

Thermal conductivity and viscosity via phonon-phonon, phonon-roton, and roton-roton scattering in thin ^4He films

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The coefficients of the thermal conductivity (κ) and first viscosity (η) in thin helium films are evaluated explicitly as a function of temperature via phonon-phonon, phonon-roton, and roton-roton scattering. Above about 0.8 K, phonon-roton scattering and five-phonon processes are the main contributors to both coefficients. Below about 0.8 K, both coefficients increase exponentially with decreasing temperature. At temperatures below 0.3 K, κ_{ph} has a T^{-5} dependence, while η_{ph} shows exponential and T^{-1} dependencies. In the case of η_{ph} , the former is due to phonon-roton scattering and the latter originates from three-phonon processes. The coefficient κ_r , from roton-roton scattering varies as T^{-1} , and the roton part η_r of the first viscosity is independent of temperature.

I. INTRODUCTION

Since the works of Landau and Khalatnikov on kinetic phenomena¹ in superfluid helium, there has been continuous interest in thermal conductivity and viscosity for bulk liquid ^4He and ^3He at low momenta and at low temperatures.² Recently Kirkpatrick and Dorfman³ obtained transport coefficients for very low temperatures ($na\lambda^2 \gg 1$) and for moderately low temperatures ($na\lambda^2 \ll 1$) on the basis of their kinetic equations for a dilute superfluid, where n , a , and λ represent the number density, the s -wave scattering length, and the thermal wave length, respectively.

In the case of thin helium films, the dissipation of superfluid flow and thermal conductance have been investigated by many authors.⁴ Ambegaokar *et al.*⁵ have predicted that the effective thermal conductance in thin helium films has exponential dependence on temperature for $T < T_c$ and diverges exponentially for $T \rightarrow T_c +$, where T_c is the thermodynamic Kosterlitz-Thouless temperature. More recently the depairing⁶ and the depinning⁷ vortices which give the power law and the exponential dependence of thermal conductance have been investigated experimentally by Gasparini *et al.*,⁸ who confirmed an exponential dependence rather than the power law for $T < T_c$. However, there is much less information about thermal conductivity and viscosity in thin helium films at low momenta and at very low and moderately low temperatures. What is more important is that an incorrect normal (downward) dispersion relation was used in Landau and Khalatnikov's well-known results,¹ whereas the correct dispersion is anomalous (upward) dispersion.⁹ For this reason and in view of recent experimental developments on helium films, we present in this paper new results on the thermal conductivity and viscosity of superfluid helium films through the theory of kinetic phenomena developed by Landau and Khalatnikov.^{1,9} We

evaluate the thermal conductivity and viscosity within three ranges of temperatures: $0.3 \text{ K} \lesssim T$, $0.3 \text{ K} \lesssim T \leq 0.8 \text{ K}$, and $T \geq 0.8 \text{ K}$. In these temperature ranges, scattering depends on the nature of interactions between elementary excitations, i.e., phonon-phonon, phonon-roton, and roton-roton interactions. The scatterings which govern the transport processes and kinetic coefficients of thermal conductivity and viscosity can be determined by the characteristic time τ of scattering.

In the present paper we shall treat a thin helium film as two dimensional (2D)—less than three atomic layers, namely one statistical layer of 3.6 \AA —and neglect substrate effects. In the calculations we shall use the 2D excitation dispersion relation obtained microscopically:¹⁰

$$\epsilon(p) = c_0 p (1 + \delta_1 p^2 - \delta_2 p^4 + \dots), \quad (1.1)$$

$$\epsilon(p) = \Delta + \frac{1}{2\mu} (P - P_0)^2, \quad (1.2)$$

where c_0 is the sound velocity, and Δ , μ , and P_0 are the roton parameters. To evaluate the coefficients of thermal conductivity and viscosity, we shall first calculate the scattering cross section for the various interactions in Sec. II. Then we shall evaluate the characteristic times corresponding to the various scatterings by solving the collision integral, and then obtain the thermal conductivity in Sec. III and the viscosity in Sec. IV. Finally, we shall give results and a discussion in Sec. V in terms of graphs and tables.

II. SCATTERING CROSS SECTIONS AND DIFFERENTIAL DECAY RATES

In this section we consider the interactions of elementary excitations by the second quantization method.¹¹ To obtain characteristic times corresponding to the three interactions, we first evaluate the scattering cross section or the differential decay rate, which are directly related to the collision integral. The collision process of phonon-

phonon interactions includes a three-phonon process (3PP, $\mathbf{p}_1 \rightleftharpoons \mathbf{p}_2 + \mathbf{p}_3$), four-phonon process (4PP, $\mathbf{p}_1 + \mathbf{p}_2 \rightleftharpoons \mathbf{p}_3 + \mathbf{p}_4$) and five-phonon process (5PP, $\mathbf{p}_1 + \mathbf{p}_2 \rightleftharpoons \mathbf{p}_3 + \mathbf{p}_4 + \mathbf{p}_5$).

The differential cross section and differential decay rate for 3PP in two dimensions are defined as

$$d\sigma = \left[\frac{2\pi}{\hbar c_0} \right] |\langle F | \mathcal{H}_3 | I \rangle|^2 \delta(E_F - E_I) \frac{1}{(2\pi\hbar)^2} d\mathbf{p}, \quad (2.1)$$

$$dw = \left[\frac{2\pi}{\hbar} \right] |\langle F | \mathcal{H}_3 | I \rangle|^2 \delta(E_F - E_I) \frac{1}{(2\pi\hbar)^2} d\mathbf{p}_2 d\mathbf{p}_1. \quad (2.2)$$

The 3PP consists of two processes: the direct process of emission of a phonon \mathbf{p}_3 by $\mathbf{p}_1 = \mathbf{p}_2 + \mathbf{p}_3$ and the reverse process of absorption of a phonon \mathbf{p}_3 by $\mathbf{p}_1 + \mathbf{p}_3 = \mathbf{p}_2$. The transition amplitude between the initial state $|I\rangle$ and the final state $|F\rangle$ is given by¹¹

$$\langle F | \mathcal{H}_3 | I \rangle = \frac{3!}{2} \frac{(2\pi\hbar)^2}{(2S)^{3/2}} \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \left[\left(\frac{c_0}{\rho_0} \frac{p_2}{p_1 p_3} \right)^{1/2} (\mathbf{p}_1 \cdot \mathbf{p}_2) + \frac{1}{3} \frac{\rho_0^2}{c_0^2} \frac{\partial}{\partial \rho} \left(\frac{c_0^2}{\rho} \right) \left(\frac{c_0}{\rho_0} p_1 p_2 p_3 \right)^{1/2} \right] \times [n_{\mathbf{p}_1} (n_{\mathbf{p}_2} + 1) (n_{\mathbf{p}_3} + 1)]^{1/2}, \quad (2.3)$$

$$\langle F | \mathcal{H}_3 | I \rangle = \frac{3!}{2} \frac{(2\pi\hbar)^2}{(2S)^{3/2}} \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \left[\left(\frac{c_0}{\rho_0} \frac{p_2}{p_1 p_3} \right)^{1/2} (\mathbf{p}_1 \cdot \mathbf{p}_3) + \frac{1}{3} \frac{\rho_0^2}{c_0^2} \frac{\partial}{\partial \rho} \left(\frac{c_0^2}{\rho} \right) \left(\frac{c_0}{\rho_0} p_1 p_2 p_3 \right)^{1/2} \right] \times [(n_{\mathbf{p}_1} + 1) n_{\mathbf{p}_2} n_{\mathbf{p}_3}]^{1/2}, \quad (2.4)$$

where $n_{\mathbf{p}}$ is the distribution function of phonons with momentum \mathbf{p} . Then the total decay rate in both processes becomes

$$W_D = \frac{\pi c_0}{2\hbar \rho_0} (u + 1)^2 \int p_1 p_2 p n_{\mathbf{p}_1} (n_{\mathbf{p}_2} + 1) \times (n_{\mathbf{p}_3} + 1) \delta(E_F - E_I) \frac{1}{(2\pi\hbar)^2} d\mathbf{p}_2, \quad (2.5)$$

$$W_R = \frac{\pi c_0}{2\hbar \rho_0} (u + 1)^2 \int p_1 p_2 p (n_{\mathbf{p}_1} + 1) n_{\mathbf{p}_2} n_{\mathbf{p}_3} \times \delta(E_F - E_I) \frac{1}{(2\pi\hbar)^2} d\mathbf{p}_2, \quad (2.6)$$

where u is the Grüneisen constant given by $(\rho_0/c_0)(\partial c_0/\partial \rho_0)$.

In 4PP the transition matrix elements¹¹ are given by

$$\langle F | \mathcal{H} | I \rangle = \langle F | \mathcal{H}_4 | I \rangle + \sum_j \frac{\langle F | \mathcal{H}_3 | J \rangle \langle J | \mathcal{H}_4 | I \rangle}{E_I - E_j}, \quad (2.7)$$

where the main contribution is due to the second term, which becomes large as a result of vanishing denominator. This corresponds to the case where δ_1 is neglected and the scattering between phonons is collinear. Therefore, we should not discard the δ_1 term in Eq. (1.1). Taking account of $\delta_1 p^2$, which is much smaller than unity, and small-angle scattering, we can find the maximum value for $\langle F | \mathcal{H} | I \rangle$. In small-angle scattering all the

phonons are moving in the same direction, and the conservation of momentum and energy flow hold. Thus the phonons moving in a given direction attain equilibrium with each other much faster than the phonons in other directions. The differential cross section [Eq. (2.1)] for 4PP becomes

$$d\sigma(\mathbf{p}, \mathbf{p}_1, \mathbf{p}', \mathbf{p}'_1) = \left[\frac{2\pi}{\hbar c_0} \right] |\langle F | \mathcal{H} | I \rangle|^2 \delta(\varepsilon + \varepsilon_1 - \varepsilon' - \varepsilon'_1) \frac{1}{(2\pi\hbar)^2} d\mathbf{p}'. \quad (2.8)$$

Under the condition $p \ll p_1$ (Ref. 1) we obtain

$$d\sigma = \frac{(u + 1)^2 p_1^2 p'_1 p'}{8\pi\hbar^3 \rho_0^2 c_0 \rho} \frac{\delta(\varepsilon(p) + \varepsilon(p_1) - \varepsilon(p') - \varepsilon(p'_1))}{(1 - \cos\theta - 3\delta_1 p_1^2)^2} d\mathbf{p}'. \quad (2.9)$$

As for 4PP above, the direction of momenta of the colliding particles is not changed in the 5PP case. The transition matrix elements in second-order perturbation are

$$\sum_{i, \Pi} \frac{\langle I | \mathcal{H}_3 | i \rangle \langle i | \mathcal{H}_3 | \Pi \rangle \langle \Pi | \mathcal{H}_3 | F \rangle}{(E_I - E_i)(E_I - E_{\Pi})}, \quad (2.10)$$

where Π means intermediate states, and some terms contain the vanishing denominators under the conditions of $\delta_1 = 0$ and collinear scattering. The 5PP has the maximum probability in small-angle scattering and leads to equilibrium for the phonons moving in a given direction. Rather than calculating Eq. (2.10) tediously, we make use of the kinetic coefficient given by Landau and Khalatnikov.¹ The rate of change per unit time in the phonon numbers is

$$\dot{N}_{\text{ph}} = - \int \int \int [n_1 n_2 n_3 (n_4 + 1)(n_5 + 1) - (n_1 + 1)(n_2 + 1)n_4 n_5] dw \frac{1}{(2\pi\hbar)^6} d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3, \quad (2.11)$$

where dw is the differential decay rate defined by Eq. (2.2), which is proportional to p^2 . \dot{N}_{ph} can be expressed in terms of the kinetic coefficient Γ_{ph} as

$$\dot{N}_{\text{ph}} = -\Gamma_{\text{ph}}\mu_{\text{ph}}, \quad (2.12)$$

where μ_{ph} is the chemical potential of phonons. The phonon distribution function n has small deviation from the equilibrium distribution function n_0 . The distribution function n can be expanded as a function of chemical potential as follows:

$$n - n_0 = n_0(n_0 + 1) \frac{\mu_{\text{ph}}}{k_B T}. \quad (2.13)$$

To simplify Eq. (2.11), we replace n_1, n_2, n_3 , and n by the equilibrium distribution n_{10}, n_{20}, n_{30} , and n_{40} , and instead of n_5 we substitute Eq. (2.13) into Eq. (2.11). Then Eq. (2.11) becomes

$$\Gamma_{\text{ph}} = \frac{1}{k_B T} \int \int \int \int n_{10} n_{20} n_{30} (n_{40} + 1)(n_{50} + 1) \times dw \frac{1}{(2\pi\hbar)^6} d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3. \quad (2.14)$$

Since n_{40} and n_{50} are much smaller than unity, they may be neglected without any disturbance in Eq. (2.14), and the integration of dw is replaced by average \bar{w} . Then Eq. (2.14) becomes

$$\Gamma_{\text{ph}} = \frac{1}{k_B T} \int \int \int n_{10} n_{20} n_{30} \bar{w} \frac{1}{(2\pi\hbar)^6} d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3. \quad (2.15)$$

Since \bar{w} and \mathcal{H}_3 are proportional to $\sim p^2$ and $\sim p^{3/2}$, respectively, the integral of Eq. (2.15) is proportional to p^8 . Averaging over the phonon momentum gives

$$\Gamma_{\text{ph}} = aT^7. \quad (2.16)$$

Here a is constant, which can be determined experimentally by the attenuation coefficient of ultrasonic waves.

Now we return to the scattering of phonons by rotons. We consider a roton in the presence of the phonon field. We can treat this roton as a particle in a moving liquid ^4He . Therefore, there appears an additional term $-\mathbf{P}\cdot\mathbf{v}$. This can be written in symmetric form

$$-\frac{1}{2}(\mathbf{P}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{P}), \quad (2.17)$$

where \mathbf{P} and \mathbf{v} are the momentum and velocity operators.¹¹ The phonon field changes the density of the medium, and thus we may expand the roton energy in terms of the density ρ' to second order ($\rho' = \rho - \rho_0$) as

$$\mathcal{H}_r = \mathcal{H}_{r0} + \frac{\partial}{\partial \rho} \left[\Delta + \frac{1}{2\mu} (P - P_0)^2 \right] \rho' + \frac{1}{2!} \frac{\partial^2}{\partial \rho^2} \left[\Delta + \frac{1}{2\mu} (P - P_0)^2 \right] \rho'^2 + \dots, \quad (2.18)$$

where \mathcal{H}_{r0} is given by Eq. (1.2). Since the value of most rotons is close to P_0 , we may neglect $P - P_0$ and replace P by P_0 . We may also drop the term $(\partial\Delta/\partial\rho)\rho'$, which is much smaller than (2.17). Then the interaction energy between phonons and rotons can be expressed as

$$V = -\frac{1}{2}(\mathbf{P}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{P}) + \frac{1}{2} \left[\frac{\partial^2 \Delta}{\partial \rho^2} + \frac{1}{\mu} \left(\frac{\partial P_0}{\partial \rho} \right)^2 \right] \rho'^2. \quad (2.19)$$

We note that the terms in the large square brackets of Eq. (2.19) have magnitude on the order of 10^{-1} to 1 in 3D liquid helium.¹²

When the roton changes momentum \mathbf{P} to \mathbf{P}' , it absorbs a phonon with momentum \mathbf{p} and emits a phonon with momentum \mathbf{p}' . In these processes we may consider two intermediate processes, i.e., (I) $\mathbf{P} + \mathbf{p} \rightarrow \mathbf{P}' = \mathbf{P} + \mathbf{p} - \mathbf{p}'$ and (II) $\mathbf{P} - \mathbf{p}' \rightarrow \mathbf{P}' = \mathbf{P} - \mathbf{p}' + \mathbf{p}$. Since the roton momentum is much larger than that of the phonon, we may view this interaction as similar to that between heavy and light particles. The momentum and the energy conservation law in collision processes can be written as

$$cp + \frac{1}{2\mu} (P - P_0)^2 = cp' + \frac{1}{2\mu} (|\mathbf{P} + \mathbf{p} - \mathbf{p}'| = P_0)^2. \quad (2.20)$$

Under the conditions $p, p' \ll P_0$ and $\epsilon = cp \ll 3\mu c^2$, Eq. (2.20) becomes

$$p - p' = \frac{1}{2\mu c P_0^2} [P_0 \hat{\mathbf{m}} \cdot (p \hat{\mathbf{n}} - p' \hat{\mathbf{n}}')]^2 = \frac{p^2}{2\mu c} [\hat{\mathbf{m}} \cdot (\hat{\mathbf{n}} - \hat{\mathbf{n}}')]^2, \quad (2.21)$$

where $\hat{\mathbf{m}}, \hat{\mathbf{n}}$, and $\hat{\mathbf{n}}'$ are unit vectors directed along \mathbf{P}_0, \mathbf{p} , and \mathbf{p}' , respectively. Therefore, energy conservation implies $p \cong p'$. This means that the light particles do not change the magnitude of momentum but change its direction. Taking account of $p, p' \ll P_0$ and $P \cong P_0$, the matrix element [Eq. (2.1)] in second-order perturbation can be obtained as

$$\mathcal{H}_{IF} = \frac{P_0 f}{2\rho_0} \left[[(\hat{\mathbf{m}} \cdot \hat{\mathbf{n}}) + (\hat{\mathbf{m}} \cdot \hat{\mathbf{n}}')] (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}') + \frac{P_0}{\mu c} (\hat{\mathbf{m}} \cdot \hat{\mathbf{n}})^2 (\hat{\mathbf{m}} \cdot \hat{\mathbf{n}}')^2 + A \right], \quad (2.22)$$

$$A = \frac{\rho^2}{P_0 c} \left[\frac{\partial^2 \Delta}{\partial \rho^2} + \frac{1}{\mu} \left[\frac{\partial P_0}{\partial \rho} \right]^2 \right]. \quad (2.23)$$

Substituting Eqs. (2.22) and (2.23) into Eq. (2.1) and per-

forming the integration over p' , we obtain

$$d\sigma = \frac{P_0^2 p^3}{8\pi \hbar^3 \rho_0^2 c^2} \left[[(\hat{\mathbf{m}} \cdot \hat{\mathbf{n}}) + (\hat{\mathbf{m}} \cdot \hat{\mathbf{n}}')] (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}') + \frac{P_0}{\mu c} (\hat{\mathbf{m}} \cdot \hat{\mathbf{n}})^2 (\hat{\mathbf{m}} \cdot \hat{\mathbf{n}}')^2 + A \right]^2 d\theta. \quad (2.24)$$

Averaging Eq. (2.24) over all directions of roton momentum, we finally get

$$d\sigma = \frac{P_0^2 p^3}{8\pi \hbar^3 \rho_0^2 c^2} \left[(1 + \cos\psi) \cos^2\psi + \frac{1}{128} \left[\frac{P_0}{\mu c} \right]^2 (35 \cos^4\psi + 3 \sin^4\psi) + 30 \cos^2\psi \sin^2\psi + \frac{1}{4} \frac{P_0 A}{\mu c} (3 \cos^2\psi + \sin^2\psi) + A^2 \right] d\psi, \quad (2.25)$$

where ψ is the angle between the incident and scattered phonons (see Fig. 1).

We shall now examine roton-roton scattering. Since the character of interaction between rotons is not known, we may assume the short-range potential given by Landau and Khalatnikov¹³ and take the roton interaction to be a δ function potential.

$$V = V_0 \delta(\mathbf{r} - \mathbf{r}_1), \quad (2.26)$$

where V_0 is the interaction constant which can be determined experimentally by viscosity measurements, and \mathbf{r} and \mathbf{r}_1 are the radius vectors of rotons. We construct the symmetrized pairwise plane waves over incoming and outgoing rotons as

$$\begin{aligned} \psi(\mathbf{P}, \mathbf{P}_1) &= \frac{1}{\sqrt{2}} \left[\exp \left[\frac{i}{\hbar} (\mathbf{P} \cdot \mathbf{r} + \mathbf{P}_1 \cdot \mathbf{r}_1) \right] + \exp \left[\frac{i}{\hbar} (\mathbf{P} \cdot \mathbf{r}_1 + \mathbf{P}_1 \cdot \mathbf{r}) \right] \right], \\ \psi(\mathbf{P}', \mathbf{P}'_1) &= \frac{1}{\sqrt{2}} \left[\exp \left[\frac{i}{\hbar} (\mathbf{P}' \cdot \mathbf{r} + \mathbf{P}'_1 \cdot \mathbf{r}_1) \right] + \exp \left[\frac{i}{\hbar} (\mathbf{P}' \cdot \mathbf{r}_1 + \mathbf{P}'_1 \cdot \mathbf{r}) \right] \right]. \end{aligned} \quad (2.27)$$

The differential decay rate from before to after collision is

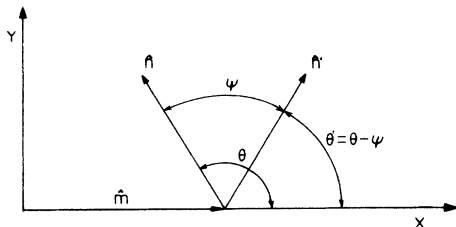


FIG. 1. Relation between incident and scattered phonons in two dimensions.

$$dw = \frac{2\pi}{\hbar} |V_{IF}|^2 \delta(E + E_1 - E' - E'_1) \frac{1}{(2\pi\hbar)^4} d\mathbf{P}' d\mathbf{P}'_1, \quad (2.28)$$

where the matrix element V_{IF} is

$$\begin{aligned} V_{IF} &= V_0 S^{-1/2} \int \psi^*(\mathbf{P}, \mathbf{P}_1) \delta(\mathbf{r} - \mathbf{r}_1) \psi(\mathbf{P}', \mathbf{P}'_1) d\mathbf{r} d\mathbf{r}_1 \\ &= 2V_0 S^{-1/2} \int \exp \left[\frac{i}{\hbar} (\mathbf{P}' + \mathbf{P}'_1 - \mathbf{P} - \mathbf{P}_1) \cdot \mathbf{r} \right] d\mathbf{r}. \end{aligned} \quad (2.29)$$

Performing integration over \mathbf{P}'_1 in Eq. (2.28), we obtain

$$dw = \frac{8\pi}{\hbar} |V_0|^2 \delta(E + E_1 - E' - E'_1) \frac{1}{(2\pi\hbar)^2} d\mathbf{P}'. \quad (2.30)$$

Dividing Eq. (2.30) by the relative velocity of the rotons given as

$$V = \left| \frac{\partial E}{\partial \mathbf{P}} - \frac{\partial E_1}{\partial \mathbf{P}_1} \right|, \quad (2.31)$$

we can obtain the differential scattering cross section.

As mentioned earlier in the discussion of the scattering of phonons by rotons, most rotons have a momentum close to P_0 . Thus the change of momentum after collision is very small in comparison with P_0 . Let us take θ as an angle between the incident rotons with momenta \mathbf{P} and \mathbf{P}_1 before collision and introduce the variable f (Fig. 2). Then the roton momenta after collision can be expressed as

$$\begin{aligned} P' &= P_0 + f_x \cos \frac{\theta}{2} + f_y \sin \frac{\theta}{2}, \\ P'_1 &= P_0 - f_x \cos \frac{\theta}{2} + f_y \sin \frac{\theta}{2}. \end{aligned} \quad (2.32)$$

Here, we have made use of $P \cong P_1 \cong P_0$ and $|f| \ll P_0$. From conservation of energy we have

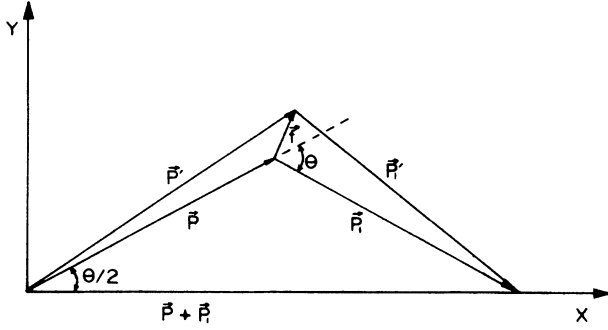


FIG. 2. Roton-roton scattering process in two dimensions.

$$f_x^2 \cos^2 \frac{\theta}{2} + f_y^2 \sin^2 \frac{\theta}{2} = \frac{1}{2}(P - P_0)^2 + \frac{1}{2}(P_1 - P_0)^2. \quad (2.33)$$

To obtain the total scattering cross section, we integrate Eq. (2.30) together with Eqs. (2.31) and (2.32) over the momentum of the scattered particles to get the total scattering cross section

$$\sigma = \frac{4\mu |V_0|^2}{\left| \frac{\partial E}{\partial \mathbf{P}} - \frac{\partial E_1}{\partial \mathbf{P}_1} \right| \hbar^3 \sin \theta}, \quad (2.34)$$

and the average collision time between rotons becomes

$$\begin{aligned} \frac{n'}{k_B T} \left\{ \left[\frac{1}{T} \frac{\partial T}{\partial \rho} \epsilon - \frac{\partial \epsilon}{\partial \rho} \right] \nabla \cdot (\mathbf{j} - \rho \mathbf{v}_n) + \left[\frac{1}{T} \left(\frac{\partial T}{\partial \rho} \rho + \frac{\partial T}{\partial S} S \right) \epsilon - \frac{\partial \epsilon}{\partial \rho} \right] \nabla \cdot \mathbf{v}_n \right. \\ \left. + \frac{\nabla T}{T} \cdot \left[\mathbf{p} \frac{ST}{\rho_n} - \epsilon \frac{\partial \epsilon}{\partial \mathbf{p}} \right] - \frac{\partial \epsilon}{\partial \mathbf{p}} \nabla (\mathbf{p} \cdot \mathbf{v}_n) \right\} = J(n_1), \end{aligned} \quad (3.2)$$

where $n' = -n(n+1)$.

When there exists a temperature gradient in superfluid ^4He , there is not only transport of heat but also an irreversible heat flow which can be expressed by the coefficient of thermal conduction given as

$$\mathbf{q} = -\kappa \nabla T. \quad (3.3)$$

Keeping only the temperature gradient in Eq. (3.2) gives

$$\frac{n'}{k_B T^2} \frac{\partial T}{\partial x} (\cos \theta) \left[p \frac{ST}{\rho_n} - \epsilon \frac{\partial \epsilon}{\partial \rho} \right] = J(n_1), \quad (3.4)$$

where θ is the angle between \mathbf{p} and ∇T . The phenomena associated with thermal conduction in bulk liquid ^4He have aspects in common with thermal transport properties of ordinary classical liquids. However, there are specific features of thermal transport which are connected with the unusual elementary excitations of liquid ^4He .

$$\frac{1}{t_r} = \sigma \left| \frac{\partial E}{\partial \mathbf{P}} - \frac{\partial E_1}{\partial \mathbf{P}_1} \right| N_r = \frac{4\mu |V_0|^2}{\hbar^3 \sin \theta} N_r, \quad (2.35)$$

where N_r is number of rotons per unit area given by

$$N_r = \left(\frac{\mu k_B T}{2\pi} \right)^{1/2} \frac{P_0}{\hbar^2} e^{-\Delta/k_B T}. \quad (2.36)$$

III. COEFFICIENT OF THERMAL CONDUCTIVITY

The equilibrium distribution function n_0 of excitation satisfies the kinetic equation

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \mathbf{r}} \cdot \frac{\partial \mathcal{H}}{\partial \mathbf{p}} - \frac{\partial n}{\partial \mathbf{p}} \cdot \frac{\partial \mathcal{H}}{\partial \mathbf{r}} = J(n), \quad (3.1)$$

with vanishing collision integral. When equilibrium is disturbed, we assume that the nonequilibrium distribution function n deviates slightly from equilibrium. The small deviation can be determined by the first derivatives of the velocities $\mathbf{v}_n, \mathbf{v}_s$ and the thermodynamic variables, since function can be written as $n = n_0 + n_1$, where $n_1 \ll n_0$. Putting n into the left-hand side in Eq. (3.1), it is sufficient to keep only the differentiation of n_0 since the derivative of n_1 makes higher derivatives which can be neglected. For the collision integral on the right-hand side we only keep the terms linear in n_1 . With the help of the continuity equation, entropy equation, and superfluid equation of motion, we can write the kinetic equation [Eq. (3.1)] as

The left-hand side of Eq. (3.4) is always zero for a pure phonon gas, and thus the corresponding thermal conductivity vanishes.

The thermal conductivity κ consists of two parts, i.e., the phonon κ_{ph} and the roton κ_r :

$$\kappa = \kappa_{\text{ph}} + \kappa_r. \quad (3.5)$$

Let us first consider the roton part. This is determined by the roton-roton scattering process. As mentioned in Sec. II, the roton-roton interaction is not well known, and we thus obtained the average collision time t_r under the assumption of a δ function potential. Since we need only the temperature dependence of κ_r , we replace the collision integral by

$$J(n) \rightarrow -\frac{n - n_0}{t_r}. \quad (3.6)$$

Substitution of Eq. (3.6) in Eq. (3.4) yields

$$n - n_0 = -\frac{n'}{k_B T^2} \nabla T \cdot \left[\mathbf{P} \frac{ST}{\rho_n} - \varepsilon \frac{\partial \varepsilon}{\partial \mathbf{P}} \right] t_r. \quad (3.7)$$

Substituting Eq. (3.7) into the expression for the energy flow

$$\mathbf{q} = \int \frac{\partial \varepsilon}{\partial \mathbf{P}} \varepsilon(\mathbf{P})(n - n_0) \frac{1}{(2\pi\hbar)^2} d\mathbf{P}, \quad (3.8)$$

and comparing this result with Eq. (3.3), we get

$$\kappa_r = t_r \frac{1}{2k_B T^2} \int n' \varepsilon \frac{\partial \varepsilon}{\partial \mathbf{P}} \cdot \left[\mathbf{P} \frac{ST}{\rho_n} - \frac{\partial \varepsilon}{\partial \mathbf{P}} \right] \frac{1}{(2\pi\hbar)^2} d\mathbf{P}. \quad (3.9)$$

With the help of $n' = -n$ and Eq. (1.2), we finally obtain the roton part of thermal conductivity as

$$\begin{aligned} \kappa_r(T) = \frac{t_r \Delta^2 N_r}{2\mu T} \left\{ 1 + \frac{3k_B T}{\Delta} + \frac{15}{4} \left[\frac{k_B T}{\Delta} \right]^2 \right. \\ \left. - \frac{2\mu S}{\rho_n k_B} \left[\frac{k_B T}{\Delta} + \frac{3}{2} \left[\frac{k_B T}{\Delta} \right]^2 \right] \right\}. \end{aligned} \quad (3.10)$$

According to the scattering processes in Sec. II, the collision integral $J(n)$ becomes

$$J(n) = J_{3PP}(n) + J_{4PP}(n) + J_{5PP}(n) + J_{ph-r}(n). \quad (3.11)$$

The 4PP do not change the total number of phonons but have a characteristic temperature T' in a given direction, which is different from the temperature T in the equilibrium state. The law of energy conservation yields

$$\int J_{4PP}(n) p dp = 0. \quad (3.12)$$

The total number of phonons traveling in a given direction is changed by small-angle 3PP and 5PP. Therefore, the distribution function, which not only depends on temperature T' but also the chemical potential α' , can be written as

$$n = \exp[(\alpha' + pc/k_B T') = 1]^{-1}. \quad (3.13)$$

Expanding Eq. (3.13) as a function of $T' - T$, we can express Eq. (3.13) in terms of equilibrium distribution function

$$\delta n = n - n_0 = -n_0(n_0 + 1) \left[\alpha' - \frac{PC}{k_B T} \frac{T' - T}{T} \right]. \quad (3.14)$$

Since the left-hand side of Eq. (3.4) is involved in θ , α' and $T' - T$ depend naturally on θ . To solve Eq. (3.4) we should take the forms

$$\alpha' = \alpha \cos\theta, \quad (T' - T)/T = \beta \cos\theta, \quad (3.15)$$

where α and β are constants to be determined by the kinetic equation. Considering the conservation of phonon numbers in a given direction and conservation of energy, the integrations

$$\int J_{3PP}(n) \varepsilon p dp, \quad \int J_{4PP}(n) \varepsilon p dp, \quad \int J_{5PP}(n) \varepsilon p dp$$

vanish and Eq. (3.4) becomes

$$\begin{aligned} \int \frac{n'}{k_B T} (\cos\theta) \frac{\partial T}{\partial x} \left[p \frac{ST}{\rho_n} - \varepsilon \frac{\partial \varepsilon}{\partial p} \right] p dp \\ = \int [J_{3PP}(n) + J_{5PP}(n) + J_{ph-r}(n)] p dp, \end{aligned} \quad (3.16)$$

$$\begin{aligned} \int \frac{n'}{k_B T} (\cos\theta) \frac{\partial T}{\partial x} \left[p \frac{ST}{\rho_n} - \varepsilon \frac{\partial \varepsilon}{\partial p} \right] \varepsilon p dp \\ = \int J_{ph}(n) \varepsilon p dp. \end{aligned} \quad (3.17)$$

From Eqs. (2.5) and (2.6) the collision integral for the 3PP becomes

$$\begin{aligned} J_{3PP}(n) = -\frac{\pi c_0}{2\hbar\rho_0} (u+1)^2 \int p_1 p_2 p \delta n(n_{p_{10}} - n_{p_{20}}) \\ \times \delta(\varepsilon_F - \varepsilon_I) \frac{1}{(2\pi\hbar)^2} d\mathbf{p}_2, \end{aligned} \quad (3.18)$$

where δn is equal to $n_p - n_{p_0}$, and n_{p_0} represents the equilibrium distribution functions for the phonons with momentum p . Making use of Eqs. (2.12) and (2.13), the collision integral for the 5PP can be written as

$$\int J_{5PP}(n) p dp = 2\pi\hbar^2 k_B T \alpha' \Gamma_{ph}. \quad (3.19)$$

We now evaluate the collision integral in the scattering of phonons by rotons. When a phonon with momentum \mathbf{p} changes to momentum \mathbf{p}' directed at angle θ' after collision by a roton, the probability per unit length that a particle undergoes collision is $N_r d\sigma$, where N_r is the roton distribution given by Eq. (2.36) and $d\sigma$ is given by Eq. (2.25). Then the collision integral $J_{ph-r}(n)$ can be written as

$$J_{ph-r}(n) = -N_r C_0 \int [n(p, \theta, T) - n(p', \theta', T')] d\sigma. \quad (3.20)$$

Here, θ is the angle of the incident phonon with momentum \mathbf{p} with respect to the x axis. Using Eqs. (3.14) and (3.15) for the difference between distributions, we obtain

$$\begin{aligned} J_{ph-r}(n) = (\cos\theta) n_0 (n_0 + 1) \left[\alpha - \beta \frac{pc}{k_B T} \right] N_r \frac{P_0^2 \rho^3}{8\hbar^3 \rho_0^2 c_0} \\ \times \left[\frac{1}{4} + \frac{9}{32} \left[\frac{P_0}{\mu c} \right]^2 + \frac{P_0}{\mu c} A + 2A^2 \right]. \end{aligned} \quad (3.21)$$

Substituting Eqs. (3.18) and (3.21) together with Eq. (3.19) and $n' = -n_0(n_0 + 1)$ in Eqs. (3.16) and (3.17), and performing the momentum integral, we obtain

$$\begin{aligned} \frac{1}{T} \frac{\partial T}{\partial x} \left[C - \frac{ST}{\rho_n C} \right] \frac{2!\xi(2)6!\xi(6)}{3!\xi(3)4!\xi(4)} \\ = \frac{1}{\tau_{ph-r}} \left[\alpha - \beta \frac{5!\xi(5)}{4!\xi(4)} \right] + \frac{\alpha}{\tau_{3PP}} + \frac{\alpha}{\tau_{5PP}}, \end{aligned} \quad (3.22)$$

and

$$\frac{1}{T} \frac{\partial T}{\partial x} \left[C - \frac{ST}{\rho_n C} \right] \frac{6! \zeta(6)}{5! \zeta(5)} = \frac{1}{\tau_{\text{ph-r}}} \left[\alpha - \beta \frac{6! \zeta(6)}{5! \zeta(5)} \right]. \quad (3.23)$$

Here $\tau_{3\text{PP}}$, $\tau_{5\text{PP}}$, and $\tau_{\text{ph-r}}$ are the characteristic times (see the Appendix), which characterize each collision process, given by

$$\frac{1}{\tau_{\text{ph-r}}} = \frac{6! \zeta(6)}{3! \zeta(3)} \frac{P_0^2 (k_B T)^3}{8 \hbar^3 \rho^2 C^4} \times N_r \left[\frac{1}{4} + \frac{9}{32} \left(\frac{P_0}{\mu c} \right)^2 + \left(\frac{P_0}{\mu c} A + 2A^2 \right) \right], \quad (3.24)$$

$$\frac{1}{\tau_{5\text{PP}}(T)} = \frac{6! \zeta(6)}{3! \zeta(3) 4! \zeta(3)} \frac{2\pi \hbar^2 C^2 a}{k_B} T^6, \quad (3.25)$$

and

$$\frac{1}{\tau_{3\text{PP}}(T)} = \frac{2! \zeta(2) 6! \zeta(6)}{4! \zeta(4)} \frac{(u+1)^2 k_B^4}{8\pi \hbar^3 \rho c^4} T^4. \quad (3.26)$$

Solving Eqs. (3.22) and (3.23) for α and β , we obtain

$$\alpha = \left[\frac{8.071}{\frac{0.186}{\tau_{\text{ph-r}}} + \frac{1}{\tau_{5\text{PP}}} + \frac{1}{\tau_{3\text{PP}}}} \right] \frac{1}{T} \frac{\partial T}{\partial x} \left[C - \frac{ST}{\rho_n C} \right] = A \frac{1}{T} \frac{\partial T}{\partial x} \left[C - \frac{ST}{\rho_n C} \right], \quad (3.27)$$

$$\beta = \left[-\tau_{\text{ph-r}} + \frac{1.372}{\frac{0.186}{\tau_{\text{ph-r}}} + \frac{1}{\tau_{5\text{PP}}} + \frac{1}{\tau_{3\text{PP}}}} \right] \frac{1}{T} \frac{\partial T}{\partial x} \left[C - \frac{ST}{\rho_n C} \right] = B \frac{1}{T} \frac{\partial T}{\partial x} \left[C - \frac{ST}{\rho_n C} \right], \quad (3.28)$$

where we introduced A and B for simplification. Substitution of Eqs. (3.27) and (3.28) in Eq. (3.14) yields

$$n - n_0 = -n_0(n_0 + 1)(\cos\theta) \left[A - B \frac{PC}{k_B T} \right] \times \frac{1}{T} \frac{\partial T}{\partial x} \left[C - \frac{ST}{\rho_n C} \right], \quad (3.29)$$

and the energy flux [Eq. (3.8)] for the phonon-roton process can be evaluated as

$$q = \int \frac{\partial \epsilon}{\partial p} \epsilon (n - n_0) (\cos\theta) \frac{1}{(2\pi \hbar)^2} d\mathbf{p} = -\frac{k_B^3 T^2}{4\pi \hbar^2 C} \left[C - \frac{ST}{\rho_n C} \right] [3.290A - 7.212B] \frac{\partial T}{\partial x}. \quad (3.30)$$

Comparison of Eq. (3.30) with Eq. (3.3) gives the coefficient of thermal conduction:

$$\kappa_{\text{ph}}(T) = \frac{1.803 k_B^3 T^2}{\pi \hbar^2 C^2} \left[C^2 - \frac{ST}{\rho_n} \right] \times \left[\tau_{\text{ph-r}} + \frac{2.310}{\frac{0.186}{\tau_{\text{ph-r}}} + \frac{1}{\tau_{5\text{PP}}} + \frac{1}{\tau_{3\text{PP}}}} \right]. \quad (3.31)$$

We note that the coefficient of the thermal conduction in bulk liquid helium is given by

$$\kappa(T) = 2 \times 10^3 \frac{1}{T} + T^{-3/2} e^{\Delta/k_B T} \left[1 - \frac{ST}{\rho_n C^2} \right] \times \begin{cases} 84 \frac{1 + 0.75 \tau_{\text{ph-r}}/\tau_{5\text{PP}}}{1 + 8 \tau_{\text{ph-r}}/\tau_{5\text{PP}}}, & T > 0.9 \text{ K} \\ 7.8, & T < 0.9 \text{ K}. \end{cases} \quad (3.32)$$

IV. VISCOSITY

In this section we investigate the first viscosity through calculations similar to those used above for the thermal conductivity. Equation (3.4) vanishes for the case of a pure phonon gas, and thus the corresponding coefficient $\kappa(T)$ becomes zero. However, the kinetic equation

$$\frac{\partial n}{\partial t} + \mathbf{v} \cdot \nabla n = J(n) \quad (4.1)$$

does not vanish for the pure phonon gas near zero temperature. Therefore, we should consider the contribution of the pure phonon gas to viscosity.

Let us consider the macroscopic motion of liquid helium (which does not depend explicitly on time) with velocity \mathbf{U} and the gradient of \mathbf{U} directed along the x axis. Then Eq. (4.1) becomes

$$\mathbf{v} \cdot \nabla n = v_x \frac{\partial n}{\partial x} = J(n). \quad (4.2)$$

The equilibrium distribution function of rotons in liquid helium with velocity \mathbf{U} can be expressed as

$$n_0 = \exp \left[-\frac{\Delta}{k_B T} - \frac{(P - P_0)^2}{2\mu k_B T} + \frac{\mathbf{P} \cdot \mathbf{U}}{k_B T} \right], \quad (4.3)$$

where we used Eq. (1.2). Substituting $n \cong n_0$ and the roton velocity $v = (P - P_0)/\mu$ in Eq. (4.2), we get

$$n_0 \frac{P_0(P - P_0)}{\mu k_B T} \frac{\partial U}{\partial x} (\cos\theta)(\sin\theta) = J_3(n), \quad (4.4)$$

where θ is the angle between the roton momentum P and the x axis. The collision integral can be replaced by Eq. (3.6) to give

$$n - n_0 = -t_r \frac{(P - P_0)P_0}{\mu k_B T} n_0 (\cos\theta)(\sin\theta) \frac{\partial U}{\partial x}. \quad (4.5)$$

Substituting Eq. (4.5) into the following stress tensor of 2D liquid helium

$$\sigma_{xy} = \int P_y v_x (n - n_0) \frac{1}{(2\pi\hbar)^2} d\mathbf{P}, \quad (4.6)$$

and performing the integral over the momentum space, we arrive at the expression

$$\sigma_{xy} = -t_r \frac{P_0^2}{8\mu} N_r \frac{\partial U}{\partial x}. \quad (4.7)$$

Comparing Eq. (4.7) with the general expression of viscosity

$$\sigma_{xy} = -\eta_r \frac{\partial U}{\partial x}, \quad (4.8)$$

we obtain the coefficient of viscosity for the roton part as

$$\eta_r = \frac{P_0^2}{8\mu} t_r N_r. \quad (4.9)$$

The equilibrium distribution function of the phonon gas is

$$n_0 = \left[\exp \left[\frac{cp - \mathbf{U} \cdot \mathbf{p}}{k_B T} \right] - 1 \right]^{-1}. \quad (4.10)$$

Let us assume the distribution function to deviate slightly from n_0 , i.e., $n = n_0 + \delta n$, which satisfies Eq. (4.1). Substituting Eq. (4.10) into Eq. (4.2) we get

$$n_0(n_0 + 1) \frac{cp}{k_B T} \frac{\partial U}{\partial x} (\cos\theta)(\sin\theta) = J(n). \quad (4.11)$$

Solving Eq. (4.11) we can obtain the phonon part of the viscosity. It is necessary to consider $J(n)$ in the various collision processes. Then the collision integral $J(n)$ in Eq. (4.11) can be replaced by Eq. (3.10). Through the same processes that we argued in Sec. III, we obtain the following two equations:

$$\begin{aligned} \int n_0(n_0 + 1) \frac{cp}{k_B T} \frac{\partial U}{\partial x} (\cos\theta)(\sin\theta) p dp \\ = \int [J_{3PP}(n) + J_{5PP}(n) + J'_{ph-r}(n)] p dp, \end{aligned} \quad (4.12)$$

$$\begin{aligned} \int n_0(n_0 + 1) \frac{cp}{k_B T} \frac{\partial U}{\partial x} (\cos\theta)(\sin\theta) \epsilon(p) p dp \\ = \int J'_{ph-r}(n) \epsilon(p) p dp. \end{aligned} \quad (4.13)$$

We can express the distribution function by Eq. (3.14), except that now the dependence of α' and $(T' - T)/T$ on the angle θ is given by

$$\alpha' = \alpha(\cos\theta)(\sin\theta), \quad (T' - T)/T = \beta(\cos\theta)(\sin\theta) \quad (4.14)$$

instead of Eq. (3.15).

In Eq. (4.12) the collision integrals $J_{3PP}(n)$ and $J_{5PP}(n)$ are given by Eqs. (3.18) and (3.19), respectively. Using Eqs. (3.15)–(3.20), we obtain the collision integral $J'_{ph-r}(n)$:

$$\begin{aligned} J'_{ph-r}(n) = (\cos\theta)(\sin\theta) N_r C_0 n_0 (n_0 + 1) \left[\alpha - \beta \frac{cp}{k_B T} \right] \\ \times \frac{P_0^2 p^3}{8\hbar^3 \rho^2 C^2} \left[\frac{1}{2} + \frac{5}{32} \left[\frac{P_0}{\mu C} \right]^2 \right. \\ \left. + \frac{3}{4} A \left[\frac{P_0}{\mu C} \right] + 2A^2 \right]. \end{aligned} \quad (4.15)$$

Here, we note that Eq. (4.15) is slightly different from Eq. (3.21), because Eq. (4.14) contains an extra $(\sin\theta)$ term. With the use of similar calculations for Eqs. (3.16)–(3.26), we obtain α , β , and $\tau'_{ph^{-1}}$ as

$$\alpha = \left[\frac{8.071}{\frac{0.186}{\tau'_{ph-r}} + \frac{1}{\tau_{5PP}} + \frac{1}{\tau_{3PP}}} \right] \frac{\partial U}{\partial x} = A' \frac{\partial U}{\partial x} \beta, \quad (4.16)$$

$$\beta = \left[-\tau'_{ph-r} + \frac{1.372}{\frac{0.186}{\tau'_{ph-r}} + \frac{1}{\tau_{5PP}} + \frac{1}{\tau_{3PP}}} \right] \frac{\partial U}{\partial x} = B' \frac{\partial U}{\partial x}, \quad (4.17)$$

and

$$\begin{aligned} \tau'_{ph-r} = \frac{6!\zeta(6)}{3!\zeta(3)} \frac{P_0^2 k_B^3 T^2}{8\hbar^3 \rho^2 C^4} N_r \left[\frac{1}{2} + \frac{5}{32} \left[\frac{P_0}{\mu C} \right]^2 \right. \\ \left. + \frac{3}{4} \left[\frac{P_0}{\mu C} \right] + 2A^2 \right], \end{aligned} \quad (4.18)$$

where Eq. (4.18) is not equal to Eq. (3.24). Substitution of Eqs. (4.16) and (4.17) in Eq. (3.14) yields

$$\delta n = -n_0(n_0 + 1) \left[A' - B' \frac{pc}{k_B T} \right] \frac{\partial U}{\partial x}. \quad (4.19)$$

Combining Eq. (4.19) with Eq. (4.16) and integrating over momentum space gives

$$\sigma_{xy} = -\frac{C}{16\pi\hbar^2} \left[\frac{k_B T}{C} \right]^3 [3.290 A' - 7.212 B'] \frac{\partial U}{\partial x}. \quad (4.20)$$

Comparing Eq. (4.20) with Eq. (4.8), we get the phonon part of the coefficient of the first viscosity:

$$\begin{aligned} \eta_{ph}(t) = \frac{7.212(k_B T)^3}{16\pi\hbar^2 C^2} \\ \times \left[\tau'_{ph-r} + \frac{2.310}{\frac{0.186}{\tau'_{ph-r}} + \frac{1}{\tau_{5PP}} + \frac{1}{\tau_{3PP}}} \right]. \end{aligned} \quad (4.21)$$

From the theory and experiments on attenuation of ultrasonic sound waves,¹⁴ we can confirm that at near-zero temperature the contribution of 3PP to the viscosity plays a main role, and 4PP establish only the equilibrium of energy. Therefore, we consider the 3PP contribution separately from other contributions. The kinetic equation for 3PP in Eq. (4.11) becomes

$$n_0(n_0 + 1) \frac{cp}{k_B T} \frac{\partial U}{\partial x} (\cos\theta)(\sin\theta) = J_{3PP}(n). \quad (4.22)$$

Since the distribution function of phonons depends on the chemical potential α , Eqs. (3.14) and (3.15) become

$$\begin{aligned} \delta n &= n - n_0 = -n_0(n_0 + 1)\alpha', \\ \alpha' &= \alpha(\cos\theta)(\sin\theta). \end{aligned} \quad (4.23)$$

With a similar calculation for Eqs. (3.16)–(3.26), we obtain the stress tensor

$$\eta_{ph}(T) = \begin{cases} 3.75 \times 10^{-8} T^{-1/2} e^{\Delta/k_B T} \frac{1 + 0.75 \tau'_{ph-r}/\tau_{ph}}{1 + 8 \tau'_{ph-r}/\tau_{ph}}, & T > 0.9 \text{ K} \\ 3.5 \times 10^{-9} T^{-1/2} e^{\Delta/k_B T} (1 + 2.15 \times 10^{-5} T^{9/2} e^{\Delta/k_B T})^{-1}, & T < 0.9 \text{ K}. \end{cases} \quad (4.26)$$

V. RESULTS AND DISCUSSION

In the previous sections we have evaluated the scattering cross sections and characteristic times for various interactions of the elementary excitations. Using these results we have obtained the coefficients of thermal conduction and first viscosity. To investigate the temperature variation of the coefficients $\kappa(T)$ and $\eta(T)$, we adopt the parameters which are determined from the specific-heat data of Bretz *et al.*¹⁵ The parameters are listed in Table I. With this choice we obtained $c = 84.06$ m/s, which is smaller than the value 157 m/s of Hipólito and Lobo,¹⁶ but is very close to the experimental value of 76 ± 2 m/s of Washburn *et al.*¹⁷ The parameters u , A , and a in Eqs. (2.9), (2.23), and (2.16) for two dimensions are assumed to be 1.8,¹⁸ 0.425,⁹ and 1.0×10^{43} (Ref. 13) used by previous workers for the bulk case.

$$\sigma_{xy} = - \frac{[2!\zeta(2)]^2 6!\zeta(6)}{3!4!\zeta(3)\zeta(4)} \frac{k_B^3 T^3}{16\pi\hbar^2 C^2} \tau_{3PP} \frac{\partial U}{\partial x}. \quad (4.24)$$

From comparison of Eq. (4.24) with Eq. (4.8), we obtain the phonon part of the coefficient η_{ph} near zero temperature as

$$\eta_{ph}(T) = \frac{2.645}{\pi^2 \hbar^2 C^2} k_B^3 T^3 \tau_{3PP}. \quad (4.25)$$

We note that the coefficient of the first viscosity in bulk liquid helium is given by

The coefficient of the thermal conduction is given by Eqs. (3.10) and (3.31). We can confirm easily that Eq. (3.4) vanishes by considering only a pure phonon gas because of the peculiar excitation of liquid helium. Therefore, the thermal conduction depends on the interactions between excitations, and it is necessary to take into account the phonon-roton and roton-roton interactions.

The roton part $\kappa_r(T)$ of the thermal conductivity is proportional to the average collision time t_r . When $\theta = 0$, t_r is zero, and for small-angle scattering t_r becomes very small. Since Eq. (5.10) is involved in the unknown interaction potential constant V_0 and we have only to know the magnitude of the temperature dependence for κ_r , we may take the maximum t_r . Taking $\theta = \pi/2$ and the numerical parameters in Table I, $\kappa_r(T)$ can be expressed as

$$\kappa_r(T) = \frac{0.191 \times 10^{-66}}{|V_0|^2} \left[\frac{1}{T} + 0.221T + 0.728 + 4.743 \left(\frac{1}{T} + 0.362 \right) \left[\frac{1 + 28.02 \left[\frac{3}{2} + \frac{4.12}{T} \right] T^{-3/2} e^{-(4.12)/T}}{1 + 4632.89 T^{-7/2} e^{-(4.12)/T}} \right] \right]. \quad (5.1)$$

Figure 3 illustrates κ_T as a function of temperature. The coefficient increases slowly as temperature decreases and is proportional to T^{-1} .

The characteristic times which determine the coefficient $\kappa_{ph}(T)$ are given by Eqs. (3.24)–(3.26). τ_{ph-r}^{-1} has the temperature dependence of $T^{7/2} e^{-\Delta/k_B T}$, which is one power lower than $T^{9/2} e^{-\Delta/k_B T}$ in bulk liquid helium. This is due to dimensionality, where τ_{5PP}^{-1} has the temperature variation of T^6 , which is a lower power than that of three dimensions. The inverse of τ_{3PP} is proportional to T^4 , which originates from the anomalous excita-

tion spectrum. Figure 4 is the temperature variation of the characteristic times. τ_{5PP}^{-1} is comparable with τ_{ph}^{-1} near about 0.8 K, and thus we can conclude that above 0.8 K the 5PP and the scattering of phonons by rotors will mainly contribute to the thermal conduction. Since 5PP appear in the inelastic collision process in the tem-

TABLE I. Theoretical parameters.

ρ (\AA^{-2})	Δ/k_B (K)	q_0 (\AA^{-1})	μ	C (m/s)
2.79×10^{-2}	4.12	1.02	$0.75 m_{\text{He}}$	164.4

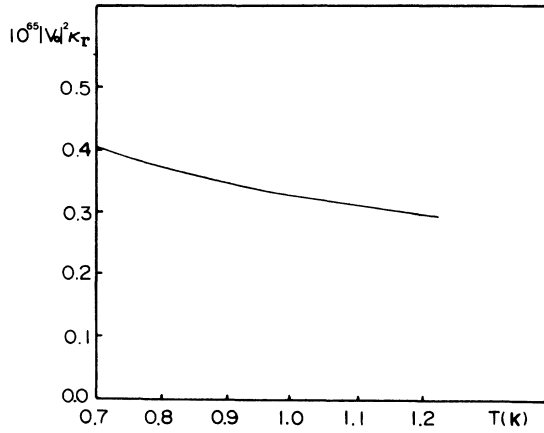


FIG. 3. Temperature variation of the roton part of $\kappa(T)$.

perature range of roton-roton collisions, we should not take into account this contribution to $\kappa_{ph}(T)$ below 0.8 K, and should consider only the phonon-roton collision process. Therefore, in accordance with the regions of temperature, we can express Eq. (3.31) as

$$\kappa_{ph}(T) = \frac{1.803k_B^3 T^2}{\pi \hbar^2 c^2} \left[C^2 - \frac{ST}{\rho_n} \right] \times \tau_{ph-r} \left[\frac{13.419 + 5.376\tau_{ph-r}/\tau_{5PP}}{1 + 5.376\tau_{ph-r}/\tau_{5PP}} \right], \quad T > 0.8 \text{ K} \quad (5.2)$$

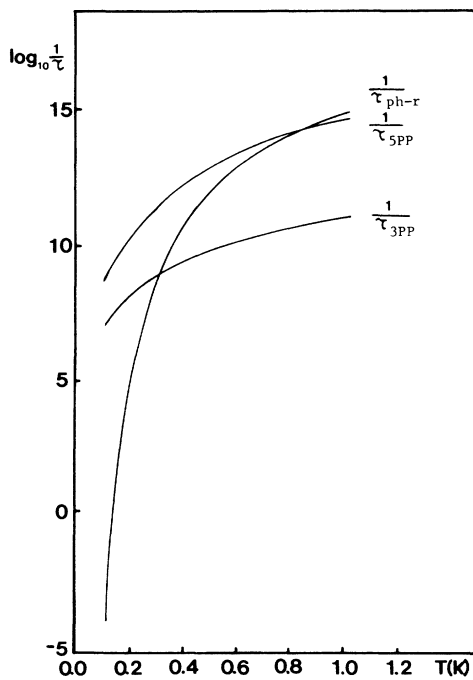


FIG. 4. Temperature variation of the various characteristic times.

and

$$\begin{aligned} \kappa_{ph}(T) &= \frac{1.803k_B^3 T^2}{\pi \hbar^2 c^2} \left[C^2 - \frac{ST}{\rho_n} \right] \tau_{ph-r} \\ &= 2.36 \times 10^{-11} \left[1 - \frac{ST}{\rho_n C^2} \right] T^{-3/2} e^{4.12/T}, \end{aligned} \quad T < 0.8 \text{ K} . \quad (5.3)$$

Since the temperature-dependent term within the parenthesis of Eq. (5.3) is much smaller than unity for temperatures below about 0.3 K, it can be neglected. Then we have

$$\kappa_{ph}(T) = 1.093 \times 10^{-7} T^{-5}, \quad T \lesssim 0.3 \text{ K} . \quad (5.4)$$

In Fig. 5 the coefficient $\kappa(T)$ is plotted as a function of temperature based on Eqs. (5.2)–(5.4). As the temperature decreases ($\lesssim 0.8$ K), Eqs. (5.3) plays a dominant role and gives the exponential increase. As yet lower temperatures (below about 0.3 K), Eq. (5.4) takes part in the thermal conductivity, and K_{ph} increases rapidly and diverges as temperature tends to zero.

Comparison of Eq. (5.3) with Eq. (3.32) reveals that the second term in Eq. (3.32) has the same factor $T^{-3/2} e^{\Delta/k_B T}$. This is due to the fact that τ_{ph}^{-1} and the thermodynamic functions are one power lower in temperature than those of three dimensions, and thus their ratios have the same temperature-dependent factor.

For viscosity we may give similar arguments as those for thermal conductivity. However, the main difference

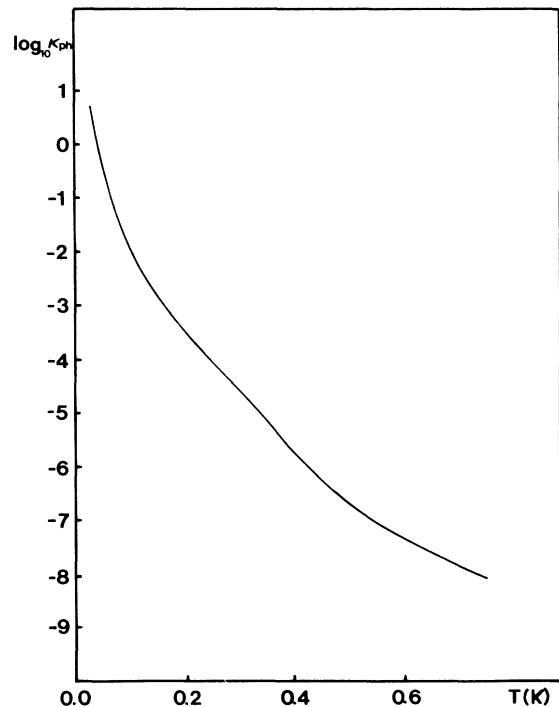


FIG. 5. The coefficient $\kappa(T)$ of thermal conduction as a function of temperatures.

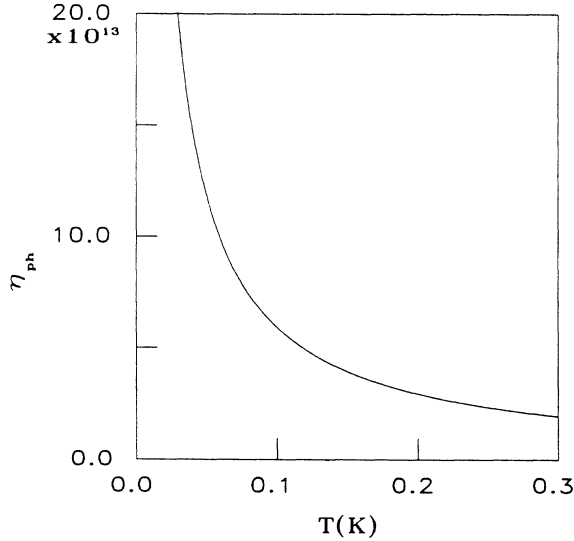


FIG. 6. The coefficient $\eta(T)$ of first viscosity vs temperature.

from thermal conduction is that the kinetic equation [Eq. (4.1)] does not vanish for a pure phonon gas near zero temperature. Therefore, we have treated separately the contribution from the pure phonon gas to viscosity [Eq. (4.25)].

Substitution of Eq. (2.35) in Eq. (4.9) and the choice of $\theta = \pi/2$ gives the roton part of the first viscosity as

$$\eta_r(T) = \frac{\hbar^3 P_0^2}{32\mu^2 |V_0|^2}, \quad (5.5)$$

which is independent of temperature. Since Eq. (4.3) decreases exponentially with temperature and the mean free path of roton increases at about same rate, the viscosity becomes independent of temperature.

At temperatures above about 0.8 K, the main contribution is due to 5PP and the phonon-roton collision process, and thus Eq. (4.21) becomes

$$\eta_{ph}(T) = \frac{7.212}{16\pi\hbar^2 C_0^2} (k_B T)^3 \tau'_{ph-r} \times \left[\frac{13.419 + 5.376\tau'_{ph-r}/\tau_{5PP}}{1 + 5.376\tau'_{ph-r}/\tau_{5PP}} \right], \quad T > 0.8 \text{ K}. \quad (5.6)$$

At temperatures below 0.8 K, we consider only the phonon-roton collision process and thus have

$$\eta_{ph}(T) = \frac{2.645}{\pi\hbar^2 C_0^2} (k_B T)^3 \tau'_{ph-r} = 2.17 \times 10^{-20} T^{-1/2} e^{4.12/T}. \quad (5.7)$$

For temperatures below about 0.3 K, we have another temperature dependence given by Eq. (4.25):

$$\eta_{ph}(T) = 5.92 \times 10^{-14} T^{-1}. \quad (5.8)$$

Figure 6 illustrates the coefficient $\eta_{ph}(T)$ of the first viscosity as a function of temperature based on Eq. (5.8). At temperatures below about 0.8 K, η_{ph} increases exponentially as temperature decreases. However, as temperature approaches absolute zero, the roton density becomes zero, and thus the contribution from the roton-phonon scattering to viscosity vanishes so that the main contribution comes from the 3PP. Therefore, the temperature dependence is changed from $T^{-1/2} e^{\Delta/k_B T}$ to T^{-1} . The coefficient increases slowly, and near zero temperature it increases very rapidly and finally diverges.

In conclusion, we remark that the behavior of the coefficients of the thermal conductivity and first viscosity are very much like that of the bulk case. At below 0.3 K, the contribution to the coefficient of first viscosity is due to the 3PP, which is shown to have a T^{-1} dependence.

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APPENDIX

Since the calculations for Eqs. (3.24)–(3.26) are very similar, we shall derive only Eq. (3.24). The collision integral [Eq. (3.20)] can be expressed as

$$J_{ph-r}(n) = N_r C \int [(n' - n_0) - (n - n_0)] d\sigma. \quad (A1)$$

Making use of Eq. (3.14), Eq. (A1) becomes

$$J_{ph-r}(n) = N_r C \int n_0(n_0 + 1) \beta \frac{pc}{k_B T} (\cos\theta' - \cos\theta) d\sigma. \quad (A2)$$

In Fig. 1 the relation between angles is given by

$$(\cos\theta') - (\cos\theta) = (\cos\theta) [(\cos\psi) - 1 + (\tan\theta)(\sin\varphi)]. \quad (A3)$$

Substituting Eqs. (2.25) and (A3) in (A2) and performing the integration over ψ , we get

$$J_{ph-r}(n) = N_r C (\cos\theta) \left[-\beta \frac{pc}{k_B T} \right] n_0(n_0 + 1) \frac{P_0^2 \rho^3}{8\hbar^3 \rho^2 C^2} \times \left[\frac{1}{4} + \frac{9}{32} \left[\frac{P_0}{\mu C} \right] + \frac{P_0 A}{\mu c} + 2A^2 \right]. \quad (A4)$$

The collision integral may be written as

$$J_{ph-r}(n) = -\frac{n - n_0}{\tau_{ph}(p)}. \quad (A5)$$

To obtain the temperature dependence of $\tau_{ph-r}(T)$, we calculate the following integral:

$$\xi = \int J_{\text{ph-}r}(n)\epsilon p \, dp / \int \frac{\partial n_0}{\partial \epsilon} \epsilon^2 p \, dp . \quad (\text{A6})$$

Performing the integrations over momentum space, we have

$$\int J_{\text{ph-}r}(n)\epsilon p \, dp = (\cos\theta)N_r C^2 \left[-\beta \frac{pc}{k_B T} \right] \times \frac{P_0^2}{8\hbar^3 \rho^2 C^2} G \left[\frac{k_B T}{C} \right]^7 6! \zeta(6) , \quad (\text{A7})$$

$$\int \frac{\partial n_0}{\partial \epsilon} \epsilon^2 p \, dp = -3! \zeta(3) C \left[\frac{k_B T}{C} \right]^3 , \quad (\text{A8})$$

where G represents the term on square brackets on the right-hand side in Eq. (A4), and then Eq. (A6) becomes

$$\xi = (\cos\theta)N_r \beta \frac{6!(6)}{3!(3)} \frac{P_0^2 k_B^3 T^3}{8\hbar^3 \rho^2 C^2} G . \quad (\text{A9})$$

Substitution of Eqs. (A5) and Eq. (3.14) in Eq. (A6) and integration over momentum space gives Eq. (A6) as

$$\xi = \beta(\cos\theta) / \tau_{\text{ph-}r}(T) . \quad (\text{A10})$$

Comparing Eqs. (A10) and (A9), we obtain the characteristic time $\tau_{\text{ph-}r}(T)$:

$$\tau_{\text{ph-}r}^{-1}(T) = \frac{6! \zeta(6)}{3! \zeta(3)} \frac{N_r P_0^2 k_B^3 T^3}{8\hbar^3 \rho^2 C^4} \left| \frac{1}{4} + \frac{9}{32} \left[\frac{P_0}{\mu C} \right]^2 + \frac{P_0 A}{\mu C} + 2A^2 \right| . \quad (\text{A11})$$

In the case of the first phonon viscosity, we should replace Eq. (A4) by Eq. (4.15). Through the same calculation we get

$$\tau_{\text{ph-}r}^{-1}(T) = \frac{6! \zeta(6)}{3! \zeta(3)} \frac{N_r P_0^2 k_B^3 T^3}{8\hbar^3 \rho^2 C^4} \times \left[\frac{1}{2} + \frac{5}{32} \left[\frac{P_0}{\mu C} \right]^2 + \frac{3}{4} \frac{P_0 A}{\mu C} + 2A^2 \right] . \quad (\text{A12})$$

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