Perturbation theory of superconducting micronetworks near the phase-transition boundary

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(Received 4 March 1988)

A perturbation theory for micronetworks is developed which is valid near the second-order phase boundary. Perturbation equations to general order in terms of the nodal order parameter are derived. Explicit corrections to first order in temperature T and magnetic flux ϕ are obtained from the latter equations. This scheme is then applied to the infinite ladder. Stability limits in terms of the Gibbs free energy are derived for different spatial vortex solutions. Furthermore, the shielding currents of the ladder for $|\phi| < 0.21\phi_0$ are calculated (ϕ_0 is the fluxoid quantum) near the phase boundary as a function of ΔT and $\Delta\phi$ and are compared to our numerical solutions of the nonlinear Ginzburg-Landau equations. The agreement is very good. Similarly, very good agreement is obtained for the order parameter near $\phi = 0.5\phi_0$. Expressions for magnetization, differential susceptibility, entropy difference, and specific-heat jump near the phase-transition boundary of the ladder are derived.

I. INTRODUCTION

Superconducting micronetworks[1–3] make a second-order phase transition in a magnetic field when going from the normal to the superconducting state. The details of this transition depend only on the topology of the network. If the network contains areas which are commensurate to each other, the phase boundary becomes a periodic function of the fluxoid quantum ϕ_0. Similarly, if the underlying flux corresponds to a periodic flux structure due to the presence of vortices, it may lock-in, for certain magnetic field values, in a state commensurate with the network, at a global energy minimum.

The mean-field theory,[1–3] based on the Ginzburg-Landau (GL) equations, assumes that we are dealing with interconnected filaments thin enough so that they can be treated as one-dimensional branches in the GL context [radius $< \xi(t)$ and $\lambda(t)$].

Vortex current patterns are predicted in a network as, for example, for the infinite ladder.[4,5] However, these loop or mesh currents are of infinitesimally small magnitude at the phase boundary and become of physical significance only in the superconducting state beyond the phase boundary.

It is the purpose of this work to develop a perturbation theory suitable for dealing with currents in thin wire structures which are a consequence of a magnetic flux or temperature change away from the phase boundary. In that case, it is appropriate to assume that the normalized order parameter increases continuously from zero amplitude, at the phase boundary, to a finite value beyond it, thus permitting a perturbation approach similar to that used by Abrikosov[6] in deriving the vortex state of a bulk type-II superconductor.

When a current from an external source is introduced, however, the transition from the superconducting to the normal state is not of second order. In that case, the full nonlinear theory has to be applied to obtain meaningful results.[7–9]

Micronetworks of micrometer and submicrometer dimensions have been studied by a number of experimentalists, in particular with regards to the details of the phase-transition diagram of periodic networks,[10–13] confirming predictions of theoretical studies.[1–4] Interesting experiments of aperiodic and disordered networks have been performed recently[14] which have been compared to theoretical predictions.[15]

This work deals with a perturbation theory for micronetworks near the second-order phase-transition boundary. In Sec. II we develop the general perturbative formalism for micronetworks, in Sec. III, specific perturbation equations are derived, and in Sec. IV, the Gibbs free energy is evaluated and applied to a ladder network in order to find the most stable spatial configuration of the order parameter and vortex current pattern. In Sec. V we compare numerical results of our perturbation theory, applied to the ladder, to our exact numerical, nonlinear calculations on the ladder for two different extreme cases, and Sec. VI is devoted to the conclusions.

II. PERTURBATION EQUATIONS OF SUPERCONDUCTING MICRONETWORKS

A. General considerations

We start our perturbation calculation at a point on the phase boundary, separating the normal from the superconducting state. When entering the superconducting state from the normal region, the phase transition is of second order and occurs at a magnetic field $H=H_0$. The latter is a function of the transition temperature T_0 so that the phase boundary is described by the pair of variables H_0, T_0. The relation between H_0 and T_0 is obtained from the eigenvalue solution of the linearized GL equations, that is, in the limit that the order parameter (OP) is vanishingly small. As a consequence, any persistent currents are infinitesimally small and the local magnetic field is the applied field.

For the vortex state of type-II superconductors, Abrikosov[6] devised a perturbation scheme near the bulk transition field H_{c2} in which the temperature is kept constant and H is varied near H_{c2}. It consists of expansions of the vector potential $\mathbf{A}(x,y)$ and the order parameter $\psi(x,y)$ of the following form:

$$\mathbf{A}=\mathbf{A}_0+\eta^2\mathbf{A}_2+\eta^4\mathbf{A}_4+\cdots , \quad (1)$$

$$\psi=\eta\psi_1+\eta^3\psi_3+\cdots . \quad (1')$$

$\mathbf{A}_0$ and ψ_1 are the dominant terms at the phase boundary at H_{c2}. $\mathbf{A}_0$ is directly related to the applied field, that is to H_{c2}, and ψ_1 is obtained from the lineared GL equations as a function of $\mathbf{A}_0$. The parameter η is dimensionless and is a measure of the order of magnitude of the correction terms in the perturbative scheme. Starting with $\mathbf{A}_0$ and ψ_1, $\mathbf{A}_2$ is obtained from the second (current) GL equation. With the corrected $\mathbf{A}$ value, a relation for ψ_3 is obtained from the first GL equation from which the vortex state near H_{c2} is calculated. We take a more general approach for micronetworks. In addition to the above deviations of $\mathbf{A}$, we permit also a variation in temperature, expressible in terms of changes in the temperature-dependent coherence length $\xi(t)$, where $t=T/T_c$ and T_c is the transition temperatures in zero magnetic field (critical temperature)

$$\xi^2(t)=\xi_0^2-\eta^2\xi_2^2-\eta^4\xi_4^2+\cdots . \quad (2)$$

ξ_0 is the value of the coherence length at the phase boundary at which $t=t_0=T_0/T_c$. The value of ξ_0 is nonzero at t_0 except in zero field where t_0 approaches one and ξ_0 infinity. $\mathbf{A}_0$ is related to $H_0(T_0)$ of the network.

An equation for networks equivalent to Abrikosov's relation for ψ_3 will be discussed below, separately, in the context of higher-order nodal equations of a network. This equivalent relation will lead to a different conclusion than that obtained by Abrikosov for the vortex state of infinite extent because of the finite length of the filaments of a micronetwork and the specific boundary conditions at the nodes (generalized Kirchhoff's current laws). The expansion of the order parameter will be done in terms of a modified order parameter which has the advantage of simplifying the starting equations and nodal relations.

B. Modified order parameter

Assume that the usual complex GL order parameter on a branch connecting nodes a and b is $\psi_{ab}(x)$, where x is the curvilinear coordinate measured from a towards b. We then define a complex modified order parameter by

$$f_{ab}(x)=e^{-i\gamma_{ax}}\psi_{ab}(x) , \quad (3)$$

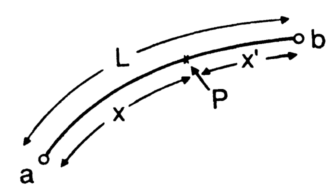
where

$$\gamma_{ax}=\frac{2\pi}{\phi_0}\int_a^x \mathbf{A}(u)\cdot d\mathbf{u} , \quad (4)$$

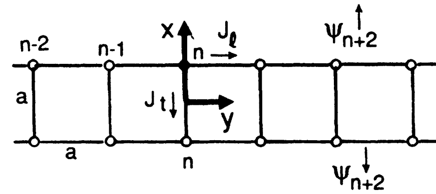
and where the path of integration is along the branch. Although $f_{ab}(x)$ is not single valued, it has obvious advantages in simplifying the starting equations.

Let the branch length between nodes a and b be L and the curvilinear coordinate be x' when measured from node b, so that $x'=L-x$ as shown in Fig. 1(a). We then simplify our notation to the following scheme:

$$\begin{aligned} f_a(x)&\equiv f_{ab}(x)=e^{i\gamma_{ba}}f_b(x')=e^{-i\gamma_{ax}}\psi_a(x)\\ &=e^{-i\gamma_{ax}}\psi_b(x')^* ,\\ \gamma_{bx'}&=\gamma_{ba}+\gamma_{ax} ,\\ \nabla_x\gamma_{ax}&=\frac{2\pi}{\phi_0}\mathbf{A}(x) ,\\ \mathbf{A}(x)&=\mathbf{A}(x') ,\\ \gamma_{ab}&=-\gamma_{ba} ,\\ f(x)&=|f|e^{i\Theta(x)} ,\\ \psi(x)&=|\psi|e^{\varphi(x)} . \end{aligned} \quad (5)$$



(a)



(b)

FIG. 1. (a) Coordinate system used in conjunction with Eq. (3) for the modified order parameter between nodes a and b. The length x to point P on the filament is measured from node a, while x' is measured from node b towards P. (b) Shown are the symbols of the infinite ladder as used in the text. Coordinate system x-y, longitudinal and transverse current densities J_l and J_t, index of node n, order parameter at upper ($\psi^\uparrow$) and lower ($\psi^\downarrow=\psi^{\uparrow *}$) branches, and lattice spacing a. The vector potential $\mathbf{A}$ is parallel to the y direction $(0,A_y,0)$.

Unit vectors along the wire are taken radially outward from nodes a and b.

C. General starting equations

When subscripts are dropped, the GL equations in the modified OP notation become

$$\xi^2\nabla^2 f+(1-|f|^2)f=0 , \quad (6)$$

$$\frac{4\pi}{c}\mathbf{J}=-\frac{\phi_0}{4\pi\lambda^2}i(f^*\nabla f-f\nabla f^*) , \quad (7)$$

where current density $\mathbf{J}$ is in Gaussian units (statamperes/cm^2), the fluxoid quantum $\phi_0=hc/2e$ and ξ and λ are the temperature dependent GL coherence length and penetration depth, respectively. The usual nodel condition[1-3]

$$\sum\left[\nabla_x-i\frac{2\pi}{\phi_0}\mathbf{A}\right]\psi_a(x)\Big|_{x=0}=0 ,$$

where the sum is over all branches connected to node a, reduces in the modified OP notation to

$$\sum\nabla_x f_a(x)\Big|_{x=0}=0 . \quad (8)$$

The gradient is measured outward from the node. Then within the context of our perturbation scheme the modified OP is

$$f_a(x)=\eta f_{a1}(x)+\eta^3 f_{a3}(x)+\eta^5 f_{a5}(x)+\cdots . \quad (9)$$

It is the object of our perturbation scheme to apply consistently Eqs. (1), (2), and (9) to Eqs. (6)–(8). Equations (6) and (7) are the starting equations and Eq. (8) is the nodal condition, with f being complex. Taking real and imaginary parts of Eq. (6) yields

$$\xi^2\nabla^2|f|+[1-|f|^2-\xi^2(\nabla\Theta)^2]|f|=0 ,$$

and

$$\nabla^2\Theta+2\nabla\Theta(\nabla|f|)/|f|=0 .$$

Equation (7) is also

$$\frac{4\pi}{c}\mathbf{J}=\nabla\times\nabla\times\mathbf{A}=\frac{\phi_0}{2\pi\lambda^2}|f|^2\nabla\Theta .$$

D. Perturbation equations

Inserting Eq. (9) into (6) and measuring lengths in units of ξ_0, we obtain the general perturbation equations for the order parameter

$$\nabla^2 f_{aj}(x)+f_{aj}(x)=\chi_{aj}(x) \text{ with } j=1,3,5,\ldots , \quad (10)$$

where

$$\chi_{a1}=0 ,$$

$$\chi_{a3}(x)=\left[\frac{\xi_2}{\xi_0}\right]^2\nabla^2 f_{a1}+|f_{a1}|^2 f_{a1}$$

$$=\left[|f_{a1}|^2-\left[\frac{\xi_2}{\xi_0}\right]^2\right]f_{a1} ,$$

$$\chi_{a5}(x)=\left[\frac{\xi_2}{\xi_0}\right]^2\nabla^2 f_{a3}+\left[\frac{\xi_4}{\xi_0}\right]^2\nabla^2 f_{a1}+(f_{a1})^2(f_{a3})^*+2|f_{a1}|^2 f_{a3} ,$$

etc.

The solution of the lowest-order equation ($j=1$) yields a modified OP to first order which is

$$f_{a1}(x)=[f_{a1}(0)\sin(L-x)+e^{-i\gamma_{ab0}}f_{b1}(0)\sin x]/\sin L , \quad (11)$$

where

$$\gamma_{abj}=\xi_0\frac{2\pi}{\phi_0}\int_a^b\mathbf{A}_j\cdot d\mathbf{x} \quad (\text{with } x\equiv x/\xi_0) .$$

Inserting $f_{a1}(x)$, Eq. (11), into the nodal condition (8) gives the usual first-order characteristic equation

$$f_{a1}(0)\sum_\beta\cot L_\beta-\sum_\beta\frac{e^{-i\gamma_{a\beta0}}}{\sin L_\beta}f_{\beta1}(0)=0 , \quad (12)$$

where the sum is over all branches connected to node a with β being a running index over all nodes β, including node b, which are connected directly to node a, and where $f_{aj}(0)\equiv f_{aj}$ $(x=0)$ and $f_{bj}(0)\equiv f_{bj}$ $(x'=0)$ are the modified OP's at nodes a and b.

The determinant of the coefficient matrix of the homogeneous set of equations of the form (12), corresponding to a given network, must be zero if the nodal OP's are to be nonzero. This leads to an eigenvalue equation which determines the phase boundary $H_0(T_0)$ of that particular network.

Higher-order solutions are found the following way: For order $j\geq3$ (odd), the function $\chi_{aj}(x)$ is known in terms of $f_{aj-2}(x)$ and the temperature difference from the phase boundary implicitly through the term ξ_{j-1}^2. Then Eq. (10) can be integrated by the variation-of-constants method. The nodal condition (8) then leads to the following nodal equation for the higher-order solutions:

$$f_{aj}(0)\sum_\beta\cot L_\beta-\sum_\beta\frac{e^{-i\gamma_{a\beta0}}}{\sin L_\beta}f_{\beta j}(0)=\sum_\beta\frac{P_{a\beta j}}{\sin L_\beta} , \quad (13)$$

where

$$P_{a\beta3}=i\gamma_{a\beta2}e^{-i\gamma_{a\beta0}}f_{\beta1}(0)+\int_a^\beta[\chi_{a\beta3}\sin(L_\beta-x)]dx ,$$

$$P_{a\beta5}=[i\gamma_{a\beta4}+\tfrac{1}{2}(\gamma_{a\beta2})^2]e^{-i\gamma_{a\beta0}}f_{\beta1}(0)+i\gamma_{a\beta2}e^{-i\gamma_{a\beta0}}f_{\beta3}(0)+\int_a^\beta[\chi_{a\beta5}(x)\sin(L_\beta-x)]dx ,$$

etc.

In evaluating Eqs. (12) and (13) it should be recognized that integration around a closed path leads to

$$\gamma_0=\xi_0\frac{2\pi}{\phi_0}\oint\mathbf{A}_0\cdot d\mathbf{x}=2\pi\frac{\phi}{\phi_0} ,$$

where ϕ is the flux at the phase boundary, and

$$\gamma_2 = \xi_0 \frac{2\pi}{\phi_0} \oint \mathbf{A}_2 \cdot d\mathbf{x} = 2\pi \frac{\Delta\phi}{\phi_0} ,$$

where $\Delta\phi$ is the flux difference in the closed contour measured with respect to the enclosed flux at the phase boundary ($x \equiv x/\xi_0$).

Consider $P_{a\beta 3}$ just below Eq. (13). It contains a first-order correction term in the magnetic flux $\Delta\phi$ (through γ_2 or A_2 if integrated around a closed loop) and a first-order correction term in the temperature, $T_0 - T = \Delta T$, through ξ_2^2 embedded in χ_3, so that the existence of the correction term f_3 depends on the first-order correction terms $\Delta\phi$ and ΔT. The condition for the existence of solutions of f_j are that *all* the characteristic determinants vanish[16] (there might be several, depending on the rank of the coefficient matrix). The requirement that f_3 exists leads to an equation containing $\Delta\phi$ and ΔT, while the postulate that f_5 exists, for example, leads to an equation which contains also $(\Delta T)^2$, $(\Delta\phi)^2$, and $(\Delta T)(\Delta\phi)$ terms. In what follows, we shall limit our considerations to first-order correction terms in $\Delta\phi$ and ΔT.

III. AMPLITUDE OF ORDER PARAMETER

For $j=1$, Eq. (10) is a homogeneous, linear differential equation and, therefore, does not fix the amplitude of the OP. To determine the latter, one has to consider at least the equation with $j=3$ which contains the inhomogenous term $\chi_{a3}(x)$. This equation is similar in form to that used by Abrikosov to fix the amplitude of the OP of the vortex state.

To simplify the following arguments, we drop the subscript of χ and let $f_{a1} = u(x)$ and $f_{a3} = v(x)$. Then Eq. (10) for $j=1$ and 3 takes on the forms

$$Lu(x) = 0 , \tag{14}$$

$$Lv(x) = \chi(x) , \tag{15}$$

where L is a self-adjoint operator, regular in the interval (ab) under consideration. In our case $L = \nabla_x^2 + 1$. Multiplying Eq. (15) by $u(x)$, integrating by parts, and making use of Eq. (14) leads to

$$\int_a^b u(x)\chi(x)dx = \left[u\frac{dv}{dx} - v\frac{du}{dx} \right]_a^b . \tag{16}$$

In Abrikosov's case,[6] the limits of integration are $\pm\infty$ and the functions $u(x)$ and $v(x)$ and their derivatives are zero there, so that the above integral is also zero. So in that case $u(x)$, the unperturbed function is orthogonal to $\chi(x)$. This holds more generally whenever the perturbed and the unperturbed solutions obey the same boundary conditions. In our case, however, the limits of integration are from node a to node b at which the perturbed and unperturbed solutions are not zero or do not obey the same boundary conditions. So the orthogonality condition, as used by Abrikosov, does not apply to micronetworks.

In order to find a new relation for a micronetwork, we multiply Eq. (15) by $[u(x)]^*$. Then in the notation of Sec. II, using the same procedure as above, this relation becomes

$$\int_a^b [f_{a1}(x)]^* \chi_{a3}(x)dx = \left[[f_{a1}(x)]^* \frac{df_{a3}}{dx} - f_{a3}(x) \left[\frac{df_{a1}(x)}{dx} \right]^* \right]_{x=0}^{x=L} \tag{17}$$

Summing Eq. (17) over all branches in the network and applying the boundary condition, Eq. (8), at each node, we arrive at

$$\sum_{\text{all branches}} \int_a^b \{ [f_{a1}(x)]^* \chi_{a3}(x) - 2\mathbf{J}_2 \cdot \mathbf{A}_2(x) \} dx = 0 , \tag{18}$$

where we have defined

$$\mathbf{J}_2 \equiv \frac{\frac{4\pi}{c}(\mathbf{J}_2)_{\text{conv}}}{\sqrt{2}H_c/\lambda} ,$$

$$\mathbf{A}_2 \equiv \frac{[\mathbf{A}_2(x)]_{\text{conv}}}{\sqrt{2}H_c\lambda} ,$$

with $\mathbf{J}_2 = \mathbf{J}_{ab2}$ being a second-order term. In the above equation, the term $\mathbf{A}_2(x)$ is made up of two contributions, the self-field $\mathbf{A}_2'(x)$ arising from the current density $\mathbf{J}_2$ in the filament, and a term $\Delta\,\mathbf{A}_2(x)$ which is due to an externally imposed magnetic field difference measured from the magnetic field H_0 at the phase boundary: $\mathbf{A}_2(x) = \mathbf{A}_2'(x) + \Delta\,\mathbf{A}_2(x)$. Assuming that the network is made up of superconducting filaments of constant cross-sectional area S, we can multiply Eq. (18) by S and obtain an integral over the volume of the superconductor. Note also that $J_2 \propto |f_1|^2$, that

$$\int dv\, \mathbf{J}_2 \cdot \Delta\,\mathbf{A}_2 = S\mathbf{J}_2 \int (\Delta\,\mathbf{A}_2) dx ,$$

that

$$\int dv\, \mathbf{J}_2 \cdot \mathbf{A}_2'(x) = -(J_2 S)^2 k \int dx ,$$

where k is a constant.

If we scale $|f_1|^2$ by $\alpha^2(\Delta T, \Delta H)$, where $\alpha^2 = 0$ at the phase boundary, then α^2 can be calculated from the volume integral of Eq. (18). After replacing $\chi_{a3}(x)$ by use of Eq. (10) we obtain

$$\alpha^2 = \frac{S(\xi_2/\xi_0)^2\beta_1 + 2S\Delta\gamma\beta_2}{S\beta_3^2 + 2S^2\beta_4^2} , \tag{19}$$

where

$$\beta_1 = \sum_{\text{all branches}} \int |f_1|^2 dx ,$$

$$\Delta\gamma\beta_2 = \sum_{\text{all branches}} \int \mathbf{J}_2 \cdot \mathbf{A}_2 dx ,$$

$$\beta_3^2 = \sum_{\text{all branches}} \int |f_1|^4 dx ,$$

$$\beta_4^2 = \sum_{\text{all branches}} \int k J_2^2 dx .$$

Here $\Delta\gamma$ is the circulation of ΔA_2 along a given closed portion of the network. We see that the second term in the denominator is of order S^2. If the condition $S \ll \lambda^2$ is satisfied we are justified in neglecting the self-field term. Thus Eq. (19) becomes

$$\alpha^2 = \frac{\Delta T}{T_C - T}\frac{\beta_1}{\beta_3^2} + 2\Delta\gamma\frac{\beta_2}{\beta_3^2} , \tag{19'}$$

where ΔT and $(\Delta\gamma)\beta_2$ are positive in the superconducting state below and second-order phase transition boundary.

In a recent paper, Ref. 17, the mixed state of a superconducting network was discussed. There, the authors substitute the linearized GL equation back into the fully nonlinear free-energy expression, thus neglecting terms which are relevant beyond the phase boundary. This procedure leads to a nonpositive definite squared order parameter when the fully nonlinear GL equation is substituted into the free-energy expression and their minimization procedure is followed.

In the next section, we shall deal with the infinite ladder in which case we can exploit the symmetry of the commensurate states and replace $\frac{1}{2}\int \Delta A_2 dx$ by

$$\xi_0 \frac{2\pi}{\phi_0}\int_n^{n+1} (\Delta A_2^{\uparrow}) d\left[\frac{y}{\xi_0}\right] = \frac{\pi\Delta\phi}{\phi_0} = \frac{\Delta\gamma}{2} ,$$

and $\mathbf{J}_2$ by

$$\sum_{\text{all upper branches}} J_l^{\uparrow} = \beta_2 ,$$

where we have assumed a vector potential $\mathbf{A} = \{0, (H_0 + \Delta H)x, 0\}$ with the y-coordinate parallel to the ladder as shown in Fig. 1(a). The symbol $\uparrow$ means the upper branch, and the last integral is

$$\int_n^{n+1} \Delta H \frac{a}{2} dy = \Delta H a^2/2 = \Delta\phi/2 ,$$

where a is the lattice spacing between nodes and $\Delta\phi$ is the flux difference in a mesh of area a^2. The vector potential parallel to the x direction is zero, so that the term $\mathbf{J}_2 \cdot \int \Delta \mathbf{A}_2 dx$ does not contain contributions from the transverse current J_t.[4,5]

IV. GIBBS FREE ENERGY OF INFINITE LADDER

The Ginzburg-Landau equations are obtained by varying the Gibbs free energy with respect to ψ, ψ^*, and $\mathbf{A}$ and finding an extremum of the free energy. If the full GL equations are substituted into the Gibbs free energy and integrated over all space, the following expression[18] remains to fourth order:

$$G_s - G_n = \alpha^4 \frac{H_c^2(0)\lambda^3}{4\pi}\int dv (\mathbf{A}_2' \cdot \mathbf{J}_2 - \tfrac{1}{2}|f_1|^4) . \tag{20}$$

Distances are normalized by λ and the self-field term $\mathbf{A}_2' \cdot \mathbf{J}_2$ is of higher order for thin filaments ($S \ll \lambda^2$). We neglect this term here and in Eq. (19). Applying Eq. (20) to the infinite square ladder we obtain

$$g \equiv \frac{\Delta G}{[H_c^2(0)/4\pi]S\xi_0} \approx -\frac{\alpha^4}{2}\sum\int |f_1|^4 d\left[\frac{x}{\xi_0}\right]$$
$$= -\frac{1}{2}\left[\frac{\Delta T}{T_c - T}\frac{\beta_1}{\beta_3} + \frac{4\pi\Delta\phi}{\phi_0}\frac{\beta_2}{\beta_3}\right]^2 . \tag{21}$$

The summation extends over all branches of the network. The ratios β_1/β_3 and β_2/β_3 can be readily calculated. Note that ΔT and $(\Delta\phi)\beta_2$ are positive. Then it is straightforward to obtain other thermodynamic quantities such as the magnetization $M(T)$, the magnetic differential susceptibility $\chi(T)$, the entropy difference $\mathcal{S}(H)$, and specific-heat jump ΔC between the superconducting and normal states at the phase boundary. Assuming the normal state is nonmagnetic, that $\Delta T = T_0 - T$ and $\Delta\phi = a^2(H - H_0) = a^2\Delta H$, then in units of $H_c^2(0)S\xi_0/4\pi$ these expressions are

$$M = -\left.\frac{dg}{dH}\right|_{T_0} = \Delta H\left[\frac{4\pi a^2}{\phi_0}\frac{\beta_2}{\beta_3}\right]^2 = \chi\Delta H , \tag{21'}$$

$$\mathcal{S} = -\left.\frac{dg}{dT}\right|_{H_0} = -\Delta T\left[\frac{1}{T_c - T}\frac{\beta_1}{\beta_3}\right]^2 = -\frac{\Delta T}{T_0}\Delta C . \tag{21''}$$

Near the phase boundary[4,5] the order parameter at node n is

$$f_{n1}^{\uparrow} = \frac{\alpha}{2}(e^{-iqn} + Be^{iqn}) = (f_{n1}^{\downarrow})^* ,$$

where

$$B = 3\cos\left[\frac{a}{\xi_0}\right] - 2\cos\left[\frac{\gamma}{2} - q\right]$$

with $\gamma = 2\pi\phi/\phi_0$, and ϕ is equal to the flux through area a^2 at the phase boundary ($\phi = a^2 H_0$). The phase boundary is obtained from the characteristic determinant of Eq. (12) which leads to the following eigenvalue equation:

$$3\cos\left[\frac{a}{\xi_0}\right] = 2\cos\frac{\gamma}{2}\cos q \pm \left[1 + \left[2\sin\frac{\gamma}{2}\sin q\right]^2\right]^{1/2} . \tag{22}$$

For a fixed value of γ (or H_0), the smallest eigenvalue $a/\xi_0(T)$ (largest $T \equiv T_0$) is chosen. This determines the appropriate wave vector q, as a function of ϕ at the phase boundary. Therefore, the transition temperature T_0 becomes a function of ϕ (or H_0) and q. For $\phi < 0.214\,824\phi_0 \equiv \phi_c$ the value of $q = 0$ and for larger values of ϕ the value of q varies from zero to $\pi/2$ at $\phi = \phi_0/2$. For still larger values of ϕ, the persistent current patterns reverse direction, but symmetry is maintained with respect to $\phi_0/2$ and the phase boundary becomes periodic with period ϕ_0. Then it follows from the linearized equations, with α arbitrary, that

$$|f_{n1}|^2 = \tfrac{1}{4}\alpha^2[1 + B^2 + 2B\cos(2qn)] , \tag{23}$$

$$J_{l2}^{\dagger}(n,n+1)=\frac{\alpha^2}{4\sin\left[\frac{a}{\xi_0}\right]}\left[\sin\left[\frac{\gamma}{2}+q\right]+B^2\sin\left[\frac{\gamma}{2}-q\right]+2B\sin\left[\frac{\gamma}{2}\right]\times\cos[q(2n+1)]\right], \quad (24)$$

$$J_{t2}(n,n)=\alpha^2 B\sin\left[\frac{\gamma}{2}\right]\sin(q)\sin(2qn)/\sin\left[\frac{a}{\xi_0}\right]. \quad (25)$$

Within our perturbation scheme, however, the scaling factor α is no longer arbitrary. Away from the phase boundary, in the superconducting region, α^2 is determined to first order in ΔT and ΔH by Eq. (19′).

With Eqs. (11), (23)–(25), the coefficients β_1 to β_3 were numerically evaluated for the ladder and the Gibbs free energy, Eq. (21), was obtained for various q intervals.

Equations (19′) in conjunction with Eqs. (11) and (23), describe completely the order parameter to first order in ΔT and ΔH inside the superconducting region of the phase diagram. However, a point inside that region can be described by first-order solutions of various q values. We assume that the most stable solution is the one with a q value which has the lowest Gibbs free energy. In order to demonstrate this clearly, we permit solutions with discrete q values by taking periodic boundary conditions as will be shown below. In that case, we obtain regions in the phase diagram where one discrete q value, q_k, has a lower free energy than the adjacent q values such as q_{k-1} or q_{k+1} or other q values.

Equation (21) is that of a parabolic cylinder in ΔT and $\Delta\phi$ (or ΔH) for a specific q value, q_k. For another q value, say q_{k+1}, the free energy is also represented by a parabolic cylinder but shifted in the space $(\Delta G,\Delta T,\Delta H)$. The curve of intersection between those two surfaces in the superconducting region are lines where the energies of the adjacent q values are equal. Figure 2 shows those curves emerging from the phase boundary of the ladder. The periodic boundary conditions imply that $\psi_n=\psi_{n+m}$ with $q=2\pi k/m$ and $0\leq k<m$. However, the relevant quantities in the free energy are related to $|\psi_n|^2$ which is periodic over a distance $m/2$ instead of m, so that we choose $q=\pi k/(m/2)$ with $0\leq k<m/2$. Figure 2 is a plot with $m/2=10$, and Fig. 3 with $m/2=30$.

Consider Fig. 2. Assume that we are at the phase boundary at point P_1 with $q=\pi/10$. Then, in order to get to point P without changing the q value, we have to lower the temperature *and* decrease the magnetic field as shown in the figure. For the path shown, the ladder would be always in the lowest energy state. However, if we were to start at P_0 with $q=0$ and decrease the temperature at constant flux, or start on the phase boundary at P_2 with $q=\pi/5$ and decrease the magnetic field, in order to reach P, and also require that we stay in the lowest energy state, then the order parameter has to change its period. Therefore, for the ladder to remain in its lowest energy state, while H and T are changed, requires that the spatial period of the order parameter changes in general. An exception to that rule is that as ΔH and ΔT are changed, the initial and final destination in the phase diagram and the path taken are in the same q region such as (P_1,P) in Fig. 2.

As can be seen from Fig. 3, as $m/2$ is increased, the constant q regions become narrower and a path taken from the phase boundary to an interior point without

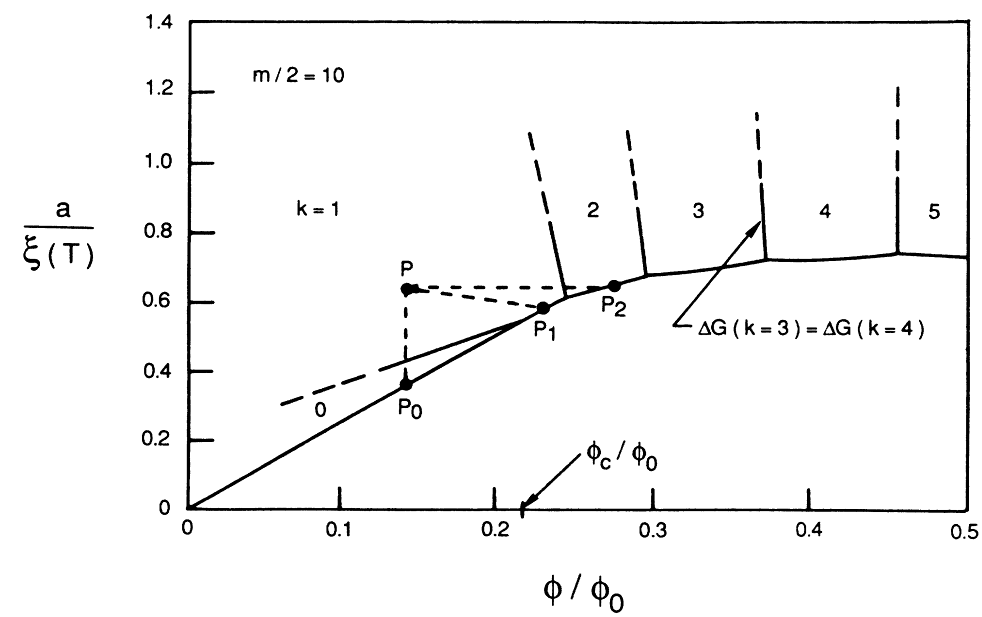


FIG. 2. Boundaries of Gibbs free energy, Eq. (21), for different wave vectors $q=2\pi k/m$ of the infinite ladder for an assumed periodicity $\psi_n=\psi_{n+m}$. The phase boundary is the lower curve for which $\Delta G=0$. The lines pointing into the superconducting regions are where $\Delta G(k)=\Delta G(k\pm1)$, indicating the energetic stability limits for various wave vectors for a period $m=20$.

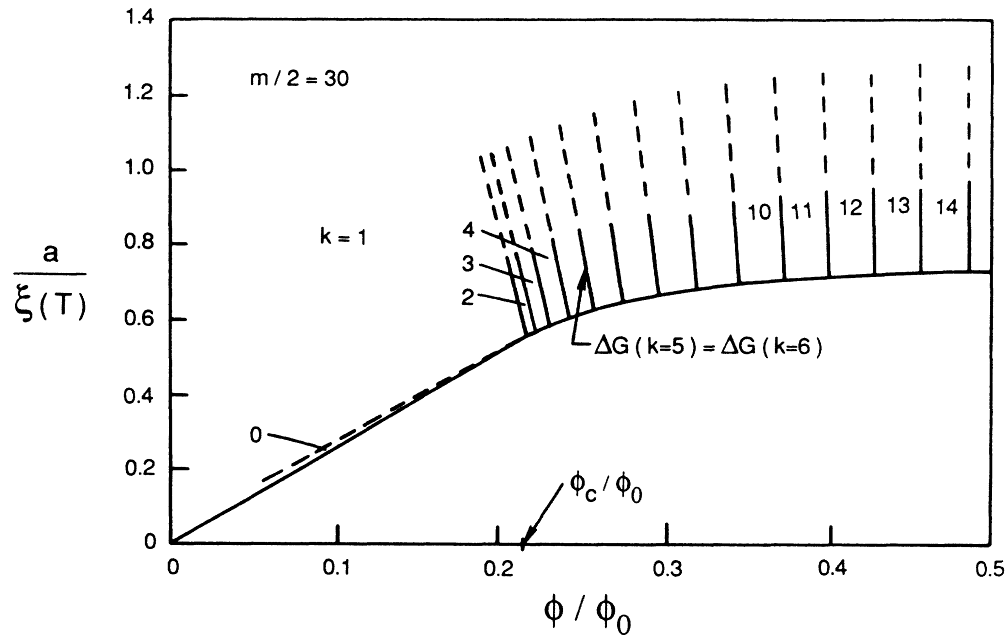


FIG. 3. Similar to Fig. 2, except that $m=60$.

changing the q value becomes more restricted. In the limit that $m/2\rightarrow\infty$ that path approaches a well-defined curve except for fluxes $\phi\lesssim\phi_c\approx 0.215\phi_0$.

Due to the perturbative character of the present approach, it should be noticed that the lines drawn in Fig. 2 and Fig. 3 are only reliable near the phase boundary where they are tangent to the exact dividing line between surfaces of different q values.

V. COMPARISON OF PERTURBATION WITH EXACT SOLUTION

In order to test our theory, we obtain exact nonlinear, numerical solutions for the ladder for two cases: (A) for $q=0$ and (B) for $\phi/\phi_0=\frac{1}{2}$, and compare these to our perturbation solutions near the phase boundary.

When $q=0$, the transverse currents are zero and the longitudinal currents act like shielding currents in a superconductor at low fields. This happens to be the case when $|n\phi_0\pm\phi|<\phi_c$, where n is a positive or negative integer or zero. The exact solution was obtained by modifying the numerical program of the SQUID circuit of Ref. 9 to that of a lasso of circumference $a/\xi(t)$ with arm length equal to one-half the circumference. Then the symmetry of the order parameter and nodal slope condition, Eq. (8), are the same for the lasso as those for the ladder with $J_t=0$ [Eq. (25)], as indicated by the insert of Fig. 4. At the antinodal point A and halfway between nodes n and $n+1$, the value of $|\psi(x)|^2$ is a minimum, while at point B, at the end of the lasso arm and in the middle of the transverse branches, it is a maximum. Figure 4 shows exact nonlinear results of the normalized shielding current J_l of the ladder as a function of magnetic flux through an area a^2 of the ladder (which corresponds to twice the flux through the lasso[3] loop) for various constant temperatures [constant $a/\xi(t)$]. To obtain a valid comparison, we compare the slope of J_l at $J_l=0$ in Fig. 4 with that obtained by our perturbation approach at the phase boundary.

For $q=0$, it follows from Eqs. (22)–(25) that $B=1$, $|f_{n1}|^2=\alpha^2$,

$$J_l^{\dagger}(n,n+1)=\alpha^2\sin(\gamma/2)/\sin[a/\xi_0(t)]\ ,$$

$$J_t=0\ .$$

The absolute values of the order parameters are the same at all nodes. However, in between they are functions of x and y. From the linearized solution, Eq. (11), it follows, in normalized notation, with $|f_n|^2=1$, that

$$|\psi(y)|^2=\frac{1}{\sin^2 a}[\sin^2(a-y)+\sin^2 y+2\sin(y)\sin(a-y)\cos(\gamma/2)]\ ,$$

for the longitudinal branch between nodes n and $n+1$, and

$$|\psi(x)|^2=\cos^2(x)/\cos^2(a/2)$$

for the transverse branch, where we have used the same coordinate system as that shown in Fig. 1(b). Also,

$$\cos(\gamma/2)=\cos(a)-\sin^2(a/2)$$

from the eigenvalue equation, Eq. (22). Because of periodicity we restrict our calculation of the β's to a unit cell. Integrating the OP on one branch from $y=0$ to a, and on the other from $x=0$ to $a/2$, for $a=\pi/8$, leads to

$$\beta_1/2=0.387\,53+0.201\,52=0.589\,05\ ,$$

and

$$\beta_3^2/2=0.382\,45+0.206\,84=0.589\,29\ ,$$

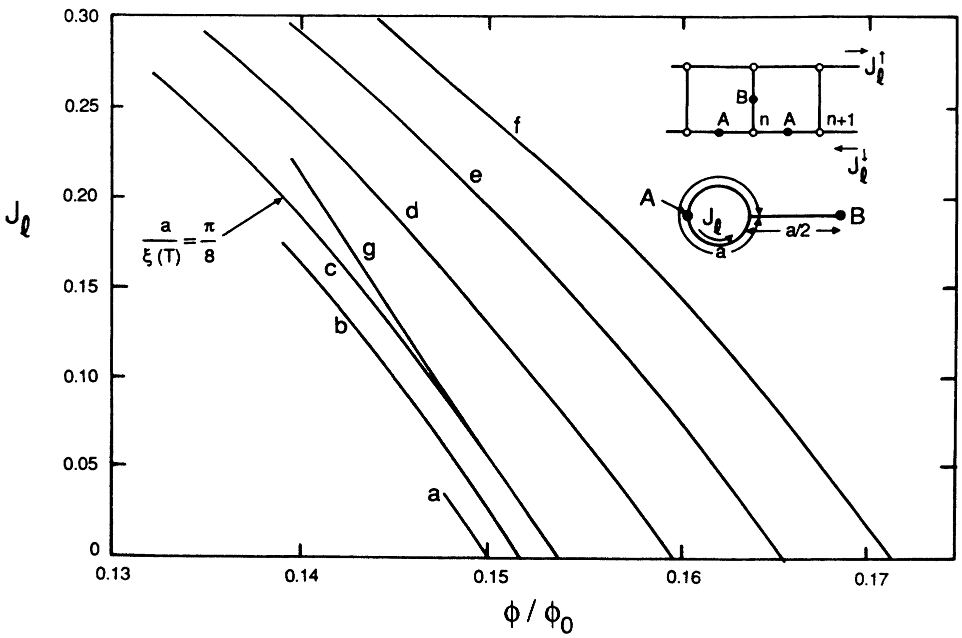


FIG. 4. Comparison of the perturbation approach, applied to the infinite ladder, using Eq. (19′), with exact nonlinear, numerical calculations as used in Ref. 9. The curves are the longitudinal currents, J_l, as a function of magnetic flux per area a^2 for various constant temperatures [constant $a/\xi(t)$]. Curve a corresponds to $a/\xi(t)=0.38359$, b to 0.38761, c to 0.39270, d to 0.40765, e to 0.42261, and f to 0.43756. The slope at $J_l=0$ of curve c [$a/\xi(t)=\pi/8$] is compared with that obtained from the perturbation results, Eq. (24′), and is shown by line g. The latter value is -15.68 while the exact value is -15.63. For inset see text.

where the first term of the sum is from the longitudinal and the second from the transverse branch. It then follows from the linearized result, Eq. (24) with $\alpha=1$, that

$$\beta_2=\sin(\gamma/2)/\sin(a) .$$

In calculating the β's, the *first-order* solutions of the OP's were taken, obtained from the *linearized* GL equations. Then, the current density, away from the phase boundary, is obtained from Eq. (24) with $q=0$ and α defined by Eq. (19′)

$$J_l=\pm\alpha^2\frac{\sin(\gamma/2)}{\sin a} .$$

When the temperature is kept constant ($\Delta T=0$), this becomes

$$J_l=\pm 2\pi\frac{\Delta\phi}{\phi_0}\frac{1}{0.58929}\left[\frac{\sin(\gamma/2)}{\sin a}\right]^2 . \qquad (24')$$

Then, for example, for $a=\pi/8$, with $\cos(\gamma/2)=1-3\sin^2(a/2)$, the slope of the normalized current with respect to the normalized flux, at the phase boundary, becomes

$$\frac{dJ_l}{d(\Delta\phi/\phi_0)}=-15.68 .$$

This compares very favorably with the result of the nonlinear calculation of -15.63. Equation (24′), the perturbation result, is also shown in Fig. 4 for $a/\xi(t)=\pi/8$.

Figure 5 shows a cross-plot of the nonlinear solutions of J_l as a function of $a/\xi(t)$ for constant flux $\phi/\phi_0=0.150$. Again, we test our perturbation equation, but this time for $\Delta\phi=0$. With

$$J_l=\pm\alpha^2\sin(\gamma/2)/\sin(a/\xi) ,$$

and

$$\alpha^2=\frac{\Delta T}{T_C-T}\frac{\beta_1}{\beta_3^2}=\frac{2\Delta[a/\xi(t)]}{a/\xi(t)}\times 0.9996$$

the slope at the phase boundary, at $a/\xi=0.3836$, becomes

$$\frac{dJ_l}{d[\Delta(a/\xi)]}=\frac{1.9992}{a/\xi}\frac{\sin(\gamma/2)}{\sin(a/\xi)}=6.33 , \qquad (24'')$$

which is shown in Fig. 5. The graphical nonlinear results indicate a value of 6.22 for the slope. The results shown in Fig. 5 are less accurate than Fig. 4 because of the graphical procedure used to construct Fig. 5. This shows that the agreement between our perturbation approach and the exact nonlinear, numerical solution is excellent for $q=0$.

B. Case $\phi=\phi_0/2$

When the flux through area a^2 of the ladder is one-half of a flux quantum ($\gamma=\pi$) then the wave vector $q=\pi/2$ and it follows from Eq. (22) that $\cos a=\sqrt{5/3}$ or $a=0.72973$. Then the value of $B=\sqrt{5}-2$. It follows from Eqs. (24) and (25) that $J_l=0$ and $J_t=0$, so that β_2,

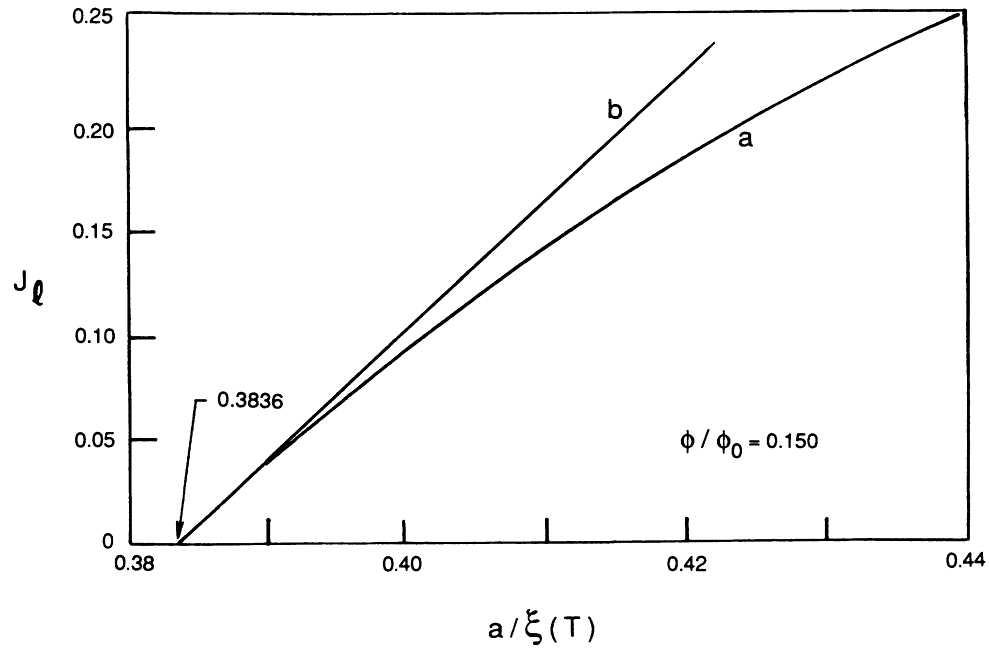


FIG. 5. Cross-plot of Fig. 4, for constant flux $\phi = 0.15\phi_0$, for various temperatures of the nonlinear results of J_l. The slope at $J_l = 0$ is equal to 6.22 for the cross-plotted results, curve a, and 6.33 for the perturbation result [from Eq. (24'')], line b.

in Eq. (19') is zero and α^2 does not contain a linear correction term in $\Delta\phi$ (or ΔH) near $\phi \approx \phi_0/2$. The only linear term which remains is in ΔT (or Δa) since $\beta_1 \neq 0$.

The functional behavior of $\psi(x)$ is explained in the insert of Fig. 6. Across branch (aa'), the phase difference of the order parameter is zero, and at point M the value of $|\psi(x)|^2$ is a maximum. The phase angle $\Theta = 0°$ at node a, $-90°$ at node b, $-180°$ at node c, $+90°$ at node d, etc., while at a', b', c', and d' it is equal to 0°, +90°, 180°, −90°. Therefore, the phase difference across the transverse branches varies as 0°, −180°, 0°, 180°, 0°, while along the upper branch, between nodes, it varies by −90°, and on the lower branch that difference is equal to +90°. The order parameter along the transverse

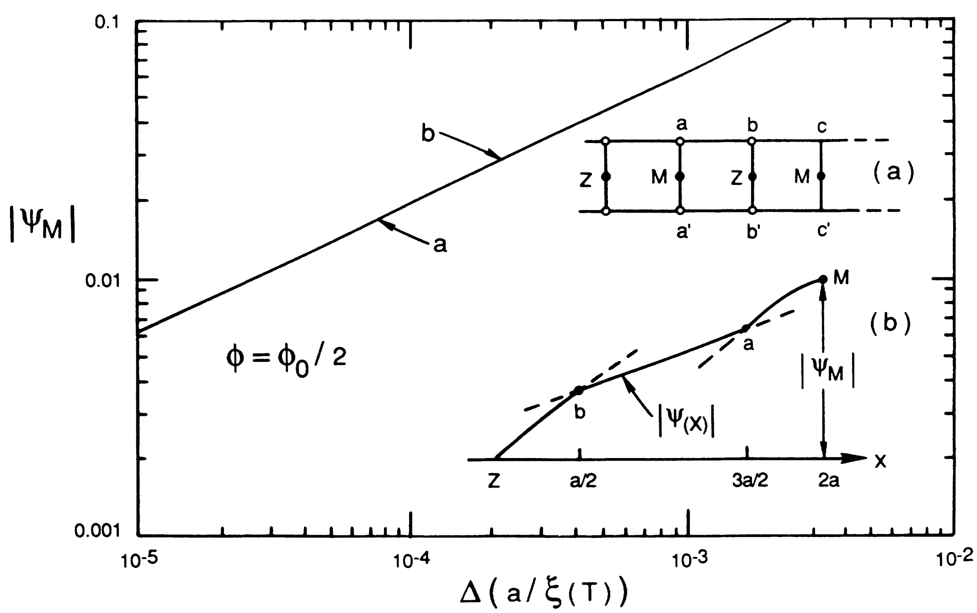


FIG. 6. Log-log plot of the maximum order parameter at position M [see inset (a)] as a function of temperature difference, measured from the phase boundary, at $\phi = \phi_0/2$. Curve a was obtained numerically from the nonlinear equations, following the procedure outlined in inset (b) [see text and Eq. (27')], and curve b, from the perturbation approach, Eq. (27).

branches, in the linearized approach, in normalized notation, is

$$\psi(x)=2\,|\,\psi_j^\dagger\,|\,[\cos\theta_j\sin(a/2)\cos x + i\,\sin\theta_j\cos(a/2)\sin x\,]/\sin a\ . \qquad (26)$$

Between a and a', where the maximum is located, one obtains from Eq. (26)

$$|\,\psi_{aa'}(x)\,|^2=|\,\psi_a\,|^2\left[\frac{\cos x}{\cos(a/2)}\right]^2 ,$$

while between b and b' it is

$$|\,\psi_{bb'}(x)\,|^2=|\,\psi_b\,|^2\left[\frac{\sin x}{\sin(a/2)}\right]^2 .$$

From the latter equation it follows that at $x=0$ the OP is zero and that the phase of the OP flips by 180° with no current passing through that transverse branch. This is what we call a *phase-flip center.* Depending on the value of q $(=\pi/n)$, the distance between such phase flip centers for a commensurate vortex state is n lattice spacings.[5]

Since the vector potential is parallel to the y direction, and $\gamma_{ab}=\pi/2$, the linearized order parameter between nodes a and b is

$$|\,\psi(y)\,|^2_{ab}=\frac{1}{4\sin^2 a}[(1+B)^2\sin^2(a-y)+(1-B)^2\sin^2(y) -2(1+B)(1-B)\sin(a-y)\sin(y)]$$

with

$$|\,\psi_a\,|^2=(1+B)^2/4$$

and

$$|\,\psi_b\,|^2=(1-B)^2/4\ .$$

It then follows from the above equations that

$$\beta_1/2=(0.146\,12+0.152\,80+0.018\,07)=0.316\,99\ ,$$

$$\beta_2/2=0\ ,$$

$$\beta_3^2/2=(0.030\,00+0.064\,09+0.001\,59)=0.095\,68\ ,$$

where the first number in brackets comes from the longitudinal branch over distance $2a$, while the second and third numbers come from the transverse branches with a maximum and a zero in the order parameter, respectively (over distance $a/2$).

With $\alpha^2=2(\Delta a)\beta_1/(a\beta_3^2)$, the maximum value of the order parameter at point M [see inset (a) of Fig. 6] is

$$|\,\psi_M\,|=\alpha\frac{|\,\psi_a\,|}{\cos(a/2)}=1.994\sqrt{\Delta(a/\xi_0)}\ , \qquad (27)$$

where the term on the right-hand side is written in conventional units.

Inset (b) of Fig. 6 shows schematically the functional dependence of $|\,\psi(x)\,|$ between point Z and M via nodes a and b, at which point the slope of $|\,\psi(x)\,|$ changes by a factor of 2 in each case due to the nodal condition (8). Using Jacobian elliptic functions and expanding these for the Jacobian parameter

$$m=|\,\psi_M\,|^2/(2-|\,\psi_M\,|^2)\ll 1\ ,$$

solutions for $|\,\psi_M\,|$ were obtained from the nonlinear equations, in the spirit of Ref. 9, and are shown in Fig. 6 on the log-log plot of $|\,\psi_M\,|$ versus $\Delta(a/\xi(t))$. The overall fit is satisfied by

$$|\,\psi_M\,|\approx 1.990\sqrt{\Delta(a/\xi)}\ , \qquad (27')$$

which compares very favorably with Eq. (27), valid near the phase boundary.

Thus, near the point $\phi=0.5\phi_0$ $(q=\pi/2)$ on the phase boundary, the order parameter can be expanded linearly in ΔT while the coefficient of the term in $\Delta\phi$ is zero. The lowest order term in $\Delta\phi$ is probably the quadratic term.

VI. CONCLUSIONS

We developed a perturbation theory which is applicable to micronetworks near the second-order phase-transition boundary. Equation (13) is the perturbation equation of the nodal order parameters. When we postulate the existence of a third-order correction to the order parameter, we obtain an equation which leads to a perturbation equation in first order of ΔT and $\Delta\phi$ measured from the phase boundary. This equation is related to but different from the orthogonality condition Abrikosov[6] uses in characterizing the vortex state of a bulk type-II superconductor, because of the unique boundary conditions in a network. Our new relation is stated by Eq. (18) which leads to Eq. (19), the amplitude equation of the perturbation. Since a spatially modulated order parameter can lead to lower free energies (for certain magnetic flux intervals) compared to the spatially uniform state, the β coefficients are functions of the wave vector q of that modulation. A consequence of this are persistent mesh and loop currents, like those in a thin film in a magnetic field tangential to its surface,[19] with the exception, that the spatial size of the vortex currents are controlled by the topology of the network. The latter circumstance leads to commensurate as well as incommensurate vortex states.

The vortex and shielding currents of the ladder are given by Eqs. (19′), (24), and (25). In order to determine which wave vectors are the appropriate ones near the phase boundary, the Gibbs free energy differences were obtained from Eq. (21) and calculated for different values of $q=2\pi k/m$. For adjacent q states, the boundary of stability of a particular spatial modulation was found by equating $\Delta G(k)$ to $\Delta G(k\pm 1)$, where stability is defined in terms of lowest Gibbs free energy. Such regions of lowest energy are shown in Figs. 2 and 3. Thus, starting from the phase boundary, if, for example, the magnetic field is changed at constant temperature (or vice versa or both) the wave vector will, in general, change when going to a point beyond the phase boundary into the superconducting state. This leads to a redistribution of the vortex currents when H or T is changed. There exist, however, very special regions or paths of ΔH and ΔT where the q value does not change.

We tested the results of our perturbation theory with the results of numerical computations, solving the nonlinear GL equations in the spirit of Ref. 9. This comparison was done for the low field case ($\phi < 0.21\phi_0$), where $q=0$, and for $\phi=0.5\phi_0$, where $q=\pi/2$. Figure 4 shows the longitudinal current J_l in the ladder as a function of flux at constant temperature and Fig. 5 shows J_l as a function of temperature for constant flux. The agreement with the exact nonlinear results is excellent in the limit that $J_l \rightarrow 0$. Similar good agreement is obtained for the order parameter as a function of ΔT (Fig. 6) when $q=\pi/2$. A change in flux, $\Delta\phi$, at $\phi=0.5\phi_0$ does not produce a change in the order parameter to first order.

The good agreement between our results, obtained from our perturbation theory, and our nonlinear numerical solutions, gives us confidence that both approaches are correct. Expressions for magnetization, differential susceptibility, entropy difference, and specific-heat jump for the infinite ladder near the phase-transition boundary are given by Eqs. (21′) and (21″). The perturbation approach developed here is general enough to be applicable to any network. The square array and other similar structures are being investigated at present.

ACKNOWLEDGMENTS

We thank Dr. V. Grünfeld and Dr. F. de la Cruz for fruitful discussions. The work of H. J. F. was supported in part by the National Science Foundation (NSF) through Grant Nos. ECS-8505627, INT-8502375, and INT-8803025.

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