. INTRODUCTION

A dielectric solid fails electrically under a large applied field when a conducting pathway is burnt through the material, turning the dielectric medium into a conductor. This phenomenon, known as dielectric breakdown, is the electrical equivalent of fracture in an elastic solid. Applied electric fields cause dielectric breakdown; applied mechanical loads cause elastic fracture. Both phenomena are of great practical and scientific interest.

The concept of *intrinsic* or theoretical strength is common to both dielectric breakdown and elastic fracture. In each case, the intrinsic or theoretical strength is defined to be that value of the applied stress (electric or elastic) that causes breakdown or fracture to occur when the material is free from defects of any kind. This strength, however, can be achieved only with great difficulty, under the most stringent sample preparation and testing conditions.[1,2] In the vast majority of materials, both the dielectric breakdown strength and fracture stress are two or more orders of magnitude smaller than the intrinsic or theoretical strength. This difference is attributed to the presence of defects (such as voids, cracks, or inclusions) in a material.

In the case of elastic fracture, there is a well-developed theory called linear elastic fracture mechanics (LEFM) for fracture caused by the propagation of existing crack-like defects. LEFM is based on the pioneering analysis of Griffith[3] of an ideal stress-free crack embedded in an isotropic elastic continuum subjected to a uniform tensile stress.

Griffith worked out the difference in elastic energy between a cracked and uncracked body (using Inglis stress analysis[4]). By defining Γ as the energy consumed in creating a unit area of new crack surface, he was able to show that there was a critical value of the applied stress at which the elastic energy released in an incremental crack extension is just able to balance the energy consumed in creating the new crack surfaces. In the problem Griffith considered, uniform tensile stress in an infinite sheet, the state of equilibrium is unstable, thereby giving a prediction for the applied stress at which the crack will freely propagate, implying fracture will spontaneously occur. This critical stress is a function of the elastic moduli, the crack length, and the fracture surface energy Γ.[5]

Although Griffith published his results in 1921, his work did not generate wide interest until after World War II, when scientific interest in crack propagation and fracture was stimulated by the many cracks that appeared in the Liberty ships that were manufactured in such large numbers by the United States (see Gordon's book in Ref. 5 for a fuller historical account). At about this time, Irwin[6] took Griffith's theory and reformulated it in terms of the parameter K_c, which is the critical value of the amplitude of the stress singularity at the crack tip. This formulation was mathematically more convenient for solving more complicated problems. It also was physically appealing, as it coupled the criterion for crack extension to a parameter K_c, which in some sense quantifies the actual conditions at the crack tip where fracture occurs.[7]

It is not well known that the same type of energy-balance calculation can be done for the case of dielectric breakdown via a conducting crack embedded in an isotropic, linear dielectric medium subjected to a uniform electric field. In fact, this calculation was done in the 1920's by Horowitz[8] shortly after Griffith published his elastic work. I have found only one reference to this work in later literature.[9] The calculation, similar to Griffith's, results in an expression for the critical applied electric field that is a function of the crack length, the dielectric constant, and a new surface energy Γ^e, which is called the breakdown surface energy and is equal to the energy consumed in creating a unit area of conducting-crack surface via dielectric breakdown.

In this paper, I rederive Horowitz's result and put it into a form I call linear dielectric-breakdown electrostatics (LDBE), in analogy with LEFM and strikingly similar in many ways. In addition, I prove the existence of a contour-independent line integral J^e, similar to the well-known J integral[10] in LEFM, which can be used to put

LDBE in the same mathematically elegant form as LEFM. Finally, in the discussion, I speculate on the physical meaning of Γ^e and on whether some of the methods that were inspired by LEFM and used to improve the fracture strength of materials have counterparts in dielectric media that can be used to improve their dielectric-breakdown strength. Also, I compare the results of LDBE with recent lattice models of dielectric breakdown.

II. HOROWITZ ENERGY-BALANCE CALCULATION

The electrostatic problem that must be solved first is the case of an elliptical inclusion with dielectric constant ϵ' placed in an isotropic linear dielectric medium with dielectric constant ϵ. The applied field is initially uniform and is parallel to the semimajor axis of the inclusion. Such an arrangement is shown in Fig. 1, with the inclusion and applied field $\mathbf{E}_0$ aligned in the x direction and the inclusion centered at the origin. The equation to be solved is Laplace's equation $\nabla^2 V=0$, subject to the following boundary conditions: the electric field $\mathbf{E}\rightarrow\mathbf{E}_0$ far from the inclusion, and the normal component of the displacement $\mathbf{D}$ continuous at the inclusion boundary. It should be noted that the latter boundary condition, in the limit $\epsilon'\rightarrow\infty$ for ϵ fixed, becomes $V=0$ at the inclusion boundary, which is the correct boundary condition for a *conducting* inclusion. This is actually the case of interest for dielectric breakdown; however, I will keep ϵ' finite for now.

The solution for this problem is readily obtained using elliptical cylindrical coordinates (u,Θ,z), where the z axis is taken normal to the plane of Fig. 1. All quantities in this problem are assumed to be uniform in the z direction. The transformation between (x,y) and (u,Θ) is given by

$$x=c\cosh u\cos\Theta\ ,$$
$$y=c\sinh u\sin\Theta\ , \tag{1}$$
$$0\le\Theta\le2\pi,\quad 0\le u<\infty\ .$$

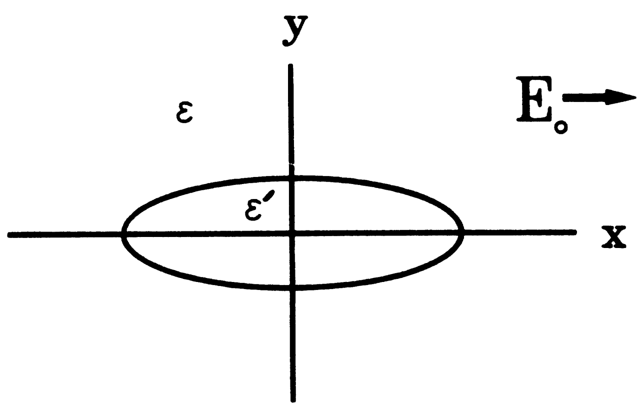

FIG. 1. An elliptical inclusion with dielectric constant ϵ' embedded in an isotropic linear dielectric medium with dielectric constant ϵ. Far from the inclusion, the electric field is uniform in the x direction, with magnitude E_0.

The surface of the inclusion is defined by $u=\beta=$constant. In the limit when $\beta\rightarrow0$, the inclusion degenerates into a crack of length $2c$, with crack tips at $x=\pm c$. The solution for $u>\beta$ is constructed[11] with a term that gives the correct uniform field far away from the inclusion plus a term that gets the boundary condition right at $u=\beta$. The result is

$$V(u,\Theta)=-cE_0\cosh u\cos\Theta+\tfrac{1}{2}(\epsilon'-\epsilon)\sinh(2\beta)\,e^{\beta-u}cE_0(\cos\Theta)/A \tag{2}$$

for $u>\beta$, and

$$V(u,\Theta)=-\epsilon E_0 c(\cosh\beta+\sinh\beta)\cosh u\cos\Theta/A \tag{3}$$

for $u<\beta$, where

$$A\equiv(\epsilon\cosh\beta+\epsilon'\sinh\beta)\ .$$

The field components are given by

$$E_u=-\tau^{-1}\partial V/\partial u \tag{4}$$

and

$$E_\Theta=-\tau^{-1}\partial V/\partial\Theta\ , \tag{5}$$

where

$$\tau\equiv c(\sinh^2u+\sin^2\Theta)^{1/2}\ .$$

If we define the *field multiplication factor* at the tip of the inclusion as the ratio

$$E_u(\beta,0)/E_0=E_x(\beta,0)/E_0\ ,$$

then this factor is

$$E_x(\beta,0)/E_0=1+(\epsilon'-\epsilon)a/(\epsilon a+\epsilon' b)\ , \tag{6}$$

where $a=c\cosh\beta$ and $b=c\sinh\beta$ are the semimajor and semiminor axes, respectively, of the inclusion. In the conducting limit, $\epsilon'\rightarrow\infty$ for fixed ϵ, Eq. (6) becomes $1+a/b$.

Using the above results, the Horowitz energy-balance calculation can now be completed. The question to be answered is the following. What is the difference in electrostatic energy between the medium *with* the inclusion and the medium *without* the inclusion? This question is answered nicely in Jackson's book,[12] for the case where the *sources* of the applied field are fixed. That is the case considered here, since one of the boundary conditions is that $\mathbf{E}\rightarrow\mathbf{E}_0$ far from the inclusion. For $\mathbf{E}=\mathbf{E}_0$ initially, and the geometry considered here, this energy difference is given by $\delta U=-\frac{1}{2}p_xE_0$, where $\mathbf{p}$ is the dipole moment of the inclusion. The dipole moment is easily found using Eqs. (4) and (5), with the result that

$$p_x=2\pi c^2\epsilon E_0(\epsilon'-\epsilon)(\cosh\beta+\sinh\beta)\sinh(\tfrac{1}{2}\beta)/A\ . \tag{7}$$

The energy difference δU then becomes

$$\delta U=-\tfrac{1}{2}p_xE_0=-\pi c^2\epsilon E_0^2(\epsilon'-\epsilon)(\cosh\beta+\sinh\beta)\sinh(\tfrac{1}{2}\beta)/A\ . \tag{8}$$

In the limit of a conducting ($\epsilon'\rightarrow\infty$ for fixed ϵ) crack ($\beta\rightarrow0$), the energy difference is

$$\delta U = -\tfrac{1}{2}\pi\epsilon c^2 E_0^2 \; . \tag{9}$$

Note that δU is negative, indicating that the introduction of a conducting crack has *lowered* the energy of the system. If one were to do the same calculation for a fixed potential, which is the more usual experimental condition, δU would have the same magnitude but opposite sign. However, in this case, one also has to consider the loss in potential energy of the external battery required to fix the potential, which *loses* $2\delta U$ of energy in this situation,[12] thereby making the net change in the electrostatic energy the same in both cases. The Appendix explicitly works out both cases for the example of a large capacitor containing the cracked dielectric medium. One should also note in (9) that the limit $\epsilon' \to \infty$ must be taken first to obtain a nonzero result.

We now define Γ^e to be the breakdown energy required to create a unit area of conducting crack (per unit length in the z direction). The surface energy of the crack is then $4\Gamma^e c > 0$. The *total* energy difference between a cracked and uncracked medium becomes

$$\delta U = -\tfrac{1}{2}\pi\epsilon E_0^2 c^2 + 4\Gamma^e c \; . \tag{10}$$

For small c, the linear term in (10) dominates, so that $\delta U > 0$, implying that it would be energetically unfavorable for a crack to exist or propagate. For large enough c, the quadratic term will make $\delta U < 0$ and decreasing with increasing c, indicating that crack propagation is energetically favorable. To find the equilibrium point, set $d(\delta U)/dc = 0$ with the result that the critical value of the applied field, $E_0 = E_0^c$, is

$$E_0^c = (4\Gamma^e/\pi c\epsilon)^{1/2} \; . \tag{11}$$

Equation (11) agrees with Horowitz's result.[8] By checking the second derivative of δU with respect to c at E_0^c, one can see that this is a point of unstable equilibrium, so that for an applied field $E_0 \geq E_0^c$, breakdown will spontaneously occur.

III. FIELD-INTENSITY FACTOR

The region around the tip of the conducting crack is of particular interest, as this is where the most intense fields are found and is the region where actual breakdown would occur. One can take Eqs. (4) and (5) and expand them in the limit $(u,\Theta) \to 0$ in order to focus on the crack tip. We will work only in the $\epsilon' \to \infty$, $\beta \to 0$ limit from now on. For definiteness, focus on the $x = c$ crack tip. Figure 2 shows the crack-tip region with the origin at the crack tip. The new coordinates are r, the radial distance from the crack tip, and Φ, the angle from the x axis. In terms of these variables, the limiting form of the electric field is

$$E_r = E_0(c/2r)^{1/2}\cos(\tfrac{1}{2}\Phi) \; ,$$
$$E_\Phi = -E_0(c/2r)^{1/2}\sin(\tfrac{1}{2}\Phi) \; . \tag{12}$$

In addition, in this limit the potential becomes

$$V = -E_0(2cr)^{1/2}\cos(\tfrac{1}{2}\Phi) \; . \tag{13}$$

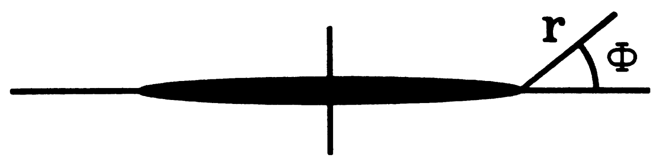

FIG. 2. The $x = c$ crack tip with new coordinates r and Φ. The origin is taken to be at the crack tip. The vertical width of the crack has been exaggerated for visibility.

A particularly useful constant to define is $K^e \equiv (\pi c)^{1/2} E_0$, called the *electric field-intensity factor* or field-intensity factor. The electric field and the potential near the crack tip then become

$$E_r = K^e \cos(\tfrac{1}{2}\Phi)/(2\pi r)^{1/2} \; ,$$
$$E_\Phi = -K^e \sin(\tfrac{1}{2}\Phi)/(2\pi r)^{1/2} \; , \tag{14}$$
$$V = -K^e (2r/\pi)^{1/2}\cos(\tfrac{1}{2}\Phi) \; .$$

K^e is seen to be essentially the amplitude of the $r^{-1/2}$ electric field singularity at the crack tip. This definition of K^e is exactly analogous to that in LEFM,[5] with stress replaced by electric field amplitude. One can define the *electrostatic energy release rate* G^e to be

$$G^e = d(\tfrac{1}{2}\pi\epsilon c^2 E_0^2)/dc = \pi\epsilon E_0^2 c \tag{15}$$

in analogy with LEFM.[5] $G^e dc$ is the amount of electrostatic energy released when the crack extends by dc. In terms of K^e, $G^e = \epsilon K^{e2}$. In LEFM, $G = K^2/E$, where E is Young's modulus. The critical value of G^e is defined via Eq. (10), so that $G_c^e = 4\Gamma^e$. The critical value of K^e is then $K_c^e = 2(\Gamma^e/\epsilon)^{1/2}$, so that Eq. (11) becomes

$$E_0^c = K_c^e/(\pi c)^{1/2} \; . \tag{16}$$

IV. CONTOUR-INDEPENDENT J^e INTEGRAL

In LEFM, Rice[10] has developed a line integral called the J integral that is contour independent. The usefulness of this integral comes about when the contour encloses the crack tip. Evaluating the J integral then gives G, the elastic energy release rate. The J integral, since it is defined for nonlinear as well as linear elasticity, is often used in elastic-plastic materials, where plastic deformation is approximately treated as nonlinear elastic displacement. An analogous integral for LDBE, denoted J^e, will be developed in the following.

The J^e integral is most easily developed by starting with the J integral from LEFM and transforming it according to a mapping between the two problems. This mapping is displayed in Table I. The elastic J integral is

$$J \equiv \int_c [-(\overleftrightarrow{\sigma}\cdot\hat{\mathbf{n}})\cdot(\partial\mathbf{u}/\partial x)ds + W dy] \; , \tag{17}$$

where $\overleftrightarrow{\sigma}$ is the stress tensor, $\hat{\mathbf{n}}$ is the unit normal to the contour, $\mathbf{u}$ is the displacement vector, and W is the elastic energy density. Using Table I, one can immediately write the form for J^e:

TABLE I. This table displays the mapping between the physical quantities and boundary conditions of LEFM and LDBE.

LEFM	LDBE
Physical quantities	
Stress $\overleftrightarrow{\sigma}$	Displacement field **D**
Strain $\overleftrightarrow{\epsilon}$	Field **E**
Displacement $\mathbf{u}$	Potential V
Moduli C_{ij}	Dielectric constant ϵ
Boundary conditions	
Fixed displacement	Fixed charge (field sources)
Fixed load	Fixed potential

$$J^e \equiv \int_c [-(\mathbf{D}\cdot\hat{\mathbf{n}})E_x ds + W^e dy] , \tag{18}$$

where W^e is the electrostatic energy density.

To show that J^e is contour independent, one first shows that J^e is zero for a region that has no field singularities. By applying Gauss's theorem[13] to the first term in (18) and Stoke's theorem[13] to the second term, one can write (for a singularity-free region):

$$J^e = \int_A da[-\mathbf{D}\cdot(\nabla E_x) + \partial W^e/\partial x] , \tag{19}$$

where the integration is now over the area enclosed in the contour and $\nabla\cdot\mathbf{D}=0$ was used. By expanding $\partial W^e/\partial x$ one can easily show that J^e is identically zero.

Note that this result holds even for a nonlinear dependence of **D** on **E**. Now consider the two contours C_1 and C_2 enclosing the crack tip in Fig. 3. Define a third contour C_3 that starts at point 1, follows C_2 counterclockwise to point 2, goes along the crack until it meets C_1 at point 4, and then follows C_1 clockwise to point 3, and finally runs along the crack back to point 1 again. This contour avoids the crack-tip singularity, so that J^e evaluated along C_3 is zero. The two pieces of C_3 that follow the crack surface make no contribution to the integral, as D_x and dy are both zero along the crack surface (away from the crack tip). Therefore, $J^e(C_1)=J^e(C_2)$ when both are transversed in the same sense. C_1 and C_2 are arbitrary, thus proving the contour independence of J^e.

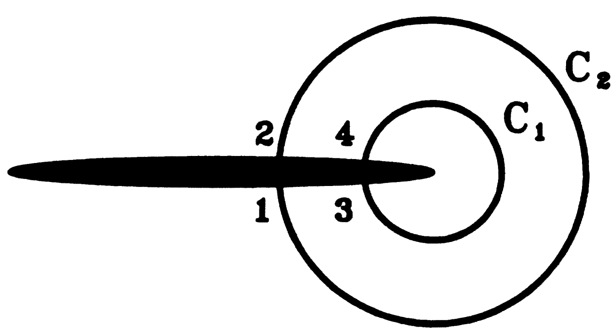

FIG. 3. Showing C_1 and C_2, the contours used for evaluating the J^e integral in Sec. IV. The marked points are the initial and final points for each contour.

To show that the value of J_e is just equal to G^e, the electrostatic energy release rate, when the contour does enclose the crack tip, take contour C_1 in Fig. 3 to be a circle of radius r centered at the crack tip, with $r<c$ small enough so that Eqs. (12) apply. Inserting (12) for the fields in the definition of J^e in Eq. (18), and taking $ds=rd\Phi$, $dy=r\cos\Phi\, d\Phi$, it is easy to see that

$$J^e = -\tfrac{1}{2}\pi\epsilon E_0^2 c = -\tfrac{1}{2}G = -\tfrac{1}{2}K^2\epsilon .$$

The factor of $\frac{1}{2}$ comes about because that is the energy release rate for *one* end of the crack. The full $G=\pi\epsilon E_0^2 c$ includes both ends of the crack. Since contour C_1 enclosed only one crack tip, we only get one half of the full electrostatic energy release rate.

It should be noted at this point that in LEFM other contour-independent line integrals similar to J have been developed by Knowles and Sternberg utilizing Noether's theorem.[14] Similarly, it is possible that J^e is not the only such integral for LDBE.

V. DISCUSSION

The two quantities that characterize the dielectric-breakdown strength of a material are, under LDBE, the critical field-intensity factor $K_c^e=2(\Gamma^e/\epsilon)^{1/2}$, and c, the length of the largest conducting cracklike flaw.

K_c^e is determined by the breakdown surface energy and the bulk dielectric constant. Clearly Γ^e must be related to the first ionization energy of the atoms or molecules making up the material, and also to the density, as Γ^e is defined per unit area. This would be the ideal surface energy, analogous to the thermodynamic surface energy Γ_T in a solid. In LEFM, however, it is known that in almost every case the surface energy that appears in the elastic K_c is always larger than Γ_T, even in "ideally brittle" solids like SiO_2 glass.[5] In some materials, like metals, the fracture surface energy includes crack-localized plastic deformation energy, making the effective Γ three or four orders of magnitude larger than Γ_T.[5] It is interesting to speculate what sorts of analogous mechanisms could exist in a dielectric. Perhaps localized Joule heating could be incorporated into an effective Γ^e. In the field of elastic fracture, artificially toughened ceramics can be made that incorporate metastable zirconia inclusions. These inclusions undergo a stress-induced phase transition to a larger-volume structure. When a propagating crack encounters such an inclusion, the transition is triggered by the high crack-tip stress fields and energy is absorbed, increasing Γ and thereby K_c.[15] Perhaps the similarities between LEFM and LDBE can fruitfully be exploited to suggest analogous ways to make dielectric materials with higher breakdown strengths.

The existence and size of cracklike conducting flaws is an important question to be resolved in order for LDBE to have application to real materials. Whitehead[9] mentions that the original Horowitz calculation, while successfully explaining experimental strengths and the existence of a discrete breakdown path, was not further applied because "no experimental evidence has been adduced as to the existence and influence of these cracks upon electrical breakdown. . . ."[9] That was the situation as of 1951; perhaps a different verdict could be given today. Certainly in the elastic case, the existence of cracks of the appropriate size to explain experimental strengths is now well established (after initial doubts).[5] If such a crack were to be filled with ionized air or water vapor, for example, it would seem to be able to act as a flaw of the right kind for LDBE to apply.

A simple experiment to perform to check the validity of LDBE would be to incorporate metallic inclusions into some kind of resin in a manner similar to that done to investigate the application of LEFM to elastic fracture.[16] Of course, one would have to duplicate the two-dimensional geometry used in the derivation of LDBE. Utilizing the usual experimental precautions to avoid electrode-induced breakdown,[1] one could then study the breakdown field E_0^c as a function of conducting crack length c. A linear fit (if appropriate) of E_0^c versus $(\pi c)^{-1/2}$ would give the value of K_c^e, which, along with a dielectric constant measurement, could be used to determine Γ^e. One could then use the measured K_c^e and careful microscopy or other techniques to analyze the breakdown strengths of materials with naturally occurring flaws. If the measured values of Γ^e were much larger than a reasonable estimate of its ideal value, one would have to analyze the difference in terms of other energy-consuming processes in the crack-tip region. There are other problems in dielectric breakdown that might be studied with LDBE, in ways analogous to LEFM. Dielectric aging[1] is the phenomenon by which breakdown can occur when a material is exposed to a subcritical field for a long period of time, or to a cyclically varying field with subcritical amplitude. The analogous phenomena in elastic fracture, static and dynamic fatigue, have been successfully studied via LEFM.[5] In addition, LDBE might also be applied to the study of treeing,[1] where the breakdown path bifurcates into a treelike structure, in a manner similar to the application of LEFM to understand crack branching.[5]

In elastic fracture, it is possible to load a cracked specimen in such a way that crack growth occurs *stably*, enabling the experimentalist to follow the crack with a microscope, an arrangement that can lead to new understanding.[17] An example is the double-cantilever beam arrangement.[5] One might speculate on the possibility of producing the same effect in a dielectric medium. To achieve this effect, one must have an arrangement of field plus crack such that G^e, the electrostatic energy release rate, is a decreasing function of the crack length c, or at most a constant independent of c.

The results of this paper can be applied to recent work on lattice models of dielectric breakdown,[18] to give a lower bound on the lattice-size dependence of the dielectric breakdown strength. In Appendix C of Ref. 18, a bond-percolation network model is considered where a fraction $p < p_c$ of the bonds are conductors and a fraction $(1-p)$ are insulators. The insulating bonds have a critical voltage V at which they break down and become conductors, which makes this network a model for dielectric breakdown. For values of p such that the linear dimension of the network, L, is much bigger than the percolation correlation length, Duxbury and co-workers found that the average dielectric breakdown field strength scales with L as

$$E_c \sim [a + b(\ln L)^\alpha]^{-1} , \qquad (20)$$

where $\alpha = \alpha_1$ for initial breakdown and $\alpha = \alpha_b$ for final breakdown, and $\frac{1}{2} \le \alpha_b \le \alpha_1 \le 1$ in two dimensions. The quantity $\ln L$ is the length of the largest critical defect, found by using a Lifshitz-type[19] argument, and a and b are size-independent constants. Equation (20) was developed using a continuum analysis to get the field concentration at the end of a long, elliptical crack, so it is not unreasonable to apply the results of the continuum analysis of this paper to the same network problem. Equation (11) states that for a single conducting crack of length c, the critical breakdown field is proportional to $c^{-1/2}$. In the network analysis,[18] $c \sim (\ln L)$, implying that the average breakdown field, at least in the dilute limit, should scale with L as

$$E_c \sim (\ln L)^{-1/2} . \qquad (21)$$

This result would imply that $\alpha_1 = \alpha_b = \frac{1}{2}$, since in the case considered here the Horowitz energy balance is at a point of unstable equilibrium, so that breakdown is spontaneous at the critical applied electric field. However, in elastic fracture, it is possible for crack propagation to change from unstable to stable for crack lengths larger than some critical value,[20] especially in a finite system when the crack length is an appreciable fraction of the system size. If this same effect exists in dielectric breakdown, it might result in apparent differences between α_1 and α_b found in finite-size computer simulations. However, recent computer simulations[21] of a similar model found that the applied voltage for breakdown initiation was almost identical over many configurations to the applied voltage required for breakdown completion for one value of the system size L. The result $\alpha_1 = \alpha_b = \frac{1}{2}$ is, of course, a lower bound for α_1 and α_b since the LDBE analysis does not consider the case of a two-crack flaw. More recent results by Duxbury and co-workers confirm that $\alpha_1 = \alpha_b = 1$ when two-crack flaws are considered.[22]

Further work in regard to the lattice model that would be interesting would be to treat ϵ and Γ in effective-medium theory as a function of p, compute c numerically, and then apply Eq. (11) to computer simulation results. Also, a similar energy-balance-type calculation could probably be done for the main problem considered in Ref. 18, that of current burnout of fuses, although the calculation would be somewhat trickier to carry out, since one has to take into account Joule heating *in* the current plane as well as energy stored in magnetic fields *out* of the current plane.

Note added in proof. After this paper was submitted, D. R. Clarke brought a paper by Hoenig[23] to my attention, in which Hoenig derived what are essentially the J^e integral and the electric field intensity factor K^e. There are some errors in his derivation and interpretation of the J^e integral, however, which I will address in a subsequent paper[24] that gives a more rigorous treatment of this integral.

ACKNOWLEDGMENTS

I would like to thank E. W. Behrens for a partial translation of Ref. 8, and M. F. Thorpe and the late D. L. Shelley for valuable discussions on the theoretical and experimental aspects of the dielectric-breakdown problem, respectively. I should also like to thank P. M. Duxbury, R. F. Cook, and M. D. Thouless for valuable discussions and encouragement.

APPENDIX

Assume a dielectric medium with an inclusion of different material is contained in a capacitor whose capacitance without the inclusion present is C. The presence of the inclusion causes $C \to C+dC$. For fixed *charge* Q on the capacitor, the change in energy is

$$\delta U_c = \tfrac{1}{2}Q^2/(C+dC) - \tfrac{1}{2}Q^2/C = -\tfrac{1}{2}(Q/C)^2 dC \ . \qquad \text{(A1)}$$

Note that if the dielectric constant of the inclusion is larger than that of the medium, $dC>0$ and so $\delta U_c<0$. This was the case covered in Sec. II. For fixed *potential* V, the change in energy of the capacitor now becomes

$$\delta U_c = \tfrac{1}{2}(C+dC)V^2 - \tfrac{1}{2}CV^2 = \tfrac{1}{2}V^2 dC \ . \qquad \text{(A2)}$$

The change in potential energy of the external battery is $\delta U_B = VdQ$, where dQ is the amount of charge transported in order to keep the potential constant. The value of dQ is found from

$$dV = 0 = (dQ/C) - (Q/C^2)dC \qquad \text{(A3)}$$

or

$$dQ = (Q/C)\, dC = V\, dC \ . \qquad \text{(A4)}$$

Therefore, $\delta U = \delta U_c + \delta U_B$ becomes

$$\delta U = \tfrac{1}{2}V^2 dC - V^2 dC = -\tfrac{1}{2}V^2 dC = -\tfrac{1}{2}(Q/C)^2 dC \ , \qquad \text{(A5)}$$

which agrees with (A1).

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*Address after Nov. 1, 1988: National Bureau of Standards, Building Materials Division, Gaithersburg, MD 20899.

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[6] G. R. Irwin, J. Appl. Mech. **24**, 361 (1957).

[7] The actual separation process in a solid at the crack tip is a subject of continuing debate. See references contained in the books listed in Ref. 5.

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