Crossover scaling in a dilute Bose superfluid near zero temperature

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The behavior of a dilute Bose superfluid at low temperatures, $T \ll T_\lambda$, is considered with emphasis on the superfluid density, $\rho_s(T)$. Many of the predictions of the Bogoliubov model are exact in this regime and are shown to lead to a scaling description of the crossover from ideal to interacting Bose gas behavior as the density is reduced. Combined with previous work on scaling near criticality, $T \simeq T_\lambda$, these results complete the crossover description of the dilute superfluid over the full temperature range. Diagrammatic perturbation theory in the superfluid phase is outlined and previous results from the helium literature are summarized. The various divergences encountered have previously been shown to arise from nonanalyticities in the self-energy functions at zero momentum, and these are related to the spin-wave nature of the superfluid phase. A full renormalization-group fixed-point picture is presented summarizing the zero-temperature and finite-temperature regimes and the flows connecting them.

I. INTRODUCTION

In Ref. 1 the crossover from ideal to interacting Bose gas critical behavior, which occurs as the overall density is reduced, was worked out in detail. From the ideal-gas relation for the transition temperature, T_0, as a function of number density, n, namely

$$n = \rho/m = \zeta(\tfrac{1}{2}d)\Lambda_{T_0}^{-d} \propto T_0^{d/2} \simeq T_c^{d/2} , \tag{1.1}$$

one sees that low densities entail low temperatures as well. Here and below d is the dimensionality, m is the particle mass, T_c is the actual transition temperature (which will be close to T_0 at low densities), $\zeta(z)$ is the Riemann ζ function, and

$$\Lambda_T = h/(2\pi m k_B T)^{1/2} \tag{1.2}$$

is the thermal de Broglie wavelength, which constitutes the most important large length at low temperatures. On the scale of T_c, however, the analysis in Ref. 1 was restricted to the critical region, $t = T/T_c \simeq 1$, and was therefore unable to treat the true neighborhood of zero temperature, $t \ll 1$. The objective of this paper is to consider this region so as to complete the crossover description of the dilute Bose fluid.

The low-temperature behavior of a dilute Bose fluid is an old and well-understood subject. The main aim of this paper is to interpret and clarify some of what is known in more modern phase transition language. It will be seen that, in many senses, $T=0$ can be viewed as a critical point with its own associated exponents; a crossover scaling description will be appropriate here, as well, in the dilute limit. Although the behavior is quite interesting near $T=0$, it will transpire that the analysis is much simpler: all exponents and scaling functions can be written *exactly* in closed form—no novel analytic techniques are required. In fact, the main results, at least for dimension $d=3$, are implicit partly in the original Landau quasiparticle model,[2] and especially in the classic work of Bogoliubov,[3] where he introduced his famous transformation (see Ref. 2 and Sec. II below). However, none of the previous work has emphasized the *critical* nature of the behavior, nor explored its dependence on the dimensionality d.

To see why some type of crossover scaling description might be appropriate in the $T=0$ region, compare the predictions of the Landau quasiparticle model with those of the ideal gas.

The Landau model treats the elementary excitations of the superfluid ground state as a gas of noninteracting, unconserved Bose quasiparticles with a dispersion relation which is asymptotically linear (phononlike) at small momenta

$$\varepsilon_{\mathbf{q}} \approx \hbar c\,|\mathbf{q}|, \quad |\mathbf{q}| \to 0 , \tag{1.3}$$

where the speed of sound, c, has a value of 238 m/s in bulk ^{4}He. From this spectrum, the low-temperature behavior of various thermodynamic quantities[2] follows immediately: for example, the "T^3 law" for the specific heat,

$$C_V \approx \frac{2\pi^2\hbar}{15c}\left[\frac{k_B T}{\hbar c}\right]^3, \quad d=3 \tag{1.4}$$

and the "T^4 law" for the normal fluid mass density,

$$\rho_n \equiv \rho - \rho_s \approx \frac{2\pi^2\hbar}{45c}\left[\frac{k_B T}{\hbar c}\right]^4, \quad d=3 . \tag{1.5}$$

In d dimensions the exponents become d and $d+1$, respectively. Contrast this with the ideal result

$$\varepsilon_{\mathbf{q}}^0 = \hbar^2\mathbf{q}^2/2m \tag{1.6}$$

implying[2,4]

$$C_V = \frac{15\zeta(5/2)}{4\zeta(3/2)} k_B (T/T_0)^{3/2}, \quad d=3 , \tag{1.7}$$

$$\rho_n = \rho (T/T_0)^{3/2}, \quad d=3 . \tag{1.8}$$

Both exponents become $d/2$ in d dimensions. The Landau results are, in fact, valid at sufficiently low temperatures in the weakly interacting limit as well (see below). Hence there is a discontinuous change in the low-temperature asymptotics as soon as interactions are introduced. Similar behavior was seen near the critical point,[1] where, for example, the exponent describing the vanishing of ρ_s near T_c changed from the ideal value $\zeta_0=1$ [as follows from (1.8)] to the interacting value $\zeta\simeq 2/3$. In that case the change was described by a scaling ansatz which took the form[1]

$$\rho_s(n;T)=\rho_s^{\text{idea}}(n;T)Y(C_d(a/\Lambda_T)^{d-2}/\bar{t}^{\phi}) \qquad \bar{t},a/\Lambda_{T_c}\ll 1 , \quad (1.9)$$

where $\bar{t}=(T_c-T)/T_c$, a is the s-wave scattering length or effective hard-core radius of the particles, $\phi=(4-d)/(d-2)$ is the crossover exponent, and C_d is a dimensionless metrical factor. The universal crossover scaling function $Y(y)$ contains all the effects of interactions and can, in general, be calculated only approximately (for example, by perturbation theory in $\epsilon=4-d$)[1]. The asymptotic behaviors $Y(0)=1$ and $Y(y)\sim y^{(\zeta_0-\zeta)/\phi}$ for large y are required in order to recover the correct limiting behavior as $a/\Lambda_T\to 0$ and as $\bar{t}\to 0$. It follows from (1.9) that the region over which the interacting value of ζ can be seen, $y\gg 1$, corresponds to a narrow temperature range

$$\bar{t}<\bar{t}_X\equiv(T_c/T_Q)^{(d-2)^2/2(4-d)} , \quad (1.10)$$

where $T_Q=h^2/2\pi m k_B a^2$ depends only on ^{4}He atomic parameters and fundamental constants. As T_c is suppressed this range becomes increasingly narrow, and the ideal behavior, $y\ll 1$, increasingly dominates. It will be seen that the same kind of mechanism operates in the low temperature, $t=(1-\bar{t})\ll 1$, region as well.

Although the Landau theory gives correct qualitative predictions, it is completely phenomenological in nature. It gives no information about the condensate fraction, nor does it provide a framework for deriving the dispersion relation (1.3) from first principles. In Sec. II the Bogoliubov model, valid at low temperatures and densities, will be introduced. This model provides a microscopic approach to the thermodynamics and yields the well-known Bogoliubov dispersion relation

$$E_q=[(\varepsilon_q^0)^2+2mc^2\varepsilon_q^0]^{1/2} \quad (1.11)$$

which reproduces (1.3) at small q with

$$c=\hbar(4\pi na)^{1/2}/m, \quad d=3 . \quad (1.12)$$

The model also allows one to construct a crossover scaling formulation near zero temperature. The detailed forms of the scaling functions for various thermodynamic quantities will be derived in Secs. III and IV, but the basic scaling variable can be inferred directly from (1.11). The linear character of the spectrum, (1.12), will cease to dominate when $k_BT\sim mc^2$. For $k_BT\gg mc^2$, the behavior will be controlled by the high-energy modes, for which $E_q\simeq\varepsilon_q^0$, resulting in ideal-gas behavior. For $k_BT\ll mc^2$ only the linear part of the spectrum will be excited, and the Landau results will be valid. It is natural, therefore, that the correct low-temperature scaling variable should be given by the ratio of the two,

$$mc^2/k_BT\propto(a/\Lambda_{T_c})^{d-2}/t^{\phi_0}, \quad \phi_0=1 , \quad (1.13)$$

where ϕ_0 is to be interpreted as the *zero-temperature crossover exponent.* The superfluid density should then take the form

$$\rho_s(n;T)=\rho_s^{\text{ideal}}(n;T)Y_0(B_d(a/\Lambda_{T_c})^{d-2}/t^{\phi_0}) , \qquad a/\Lambda_{T_c},t\ll 1 , \quad (1.14)$$

where $Y_0(y_0)$ is the zero-temperature crossover scaling function and B_d is another metrical factor. By analogy to (1.10), this form implies an *interacting* zero-temperature region defined by

$$t\ll t_0\equiv(T_c/T_Q)^{(d-2)/2} . \quad (1.15)$$

In the region $t_0\ll t\ll t_X\equiv(1-\bar{t}_X)$, the behavior will be ideal-gas-like, with small corrections.

All of the preceding results, valid for the range of dimensionalities $2<d<4$, are summarized pictorially in Fig. 1, where a generic superfluid density profile is shown. The various crossover regions at high and low

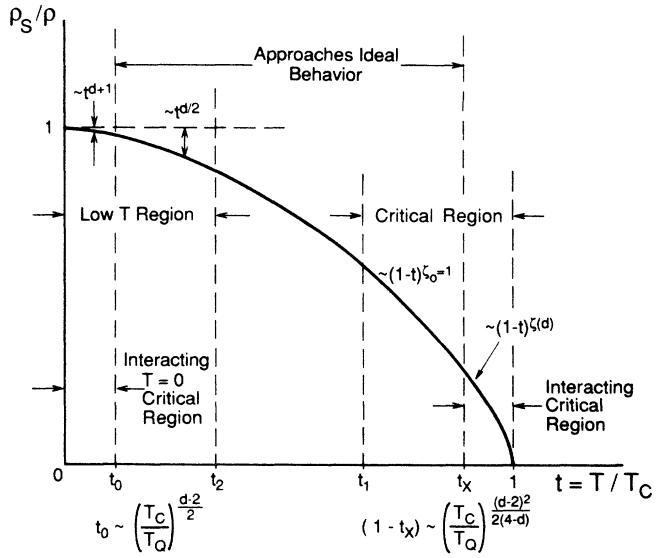

FIG. 1. Schematic plot of a generic low density ρ_s vs T curve for $2<d<4$, showing the various crossover regions discussed in the text. The reduced temperatures t_1 and t_2 are properties only of the ideal Bose gas and are therefore fixed. The crossover temperatures t_0 and t_X are driven toward 0 and 1, respectively, as the density, ρ, or transition temperature, T_c, is reduced, at a rate determined by the exponents shown on the figure. The temperature T_Q is given by $h^2/2\pi m^* k_B a^{*2}$ and is set by the ^{4}He atomic parameters and fundamental constants. For $m^*=m_{\text{He}}$ and $a^*=a_{\text{gas}}$, one finds $T_Q\approx 15.6$ K. For $t\ll t_0$ one has $\rho-\rho_s\sim t^{d+1}$, while for $(1-t)\ll(1-t_X)$ one has $\rho_s\sim(1-t)^{\zeta(d)}$ with $\zeta(3)\simeq\frac{2}{3}$. The crossovers are to the ideal results $\rho-\rho_s\sim t^{d/2}$ and $\rho_s\sim(1-t)^{\zeta_0=1}$, respectively.

temperatures are illustrated schematically, and the rate at which they shrink with T_c is indicated.

It is worth discussing briefly the behavior outside the dimensionality range $2<d<4$. In dimensions $d\geq 4$, the low-temperature end of Fig. 1 remains as shown, but the critical behavior becomes trivial: interactions are now irrelevant[5] and ideal behavior (with, at most, logarithmic corrections *in* $d=4$) results over the entire critical region.

In two dimensions the behavior is much more interesting.[6] The ideal gas no longer possesses a finite-temperature transition; interactions must be included from the outset in order to generate a superfluid phase. With interactions, the transition will be of the Kosterlitz-Thouless type.[7] Nevertheless, it is interesting to pose such questions as how the transition temperature varies with density. and, on the scale of the transition temperature, how various quantities (for example, the superfluid fraction) behave. The answers to these questions can be summarized as follows:[6] at *very* low densities, n, such that

$$\ln\ln(1/na^2)\gg 1 , \tag{1.16}$$

there are still two well-separated crossover regions, one near $T=0$, the other near $T=T_c$, with widths

$$t_0\sim 1/\ln(1/na^2) \tag{1.17}$$

and

$$\bar{t}_X\sim 1/\ln\ln(1/na^2) . \tag{1.18}$$

For $\bar{t}\ll\bar{t}_X$ the characteristic Kosterlitz-Thouless jump discontinuity in ρ_s must appear. For $t\ll t_0$, ρ_s obeys a T^3 law. In the intermediate regime, $t_0\ll t\ll t_X$, an ideal gas-like law is obeyed, namely

$$\rho_s\approx\rho(1-T/T_c) \tag{1.19}$$

[compare Eq. (1.8)]. Finally, the transition temperature is defined by the relation

$$n=\ln\ln(1/na^2)\Lambda_{T_c}^{-2} \tag{1.20}$$

[compare Eq. (1.1)]. The appearance of some sort logarithmic dependence is not unexpected since both crossover exponents ϕ and ϕ_0 vanish when $d=2$, but the double logarithms are rather unique.

The final section of this paper is devoted to a discussion of various other results from the helium literature. A method for going beyond the Bogoliubov approximation by way of perturbation theory will be indicated, and some of the problems found in perturbation theory will be related to the spin-wave nature of the superfluid phase. The section will close with a more formal discussion of the various crossovers in the language of renormalization-group flows and fixed points. Several appendices are devoted to technical points referred to in the body of the paper.

II. THE BOGOLIUBOV APPROXIMATION

Discussion of the interacting Bose gas begins with the second quantized Hamiltonian

$$H=\int d^dx\,\psi^\dagger(\mathbf{x})\left[-\frac{\hbar^2}{2m}\nabla^2-\mu\right]\psi(\mathbf{x}) -\tfrac{1}{2}\int d^dx\,[\nu^*\psi(\mathbf{x})+\nu\psi^\dagger(\mathbf{x})] +\tfrac{1}{2}\int d^dx\int d^dx'\,\psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{x}')v(\mathbf{x}-\mathbf{x}')\psi(\mathbf{x}')\psi(x) , \tag{2.1}$$

where $v(\mathbf{x})$ is the pair potential, $\psi(\mathbf{x})$ is the Bose field operator, μ is the chemical potential, and ν is the Bose symmetry breaking field. Beneath the transition temperature, T_0, spontaneous symmetry breaking occurs. This is signaled by the nonvanishing of the average

$$\langle\psi(\mathbf{x})\rangle=\psi_0\neq 0 \tag{2.2}$$

even in the limit of zero off-diagonal field, ν. It is then convenient to define new fields

$$\varphi(\mathbf{x})=\psi(\mathbf{x})-\psi_0 , \tag{2.3}$$

which have the property that

$$\langle\varphi(\mathbf{x})\rangle=\langle\varphi^\dagger(\mathbf{x})\rangle=0 \tag{2.4}$$

and rewrite the Hamiltonian in terms of these fields. The result is

$$H=H_1+H_2+H_3+H_4+C , \tag{2.5}$$

where

$$H_1=\int d^dx\,\{[(-\mu+v_0|\psi_0|^2)\psi_0^*-\tfrac{1}{2}\nu^*]\varphi(\mathbf{x})+\text{H.c.}\} , \tag{2.6}$$

$$H_2=\int d^dx\,\varphi^\dagger(\mathbf{x})\left[-\frac{\hbar^2}{2m}\nabla^2-\mu+v_0|\psi_0|^2\right]\varphi(\mathbf{x}) +\tfrac{1}{2}\int d^dx\int d^dx'[(\psi_0^*)^2\varphi(\mathbf{x}')\varphi(\mathbf{x}) +2|\psi_0|^2\varphi^\dagger(\mathbf{x}')\varphi(\mathbf{x}) +\psi_0^2\varphi^\dagger(\mathbf{x}')\varphi^\dagger(\mathbf{x})]v(\mathbf{x}-\mathbf{x}') , \tag{2.7}$$

$$H_3=\int d^dx\int d^dx'[\psi_0^*\varphi^\dagger(\mathbf{x})\varphi(\mathbf{x})\varphi(\mathbf{x}')v(\mathbf{x}-\mathbf{x}')+\text{H.c.}] , \tag{2.8}$$

$$H_4=\tfrac{1}{2}\int d^dx\int d^dx'\,\varphi^\dagger(\mathbf{x})\varphi^\dagger(\mathbf{x}')v(\mathbf{x}-\mathbf{x}')\varphi(\mathbf{x}')\varphi(\mathbf{x}) , \tag{2.9}$$

$$C=[-\mu|\psi_0|^2+\tfrac{1}{2}v_0|\psi_0|^4-\tfrac{1}{2}(\nu^*\psi_0+\nu\psi_0^*)]V , \tag{2.10}$$

where $v_0=\int d^dx\,v(\mathbf{x})$, V is the volume of the system, and H.c. stands for Hermitian conjugate. The value of $\psi_0(\mu,\nu)$ is determined self-consistently via (2.4). An equivalent method to determine $\psi_0(\mu,\nu)$ is to minimize the free energy:

$$\frac{\partial F}{\partial\psi_0}[\mu,\nu;\psi_0]\,|_{\psi_0(\mu,\nu)}=0 . \tag{2.11}$$

In momentum space one has

$$H_1=\sqrt{V}\{[(-\mu+\nu_0|\psi_0|^2)\psi_0^*-\tfrac{1}{2}\nu^*]a_0+\text{H.c.}\}\ , \quad (2.12)$$

$$H_2=\sum_{\mathbf{k}}\{[\varepsilon_{\mathbf{k}}^0-\mu+(\nu_0+\nu_{\mathbf{k}})|\psi_0|^2]a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \tfrac{1}{2}\nu_{\mathbf{k}}[(\psi_0^*)^2 a_{\mathbf{k}}a_{-\mathbf{k}}+\psi_0^2 a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger]\}\ , \quad (2.13)$$

$$H_3=V^{-1/2}\sum_{\mathbf{k},\mathbf{q}}\nu_{\mathbf{q}}(\psi_0^* a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{k}}a_{\mathbf{q}}+\psi_0 a_{\mathbf{q}}^\dagger a_{\mathbf{k}}^\dagger a_{\mathbf{k}+q})\ , \quad (2.14)$$

$$H_4=V^{-1}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}\nu_{\mathbf{q}}a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{k}'-\mathbf{q}}^\dagger a_{\mathbf{k}'}a_{\mathbf{k}}\ , \quad (2.15)$$

where

$$\nu_{\mathbf{q}}=\int d^dx\ \nu(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}} \quad (2.16)$$

$$a_{\mathbf{k}}=V^{-1/2}\int d^dx\ e^{-i\mathbf{k}\cdot\mathbf{x}}\varphi(\mathbf{x})\ , \quad (2.17)$$

and (2.4) is equivalent to

$$\langle a_0\rangle=\langle a_0^\dagger\rangle=0\ . \quad (2.18)$$

The Hamiltonian (2.5) is slightly different from that obtained from the so-called *Bogoliubov prescription*, which entails dropping the operators a_0 and $a_0^\dagger$ entirely. Thus the term H_1, and all terms in H_2, H_3, and H_4 containing the operator a_0 or $a_0^\dagger$, are eliminated. The fact that this leads to precisely the same physics is a consequence of the thermodynamic limit and Eq. (2.18): In diagrammatic language (2.18) entails the vanishing of all zero-momentum insertions (or "tadpole" diagrams).[8] Similarly, the thermodynamic limit ensures that a zero-momentum term appearing inside a convergent integral carries negligible weight in the infinite-volume limit. The effect of these observations is equivalent to leaving out a_0 and $a_0^\dagger$ from the outset. Note that if one uses the Bogoliubov prescription one is forced to use (2.11) to determine ψ_0.

In order to obtain a tractable model, define the *Bogoliubov Hamiltonian*, H_0, obtained by dropping H_3 and H_4 from (2.5)

$$H_0=H_1+H_2+C\ . \quad (2.19)$$

This model, in fact, contains much of the essential physics at low temperatures, $T/T_c\ll 1$. The reasons for this will be discussed in Sec. V. H_2 can be diagonalized by a Bogoliubov transformation. In order to better elucidate the physical meaning of the transformation, it will be carried through in a slightly different manner than is standard. The method is motivated by a close analogy to the magnetic transition in the classical spin s^4 model (see Appendix C). To begin with $\varphi(\mathbf{x})$ is divided into components parallel and transverse to ψ_0,

$$\varphi_\parallel(\mathbf{x})=[\psi_0^*\varphi(\mathbf{x})+\psi_0\varphi^\dagger(\mathbf{x})]/2|\psi_0|\ ,$$
$$\varphi_\perp(\mathbf{x})=[\psi_0^*\varphi(\mathbf{x})-\psi_0\varphi^\dagger(\mathbf{x})]/2i|\psi_0|\ , \quad (2.20)$$

or, in Fourier transform,

$$a_{\parallel,\mathbf{k}}=(\psi_0^* a_{\mathbf{k}}+\psi_0 a_{-\mathbf{k}}^\dagger)/2|\psi_0|\ ,$$
$$a_{\perp,\mathbf{k}}=(\psi_0^* a_{\mathbf{k}}-\psi_0 a_{-\mathbf{k}}^\dagger)/2i|\psi_0|\ . \quad (2.21)$$

The only important commutation relation is

$$[a_{\parallel,\mathbf{k}},a_{\perp,-\mathbf{k}}]=i/2\ , \quad (2.22)$$

all others vanish. In terms of these new operators H_2 becomes

$$H_2=\sum_{\mathbf{k}}\{[\varepsilon_{\mathbf{k}}^0-\mu+|\psi_0|^2(\nu_0+2\nu_{\mathbf{k}})]a_{\parallel,\mathbf{k}}a_{\parallel,-\mathbf{k}} +(\varepsilon_{\mathbf{k}}^0-\mu+|\psi_0|^2\nu_0)a_{\perp,\mathbf{k}}a_{\perp,-\mathbf{k}} +\tfrac{1}{2}[\varepsilon_{\mathbf{k}}^0-\mu+|\psi_0|^2(\nu_0+\nu_{\mathbf{k}})]\}\ . \quad (2.23)$$

Note that if $|\psi_0|^2=0$, the coefficients of the first two terms are identical. The broken symmetry, therefore, manifests as an inequivalence between the transverse and longitudinal directions. Furthermore, within the Bogoliubov approximation one has $\mu=\nu_0|\psi_0|^2$ (see below). Therefore, the *transverse* fluctuations are, in fact *massless*, and represent the usual Goldstone modes. The Bogoliubov transformation is a rescaling of the operators $a_{\parallel,k}$ and $a_{\perp,k}$ so as to artificially restore the symmetry of the two directions: define

$$\alpha_{\parallel,\mathbf{k}}=f_{\mathbf{k}}^{1/4}a_{\parallel,\mathbf{k}},\quad \alpha_{\perp,\mathbf{k}}=f_{\mathbf{k}}^{-1/4}a_{\perp,\mathbf{k}}\ , \quad (2.24)$$

where

$$f_k=[\varepsilon_{\mathbf{k}}^0-\mu+|\psi_0|^2(\nu_0+2\nu_{\mathbf{k}})]/(\varepsilon_{\mathbf{k}}^0-\mu+|\psi_0|^2\nu_0) \quad (2.25)$$

is the ratio of the two coefficients in (2.23). This transformation preserves (2.22), and is therefore canonical. The Hamiltonian H_2 becomes

$$H_2=\sum_{\mathbf{k}}\{E_{\mathbf{k}}(a_{\parallel,\mathbf{k}}\alpha_{\parallel,-\mathbf{k}}+\alpha_{\perp,\mathbf{k}}\alpha_{\perp,-\mathbf{k}}) +\tfrac{1}{2}[\varepsilon_{\mathbf{k}}^0-\mu+|\psi_0|^2(\nu_0+\nu_{\mathbf{k}})]\}\ , \quad (2.26)$$

which, upon converting back to the usual creation and annihilation operators

$$\alpha_{\mathbf{k}}=\alpha_{\parallel,\mathbf{k}}+i\alpha_{\perp,\mathbf{k}}\ ,$$
$$\alpha_{\mathbf{k}}^\dagger=\alpha_{\parallel,-\mathbf{k}}-i\alpha_{\perp,-\mathbf{k}}\ , \quad (2.27)$$

reads

$$H_2=\sum_{\mathbf{k}}E_{\mathbf{k}}(\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}}+c_{\mathbf{k}})\ , \quad (2.28)$$

where

$$E_{\mathbf{k}}^2=[\varepsilon_{\mathbf{k}}^0-\mu+|\psi_0|^2(\nu_0+2\nu_{\mathbf{k}})](\varepsilon_{\mathbf{k}}^0-\mu+|\psi_0|^2\nu_0) \quad (2.29)$$

is the product of the two coefficients in (2.23), and the zero point shifts are

$$c_{\mathbf{k}}=-\frac{1}{2}\left[\frac{\varepsilon_k^0-\mu+|\psi_0|^2(\nu_0+\nu_{\mathbf{k}})}{E_{\mathbf{k}}}-1\right]\ . \quad (2.30)$$

Using (2.27), (2.24), and (2.21) one can write the a's directly in terms of the α's as

$$a_{\mathbf{k}}^{\dagger}=u_{\mathbf{k}}\alpha_{\mathbf{k}}^{\dagger}-v_{\mathbf{k}}\alpha_{-\mathbf{k}}\ ,$$
$$a_{\mathbf{k}}=u_{\mathbf{k}}^{*}\alpha_{\mathbf{k}}-v_{\mathbf{k}}^{*}\alpha_{-\mathbf{k}}^{\dagger}\ , \tag{2.31}$$

where one finds

$$|u_{\mathbf{k}}|^2=1+|v_{\mathbf{k}}|^2=\tfrac{1}{2}(\xi_{\mathbf{k}}/E_{\mathbf{k}}+1)\ ,$$
$$\arg(u_{\mathbf{k}})=\arg(v_{\mathbf{k}})=\arg(\psi_0^{*})=\arg(\nu^{*})\ , \tag{2.32}$$

where $\arg(z)$ is the phase of the complex number z and

$$\xi_{\mathbf{k}}=\varepsilon_{\mathbf{k}}^{0}-\mu+(\mathscr{v}_0+\mathscr{v}_{\mathbf{k}})|\psi_0|^2\ . \tag{2.33}$$

These are the standard results, more often obtained by positing (2.31) directly and choosing $v_{\mathbf{k}}$ and $u_{\mathbf{k}}$ in such a manner that the diagonal form (2.28) results.[2,3]

In terms of the α's, H_1 becomes

$$H_1=f_0^{-1/4}\sqrt{V}\,[(-\mu+|\psi_0|^2\mathscr{v}_0)|\psi_0|-\tfrac{1}{2}|\nu|](\alpha_0+\alpha_0^{\dagger}) \tag{2.34}$$

and a simple calculation gives

$$0=\langle a_0\rangle_0$$
$$=(-\mu+3\mathscr{v}_0|\psi_0|^2)^{-1}[\tfrac{1}{2}\nu-(-\mu+\mathscr{v}_0|\psi_0|^2)\psi_0]\ , \tag{2.35}$$

where the average is with respect to H_0. The equation of state is therefore

$$\nu=2(-\mu+\mathscr{v}_0|\psi_0|^2)\psi_0\ , \tag{2.36}$$

which is of the usual Landau form. The essential results are obtained by setting $\nu=0$, yielding

$$n_0\equiv|\psi_0|^2=\mu/\mathscr{v}_0 \tag{2.37}$$

while the energy spectrum becomes

$$E_{\mathbf{k}}=[(\varepsilon_{\mathbf{k}}^{0})^2+2n_0\mathscr{v}_{\mathbf{k}}\varepsilon_{\mathbf{k}}^{0}]^{1/2} \tag{2.38}$$

with an asympotically linear slope

$$c=(n_0\mathscr{v}_0/m)^{1/2}=\hbar(\mu/m)^{1/2} \tag{2.39}$$

as $|\mathbf{q}|\to 0$. To demonstrate the equivalence of (1.12) and (2.39) the relationship between $\mathscr{v}_0$ and a is needed. This raises a technical point hinging on the question of what one means by weakly interacting. A dilute hard-core Bose gas ought to be considered weakly interacting, but there one actually has $\mathscr{v}_0=\infty$. The solution to the problem involves converting the pair potential to the scattering matrix by a diagrammatic resummation method.[2,8] The result is that in low orders in perturbation theory one may replace $\mathscr{v}_0$ by the effective potential $4\pi\hbar^2a/m$ [$d=3$; see Eq. (2.17) in Ref. 1(b) for the general d result]; the system may then be considered weakly interacting so long as $na^d\ll 1$. These considerations along with the fact that $n_0\simeq n$ at low temperatures and low densities (see Sec. III), lead immediately to (1.11) and (1.12).

As a final point, note that when $\nu\neq 0$ the energy spectrum takes the form

$$E_{\mathbf{k}}=\{[\varepsilon_{\mathbf{k}}^{0}+|\nu|/2|\psi_0|]^2+2n_0\mathscr{v}_{\mathbf{k}}[\varepsilon_{\mathbf{k}}^{0}+|\nu|/2|\psi_0|]\}^{1/2}\ . \tag{2.40}$$

The field ν, therefore, generates a gap in the excitation spectrum:

$$E_0=[|\nu|^2/4|\psi_0|^2+\mathscr{v}_0|\psi_0||\nu|]^{1/2}\approx(\mathscr{v}_0|\psi_0||\nu|)^{1/2},\quad \nu\to 0\ . \tag{2.41}$$

III. THERMODYNAMICS

A. Free energy

In order to discuss thermodynamics in a unified way, one must introduce a free energy. An obvious possibility is

$$F_2\equiv-(\beta V)^{-1}\ln \mathrm{tr}\,e^{-\beta H_0}$$
$$=\int_k[\beta^{-1}\ln(1-e^{-\beta E_{\mathbf{k}}})-E_{\mathbf{k}}|v_{\mathbf{k}}|^2]$$
$$-\mu n_0+\tfrac{1}{2}\mathscr{v}_0n_0^2-|\nu||\psi_0| \tag{3.1}$$

with $E_{\mathbf{k}}$ defined by (2.25), and the convenient notation

$$\frac{1}{V}\sum_{\mathbf{k}}\to\int_k\equiv\int d^dk\frac{1}{(2\pi)^d} \tag{3.2}$$

has been adopted. Unfortunately this choice creates an inconsistency. Given a free energy, (2.11) should be used to determine ψ_0. However, applying this to (3.1) yields

$$2(-\mu+n_0\mathscr{v}_0)\psi_0-\nu+2\psi_0I[\mu,\nu;\psi_0]=0 \tag{3.3}$$

in which

$$I[\mu,\nu;\psi_0]=\int_k\left[\frac{\xi_{\mathbf{k}}(\mathscr{v}_0+\mathscr{v}_{\mathbf{k}})-n_0\mathscr{v}_{\mathbf{k}}^2}{E_{\mathbf{k}}}n_B(E_{\mathbf{k}})+\frac{1}{2}\left[\frac{\xi_{\mathbf{k}}(\mathscr{v}_0+\mathscr{v}_{\mathbf{k}})-n_0\mathscr{v}_{\mathbf{k}}^2}{E_{\mathbf{k}}}-(\mathscr{v}_0+\mathscr{v}_{\mathbf{k}})\right]\right]\ . \tag{3.4}$$

This differs from (2.36), which is apparently embodied in the Landau term, C, alone, by the additional term I. If the solution to (3.3) is substituted back into $E_{\mathbf{k}}$, one finds a gap in the excitation spectrum even when $\nu=0$, a possibility which is ruled out on general grounds by the spin-wave character of the low-temperature phase (see Sec. V and below). In the context of helium, the nonexistence of a gap is known as the Hugenholtz-Pines theorem,[9] and follows more generally from Goldstone's theorem.[10] In fact, disagreements with spin-wave theory already appear in Eq. (2.36): one knows, for example, that the longitudinal susceptibility

$$\chi_{\nu}=(\partial|\psi_0|/\partial|\nu|)_{\mu,T} \tag{3.5}$$

should diverge as $|\nu|^{-\epsilon/2}$ ($\epsilon=4-d$) as $|\nu|\to 0$,[11] whereas (2.36) predicts a finite value, $\chi_{\nu}=(4\mu)^{-1}$, at $\nu=0$. These inconsistencies should not be too surprising in retrospect: the k sum in (3.1) represents the first fluctuation correction to the free energy, while (2.36) is the Landau result for ψ_0 which ignores fluctuations. The

term $I[\mu,\nu;\psi_0]$ in (3.3) is the first fluctuation correction to ψ_0. The inconsistencies in the Bogoliubov free energy, F_2, result therefore from an improper mixing of mean-field results with fluctuation results.

These problems can be avoided by identifying the parameter ψ_0 appearing in H_0, not as the actual long-range order, but rather as a parameter determined in this approximation from (2.36). To avoid confusion, this Landau result for ψ_0 will be denoted by $\bar{\psi}_0(\mu,\nu)$, an explicit function of the other parameters. In this approximation the actual result for ψ_0 is determined in the usual way as a derivative with respect to the off-diagonal field

$$\psi_0=-2\frac{\partial \bar{F}_2}{\partial \nu^*}(\mu,\nu,\nu^*) \tag{3.6}$$

in which $\bar{F}_2$ is given by (3.1) but with ψ_0 replaced by $\bar{\psi}_0$ [similarly, the parameters $u_{\mathbf{k}}$, $v_{\mathbf{k}}$, $\xi_{\mathbf{k}}$, and $E_{\mathbf{k}}$ are defined by (2.29)–(2.33) with $\bar{\psi}_0$ replacing ψ_0]. The result of (3.6) is

$$\psi_0=\bar{\psi}_0\{1-(-\mu+3\nu_0|\bar{\psi}_0|^2)^{-1}I[\mu,\nu;\bar{\psi}_0]\} \tag{3.7}$$

which also represents the solution of (3.3) to first order in ν_0. This result is now consistent with spin-wave predictions: the low-field susceptibility (dominated by the Bose factor part of the integral I) is

$$\chi_\nu(\mu,T)\approx\frac{\beta\Gamma(\frac{1}{2}\epsilon)}{2^{(d+2)/2}(\beta|\nu|/|\psi_0|)^{\epsilon/2}n_0\Lambda_T^d}\ ,$$

$$\beta|\nu|/|\psi_0|\ll\min(\beta\mu,1/\beta\mu),\quad 2<d<4 \tag{3.8}$$

which has the requisite $|\nu|^{-\epsilon/2}$ divergence at finite temperature. At zero temperature, only the second term in (3.4) survives, and one has instead (for $|\nu|/|\psi_0|\ll\mu$)

$$\chi_\nu(\mu,T=0)\approx\begin{cases}\dfrac{\Gamma(\frac{1}{2}(3-d))}{(4\pi)^{(d+1)/2}2^{(d+2)/2}\mu n_0\xi_0^d}(\mu|\psi_0|/|\nu|)^{(3-d)/2}, & 2<d<3\ ,\\ \dfrac{1}{64\sqrt{2}\pi^2}\dfrac{1}{\mu n_0\xi_0^3}\ln(\mu|\psi_0|/|\nu|), & d=3\ ,\\ \text{const}, & d>3\ ,\end{cases} \tag{3.9}$$

where

$$\xi_0=\hbar/(2m\mu)^{1/2} \tag{3.10}$$

is the important characteristic length at $T=0$, and diverges in the weakly interacting limit. The divergence in the low-field susceptibility at $T=0$ is weaker than at $T>0$ reflecting the fact that the quantum degrees of freedom dominate at zero temperature and give rise to behavior characteristic of a classical system in one higher dimension (see Sec. V).

Other thermodynamic properties will now be considered. The expression for the number density is

$$\begin{aligned}n&=-\left[\frac{\partial\bar{F}_2}{\partial\mu}\right]_{\nu,T}\\&=|\bar{\psi}_0|^2\left[1-\frac{2}{(-\mu+3\nu_0\bar{n}_0)}I[\mu,\nu;\bar{\psi}_0]\right]\\&\quad+\int_k\left[\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}n_B(E_{\mathbf{k}})+\frac{1}{2}\left[\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}-1\right]\right].\end{aligned} \tag{3.11}$$

To first order in ν_0, the first term is simply $|\psi_0|^2$. Equation (3.11) therefore becomes

$$n=|\psi_0|^2+\int_k\left[\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}n_B(E_{\mathbf{k}})+\frac{1}{2}\left[\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}-1\right]\right] \tag{3.12}$$

which is more commonly obtained from the alternative definition

$$n=\frac{1}{V}\sum_{\mathbf{k}}\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\rangle+|\psi_0|^2\ . \tag{3.13}$$

However, if this average is evaluated with $H_0[\bar{\psi}_0]$ as the Hamiltonian, (3.12) is recovered, but with the incorrect result, $|\bar{\psi}_0|^2$, as the second term on the right-hand side. This difficulty arises because the first term in (3.12) represents the fluctuation part of the density, hence, for consistency, $|\psi_0|^2$ should also contain a fluctuation correction. This correction is embodied in (3.7), and also appears in (3.11). Therefore, in order to use (3.13) consistently, the second term on the right-hand side should be *left as* $|\psi_0|^2$. This distinction between $|\psi_0|^2$, as it appears in (3.13), and $|\bar{\psi}_0|^2$, as it appears as a parameter in $H_0[\bar{\psi}_0]$, is not generally made explicit. The definition (3.13) involves the Green's function, rather than the free energy—a quantity which contains further subtleties which will be discussed in Sec. V.

B. Zero-temperature crossover scaling

The low-temperature scaling forms alluded to in the Introduction will now be derived with emphasis on the condensate fraction. Other properties follow in a simple way as well. The superfluid density will be addressed in

Sec. IV.

At $T=0$, the Bose occupation factor gives no contribution and (3.12) yields

$$n = n_0 + \tfrac{1}{2}\int_k (\xi_{\mathbf{k}}/E_{\mathbf{k}} - 1) \ . \tag{3.14}$$

It is convenient to define the *shape* of the potential

$$\phi(a_0^2 k^2) \equiv \omega_{\mathbf{k}}/\omega_0 \ , \tag{3.15}$$

where a_0 measures the spatial range of the interactions. With this definition the *condensate deficit* is defined as

$$\begin{aligned} n - n_0(T=0) &\simeq \tfrac{1}{2} K_d \xi_0^{-d} \int_0^\infty dy\, y^{d-1} \left[\frac{y^2+\phi}{y(y^2+2\phi)^{1/2}} - 1 \right] \\ &= \frac{1}{d} K_d \xi_0^{-d} \Gamma(\tfrac{1}{2}(d-1)) \Gamma(\tfrac{1}{2}(4-d)) 2^{(d-4)/2} \pi^{-1/2} [1 + O((a_0/\xi_0)^{4-d})] \ , \end{aligned} \tag{3.16}$$

where $K_d = 2/\Gamma(\frac{1}{2}d)(4\pi)^{d/2}$ is the area of the unit sphere in d dimensions divided by $(2\pi)^d$, and $\phi = \phi(a_0^2 y^2/\xi_0^2)$. The various Γ functions arise from the use of the identity[12]

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^\infty \frac{s^{x-1}}{(1+s)^{x+y}} ds \ . \tag{3.17}$$

For $d > 4$ the momentum cutoff inherent in ϕ is required for convergence, and one will have

$$n - n_0(0) \sim (a_0/\xi_0)^4 a_0^{-d}, \quad d > 4 \tag{3.18}$$

with a nonuniversal coefficient. From (3.14), for large ξ_0, the zero-temperature condensate fraction becomes

$$n_0(T=0) \simeq \hbar^2/2m\omega_0\xi_0^2 \ . \tag{3.19}$$

This is much larger than $n - n_0(T=0)$, hence $n_0 \simeq n$ and (3.16) becomes

$$[n - n_0(T=0)]/n \approx A_d (n a^d)^{(d-2)/2}, \quad 2 < d < 4 \tag{3.20}$$

in which A_d ($=8/3\pi$ for $d=3$) is a constant and a is the s-wave scattering length.

At low temperatures, the essential temperature dependence of n_0 originates from the Bose factor in (3.12). The other term in the integrand also depends implicitly on temperature at fixed n through $\mu(T)$, but this dependence is of higher order in the interactions. Therefore,

$$\begin{aligned} \Delta n_0(T) \equiv n_0(0) - n_0(T) &\simeq \int_k \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}} n_B(E_{\mathbf{k}}) \\ &= \xi_0^{-d} K_d \int_0^\infty dy\, y^{d-1} \frac{y^2+\phi}{y(y^2+2\phi)^{1/2}} \\ &\quad \times n_B[\mu y (y^2+2\phi)^{1/2}] \ . \end{aligned} \tag{3.21}$$

Evidently $\xi_0^d \Delta n_0(T)$ is a function only of the scaled variables a_0/ξ_0 and $\beta\mu = \Lambda_T^2/\xi_0^2 \simeq n\omega_0/k_B T$. For weak interactions the shape of the potential becomes unimportant, and one can simply take $\phi \equiv 1$. For ease of comparison to the ideal gas, define $r = \beta\mu$, change variables to $x = \beta\mu y (y^2+2)^{1/2}$ and write

$$\Delta n_0(T) \simeq \zeta(\tfrac{1}{2}d) \Lambda_T^{-d} N(r) \ , \tag{3.22}$$

where the prefactor is the ideal result, and

$$N(r) = [\zeta(\tfrac{1}{2}d)\Gamma(\tfrac{1}{2}d)]^{-1} \times \int_0^\infty dx\, [(r^2+x^2)^{1/2} - r]^{(d-2)/2} / (e^x - 1) \tag{3.23}$$

is the zero-temperature crossover scaling form. The argument, r, can be written in the more transparent form

$$r \simeq B_d (a/\Lambda_{T_c})^{d-2}/t, \quad t = T/T_c \ll 1 \ , \tag{3.24}$$

where B_d is a d-dependent constant, and the critical temperature $T_c \simeq T_0$ has been introduced via (1.1). The numerator of (3.24), which may also be written as $B_d'(n a^d)^{(d-2)/d}$, represents the dimensionless small parameter in the problem. As discussed in the Introduction, this parameter has precisely the same form as that appearing in the *critical* crossover region.[1] This confirms the proposition (1.13) for the form of the crossover variable, as well as the various exponent values.

The scaling function $N(r)$ yields very different behaviors, depending on the size of r. Asymptotically one has

$$N(r) \simeq \begin{cases} 1 + O(r^{(d-2)/2}), & r \to 0 \ , \\ \dfrac{\Gamma(d-1)\zeta(d-1)}{\Gamma(\frac{1}{2}d)\zeta(\frac{1}{2}d)} (2r)^{(2-d)/2} [1 + O(r^{-2})] \ , & \\ \qquad\qquad r \to \infty \ . \end{cases} \tag{3.25}$$

This yields

$$\frac{\Delta n_0(T)}{n} \simeq \begin{cases} t^{d/2}, & (n a^d)^{(d-2)/d} \ll t \ll 1 \ , \\ N_d (n a^d)^{-(d-2)^2/2d} t^{d-1} \ , & \\ \qquad\qquad t \ll (n a^d)^{(d-2)/d} \ , \end{cases} \tag{3.26}$$

where $N_d = \pi^{3/2}/6\zeta(\frac{3}{2})^{4/3}$ in $d=3$. The first line of (3.26) represents the ideal-gas result; the second line represents the interacting result. The function $N(r)$ interpolates between the two results and demonstrates the mechanism for the discontinuous change in exponents discussed in the Introduction. It should be emphasized that (3.22) and (3.23) are asymptotically exact only in the scaling limit (i.e., for small a/Λ_{T_c} and T/T_c, but arbitrary r). Thus, for example, given T_c (or n), the replacement of n_0 by n is valid only for $t \ll 1$; hence, the range of r is re-

stricted to $r \gg (na^d)^{(d-2)/d}$. Therefore, the entire domain of the scaling function is explored only in the limit $na^d \to 0$.

Clearly, other thermodynamic quantities will scale in a similar fashion—the variable $r = \beta\mu$ being the only important quantity in the region of interest. Corresponding scaling functions may be readily exhibited. The results are simply a restatement of standard Bogoliubov results.

It is interesting to inquire if the Bogoliubov free energy can yield information about the *critical* region. Of course, all standard approximations break down in this region. Nevertheless, as shown in Ref. 1(b), crossover exponents and scaling fields can be obtained by matching the results to a critical scaling form. In Ref. 1(b) this matching procedure was carried through in the disordered phase, $T > T_c$. In Appendix A the same procedure is applied to the ordered phase. The fact that identical answers are obtained is a further check on the validity of the scaling approach.

IV. THE SUPERFLUID DENSITY

A. Definitions

The superfluid density is a torsional spring constant which provides a response to helical twists of the order parameter,

$$\psi_0(\mathbf{r}) \equiv \langle \psi(\mathbf{r}) \rangle = \psi_0 e^{i\mathbf{k}_0 \cdot r} . \tag{4.1}$$

Such a twist can be imposed in several ways. The method most commonly used in the helium literature involves adding a term

$$-\frac{\hbar}{m}\mathbf{k}_0 \cdot \mathbf{P} = \int d^d r \psi^\dagger(\mathbf{r}) i \frac{\hbar^2}{m} \mathbf{k}_0 \cdot \nabla \psi(\mathbf{r})$$

$$= -\frac{\hbar}{m}\mathbf{k}_0 \cdot \sum_{\mathbf{k}} \hbar \mathbf{k} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \tag{4.2}$$

to the Hamiltonian (2.1), where $\mathbf{P}$ is the momentum operator. The action of this term is to impose a uniform drift velocity $v_n = (\hbar/m)\mathbf{k}_0$ on the quasiparticle excitations,[2] giving rise to a net momentum flux $\langle \mathbf{P} \rangle / V = \rho_n v_n$. The proportionality constant ρ_n defines the normal fluid mass density, and the superfluid mass density is then obtained from

$$\rho_s \equiv \rho - \rho_n . \tag{4.3}$$

It is important to realize, however, that this definition relies heavily on Galilean invariance (hence, on momentum conservation), and therefore fails in nontranslation invariant (e.g., lattice) systems where ρ_n is not defined.

An alternative formulation,[13,14] which has a firmer mathematical basis, enables calculation of ρ_s directly without reference to ρ_n. When Galilean invariance is satisfied, this definition will be seen to be equivalent to the one above. The superfluid density is calculated by comparing two systems with different boundary conditions. Let F^θ, $-\pi \le \theta \le \pi$, be the free-energy density for a finite system of length L and cross sectional area A, defined with the following boundary conditions: if $\hat{\mathbf{e}}_0$ is a unit vector pointing along the length L, the operator $e^{-i\theta \hat{\mathbf{e}}_0 \cdot \mathbf{r}/L}\psi(\mathbf{r})$ should obey periodic boundary conditions in all directions. This boundary condition imposes a twist of angle θ along the length of the system. Equation (4.1) therefore should result with $\mathbf{k}_0 = \theta \hat{\mathbf{e}}_0 / L$. The superfluid density is defined then in terms of the asymptotic difference between the two free-energy densities

$$\rho_s \equiv \lim_{\substack{A \to \infty \\ L \to \infty}} \frac{2L^2 m^2}{\hbar^2 \theta^2} [F^\theta(\mu, T) - F^0(\mu, T)] . \tag{4.4}$$

The answer should be independent of θ. This equation simply says that such a twist gives a free-energy increment $\frac{1}{2}\rho_s v_s^2$, where $v_s = \hbar \mathbf{k}_0 / m$.

Parenthetically, this definition shows that ρ_s is the analog of the *surface tension* for systems with a discrete symmetry (such as fluids or Ising magnets). The surface tension is defined also in terms of the asymptotic difference between two free energies, in this case F^0 (periodic boundary conditions) and F^π (antiperiodic boundary conditions). Here, however, the result is proportional to L^{-1}, rather than L^{-2}; and the width over which the order parameter varies is restricted to a surfacelike region whose thickness is much smaller than L.

Use of the definition (4.4) requires the ability to calculate free energies for finite systems. In general such a calculation is extremely difficult. A third definition—which avoids this difficulty, at the expense, perhaps, of some mathematical rigor—would be more convenient. This requires the imposition of a finite twist in the thermodynamic limit and is accomplished by the application of a rotating external field. Thus the usual constant field term H_1 in (2.1) is replaced by the rotating field term[14]

$$H_1(\mathbf{k}_0) = -\tfrac{1}{2} \int d^d r [\nu_0^* e^{-i\mathbf{k}_0 \cdot \mathbf{r}} \psi(\mathbf{r}) + \nu_0 e^{i\mathbf{k}_0 \cdot \mathbf{r}} \psi^\dagger(\mathbf{r})] . \tag{4.5}$$

The superfluid density is then defined as

$$\rho_s = \lim_{\nu_0 \to 0} \left[\lim_{k_0 \to 0} \frac{m^2}{\hbar^2} \frac{\partial^2}{\partial k_0^2} F[\mu, T, \mu_0; k_0] \right] , \tag{4.6}$$

where F is the bulk free-energy density. The loss of mathematical rigor originates in the fact that in (4.6) k_0 is permitted to vanish only after the limit $L \to \infty$ is taken, rather than as the limit is taken. This is a mathematically nontrivial difference, but should not be important in any reasonable model.[15]

B. The Bogoliubov transformation

Since the second and third definitions above for the superfluid density will turn out to be notationally identical, it is convenient to proceed with both of them. Differences will be pointed out along the way, and the relation to the first definition will be indicated at the end.

Begin once more with the Hamiltonian (2.1), but with the last term replaced by (4.5). Since a twist in ψ_0 is anticipated, in place of (2.3) the substitution

$$\psi^\dagger(\mathbf{r})=\psi_0^* e^{-i\mathbf{k}_0\cdot\mathbf{r}}+\varphi^\dagger(\mathbf{r}) ,$$
$$\psi(\mathbf{r})=\psi_0 e^{i\mathbf{k}_0\cdot\mathbf{r}}+\varphi(\mathbf{r}) \tag{4.7}$$

is made. The Fourier decomposition of $\varphi(\mathbf{r})$ is defined by

$$\varphi(\mathbf{r})=V^{-1/2}\sum_{\mathbf{k}} a_{\mathbf{k}} e^{i(\mathbf{k}+\mathbf{k}_0)\cdot\mathbf{r}} ,$$
$$\varphi^\dagger(\mathbf{r})=V^{-1/2}\sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger e^{-i(\mathbf{k}+\mathbf{k}_0)\cdot\mathbf{r}} , \tag{4.8}$$

where $\mathbf{k}_0=\theta\hat{\mathbf{e}}_0/L$ for the second definition, or some finite value, independent of L, for the third definition. The summation is always over the usual set of $\mathbf{k}$ consistent with *periodic* boundary conditions. The functions $V^{-1/2}e^{i(\mathbf{k}+\mathbf{k}_0)\cdot\mathbf{r}}$ form an orthonormal basis for functions with phase change θ boundary conditions, and the mapping $\psi\rightarrow e^{-i\mathbf{k}_0\cdot\mathbf{r}}\psi$ provides a one-to-one correspondence between these functions and periodic functions. It is crucial to note that, for each θ, these functions form a linear space (i.e., the θ boundary condition is preserved under linear transformations).

In momentum space the Hamiltonian becomes

$$H^{k_0}=H_1^{k_0}+H_2^{k_0}+H_3^{k_0}+H_4^{k_0}+C^{k_0} , \tag{4.9}$$

where $H_3^{k_0}$ and $H_2^{k_0}$ are identical to (2.14) and (2.15), while, with the definition $\varepsilon_0=\hbar^2k_0^2/2m$,

$$H_1^{k_0}=V^{1/2}\{[(\nu_0\,|\,\psi_0\,|^2+\varepsilon_0-\mu)\psi_0-\tfrac{1}{2}\nu_0]a_0^\dagger+\text{H.c.}\} , \tag{4.10}$$

$$H_2^{k_0}=\sum_{\mathbf{k}}\{[\varepsilon_{\mathbf{k}+\mathbf{k}_0}^0-\mu+(\nu_0+\nu_{\mathbf{k}})\,|\,\psi_0\,|^2]a_{\mathbf{k}}^\dagger a_{\mathbf{k}} +\tfrac{1}{2}\nu_{\mathbf{k}}[(\psi_0^*)^2a_{-\mathbf{k}}a_{\mathbf{k}}+\text{H.c.}]\} , \tag{4.11}$$

and

$$C^{k_0}=[(\varepsilon_0-\mu)\,|\,\psi_0\,|^2+\tfrac{1}{2}\nu_0\,|\,\psi_0\,|^4-\tfrac{1}{2}(\nu_0^*\psi_0+\nu_0\psi_0^*)]V . \tag{4.12}$$

In (4.9)–(4.12) $\nu_{\mathbf{k}}$ is defined always by (2.16), independently of the choice for $\mathbf{k}_0$.

Neglecting $H_3^{k_0}$ and $H_4^{k_0}$, the condition $\langle a_0\rangle=0$ now leads to

$$\nu_0=2\psi_0(\varepsilon_0-\mu+\nu_0\,|\,\psi_0\,|^2) \tag{4.13}$$

which generalizes (2.36). The solution to this equation will be denoted by $\bar{\psi}_0^{k_0}(\mu,\nu_0)$.

Since $\varepsilon_{\mathbf{k}+\mathbf{k}_0}^0\neq\varepsilon_{-\mathbf{k}+\mathbf{k}_0}^0$, $\mathbf{k}$ and $-\mathbf{k}$ are no longer equivalent, and the Bogoliubov diagonalization, (2.31), of $H_2^{k_0}$ becomes more complicated. However, the following observation simplifies the problem considerably. Let $p_{\mathbf{k}}$ be an odd function of $\mathbf{k}$. Then, assuming that $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are even, note that

$$\sum_{\mathbf{k}} p_{\mathbf{k}}a_{\mathbf{k}}^\dagger a_{\mathbf{k}}=\sum_{\mathbf{k}} p_{\mathbf{k}}(\,|\,u_{\mathbf{k}}\,|^2-|\,v_{\mathbf{k}}\,|^2)\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}} =\sum_{\mathbf{k}} p_{\mathbf{k}}\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}} . \tag{4.14}$$

This shows that odd operators of this type are invariant under the transformation, and therefore remain diagonal. In particular,

$$\mathbf{P}=\sum_{\mathbf{k}}\hbar\mathbf{k}a_{\mathbf{k}}^\dagger a_{\mathbf{k}}=\sum_{\mathbf{k}}\hbar\mathbf{k}\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}} . \tag{4.15}$$

It is therefore only the even part of $H_2^{k_0}$ which determines $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$. This in turn verifies self-consistently that $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are indeed even.[16]

The even part of $H_2^{k_0}$ differs from (2.13) only in that μ is replaced by $\mu-\varepsilon_0$. The odd part of $H_2^{k_0}$ is $(\hbar\mathbf{k}_0/m)\cdot\mathbf{P}$. One has immediately then

$$H_2^{k_0}=\sum_{\mathbf{k}}\left[\left[E_{\mathbf{k}}^{k_0}+\frac{\hbar^2}{m}\mathbf{k}_0\cdot\mathbf{k}\right]\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}}+\frac{1}{2}(E_{\mathbf{k}}^{k_0}-\xi_{\mathbf{k}}^{k_0})\right] , \tag{4.16}$$

where

$$\xi_{\mathbf{k}}^{k_0}=\varepsilon_{\mathbf{k}}^0+\varepsilon_0-\mu+(\nu_0+\nu_{\mathbf{k}})\,|\,\bar{\psi}_0^{k_0}\,|^2 \tag{4.17}$$

and

$$E_{\mathbf{k}}^{k_0}=[(\xi_{\mathbf{k}}^{k_0})^2-(\nu_{\mathbf{k}}\,|\,\bar{\psi}_0^{k_0}\,|^2)^2]^{1/2} . \tag{4.18}$$

For later reference, one can follow through the *first* definition for ρ_s, using (4.2) and (4.3). The final result for H_2 is the same as (4.16) except that the superscript k_0 should be dropped and the sign of the momentum term reversed:[17]

$$H_2=\sum_{\mathbf{k}}\left[\left[E_{\mathbf{k}}-\frac{\hbar^2}{m}\mathbf{k}_0\cdot\mathbf{k}\right]\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}}+\frac{1}{2}(E_{\mathbf{k}}-\xi_{\mathbf{k}})\right] , \tag{4.19}$$

where $E_{\mathbf{k}}$ and $\xi_{\mathbf{k}}$ are defined in (2.29) and (2.33). Alternatively, $H_2^{k_0}$ is obtained from H_2 simply by reversing the sign of $\mathbf{k}_0$ and replacing μ by $\mu-\varepsilon_0$. The significance of this will be discussed below.

C. The superfluid density

The free energy follows in a straightforward manner from (4.16) as

$$F^{k_0}=V^{-1}\sum_{\mathbf{k}}\left[\frac{1}{\beta}\ln(1-e^{-\beta[E_{\mathbf{k}}^{k_0}+(\hbar^2/m)\mathbf{k}_0\cdot\mathbf{k}]})-(E_{\mathbf{k}}^{k_0}-\xi_{\mathbf{k}}^{\mathbf{k}})\right] +(\varepsilon_0-\mu)\,|\,\bar{\psi}_0^{k_0}\,|^2+\tfrac{1}{2}\nu_0\,|\,\bar{\psi}_0^{k_0}\,|^2-|\,\nu_0\,|\,|\,\bar{\psi}_0^{k_0}\,|^2 . \tag{4.20}$$

There will be no attempt to carry through the second definition for ρ_s (a nontrivial calculation is involved even in the ideal-gas case[4]). The third definition, (4.6), will be used instead [see, however, the paragraph beginning after Eq. (4.24)]. The calculation is simplified greatly by noting that $\mathbf{k}_0$ appears in (4.16) and (4.20) either with the momentum operator as $\mathbf{k}_0\cdot\mathbf{k}$, or with μ as $\mu-\varepsilon_0$. Since ε_0 is of second order in $\mathbf{k}_0$, and the derivatives with respect to $\mathbf{k}_0$ will be evaluated at $\mathbf{k}_0=0$, these two sources of $\mathbf{k}_0$ dependence will not mix in the final result. Thus

$$\rho_s = \frac{m^2}{\hbar^2} \lim_{v_0 \to 0} \left[\frac{\partial^2 F^{k_0}}{\partial k_0^2} \right]_{k_0=0}$$
$$= -m \left[\frac{\partial F^0}{\partial \mu} \right]_{v_0=0}$$
$$+ \frac{m}{V} \sum_{\mathbf{k}} \frac{\partial^2}{\partial E_{\mathbf{k}}^2} \frac{1}{\beta} \ln(1 - e^{-\beta E_{\mathbf{k}}}) \frac{\hbar^2}{m} (\mathbf{k} \cdot \hat{\mathbf{e}}_0)^2$$
$$= \rho - \frac{m}{V} \sum_{\mathbf{k}} (2\varepsilon_{\mathbf{k}}/d)[-dn_B(E_{\mathbf{k}})/dE_{\mathbf{k}}] , \quad (4.21)$$

where, in the last line, isotropy is used to replace $(\mathbf{k} \cdot \hat{\mathbf{e}}_0)^2$ by k^2/d. The second term is apparently, by (4.3), the normal fluid density

$$\rho_n = \frac{m}{V} \sum_{\mathbf{k}} (2\varepsilon_{\mathbf{k}}/d)[-dn_B(E_{\mathbf{k}})/dE_{\mathbf{k}}] . \quad (4.22)$$

To see that this agrees with the *first* definition obtained via (4.2), note that from (4.19) the momentum flux is

$$\mathbf{p} = V^{-1} \langle \mathbf{P} \rangle = -\frac{m}{\hbar} \frac{\partial F}{\partial \mathbf{k}_0} \quad (4.23)$$

so that

$$\rho_n = \frac{m}{\hbar} \lim_{k_0 \to 0} |\mathbf{p}| / |\mathbf{k}_0| = -\frac{m^2}{\hbar^2} \left[\frac{\partial^2 F}{\partial k_0^2} \right]_{k_0=0} . \quad (4.24)$$

As mentioned at the end of Sec. IV B, the essential difference between F^{k_0} and F is the appearance of ε_0 with μ in the former. Hence, it is apparent that the first term in (4.21) is missing in (4.24), while the second term has opposite sign. The result, (4.24), is therefore precisely (4.22). The correspondence also follows from Galilean invariance: in (4.2) and (4.3) one has $v_n = \hbar \mathbf{k}_0/m$ and $v_s = 0$. By transforming to the frame in which $v_n = 0$ and $v_s = -\hbar k_0/m$, the Hamiltonian becomes precisely (4.9) with the sign of $\mathbf{k}_0$ reversed (see Ref. 14).

Before abandoning the second definition for ρ_s entirely, it is worth noting that if one is permitted to use *infinitesimal* values of the angle θ, yet another candidate for ρ_s is the definition

$$\rho_s \equiv \lim_{\substack{A \to \infty \\ L \to \infty}} \frac{m^2 L^2}{\hbar^2} \left[\frac{\partial^2 F^\theta}{\partial \theta^2} \right] , \quad (4.25)$$

which is the $\theta \to 0$ limit of (4.4). Since $k_0 = \theta/L$, this is just (4.6), but with the thermodynamic limit taken last instead of first. It is easy to see that this definition also yields (4.21).

Equation (4.22) for the normal fluid density agrees precisely with the Landau form.[2] There the excitation spectrum is given *a priori* and its dependence on $\mathbf{k}_0$ is argued from Galilean invariance. Here the Bogoliubov spectrum is derived microscopically, and its dependence on $\mathbf{k}_0$ results directly from the Bogoliubov transformation.

Still other definitions for the superfluid density exist in the literature. A definition in terms of the superfluid response to a rotating current field was used by de Pasquale and Tabet.[18] The definition is, in general, very complicated, and involves heavy use of diagrammatic concepts. However, within the Bogoliubov approximation the calculations can be carried through exactly,[18] and when generalized to d dimensions, the result is precisely (4.21). This agreement with previous phenomenological results does not seem to have been pointed out by these authors.

From (4.22) it is straightforward to exhibit the zero-temperature superfluid density scaling function. Once again, to obtain the important dependence at low densities a wave-vector independent potential, $v_{\mathbf{k}} \equiv v_0$ ($\phi = 1$), is taken. From the same changes of variable used to get (3.23) from (3.21) one obtains

$$\rho_n = K_d \xi_0^{-d} (2m\beta\mu/d)$$
$$\times \int_0^\infty dy\, y^{d+1} \left[-\frac{1}{\beta} n_B'[\mu y (y^2+2)^{1/2}] \right]$$
$$= m \zeta(\tfrac{1}{2}d) \Lambda_T^{-d} R(r) , \quad (4.26)$$

where the prefactor is again the ideal result. The scaling function R is given by

$$R(r) = \frac{r^{d/2}}{\Gamma(\frac{1}{2}(d+2)) \zeta(\frac{1}{2}d)}$$
$$\times \int_0^\infty dx \frac{x/r}{(1 + x^2/r^2)^{1/2}}$$
$$\times [(1 + x^2/r^2)^{1/2} - 1]^{d/2} \frac{e^x}{(e^x - 1)^2} \quad (4.27)$$

and the variable $r = \beta\mu$ was displayed in (3.24). This function has the asymptotic behaviors

$$R(r) = \begin{cases} 1 + O(r^{(d-2)/2}), & r \to 0 , \\ 4 \left[\dfrac{d+1}{d} \right] \dfrac{\zeta(d+1)\Gamma(d+1)}{\zeta(\frac{1}{2}d)\Gamma(\frac{1}{2}d)} (2r)^{-(d+2)/2} , & \\ \qquad\qquad\qquad\qquad\qquad\qquad r \to \infty , \end{cases} \quad (4.28)$$

so that for $t = T/T_c \ll (na^d)^{(d+2)/2} (r \gg 1)$,

$$\rho_n/\rho \simeq R_d (na^d)^{(4-d^2)/2d} (T/T_c)^{d+1} , \quad (4.29)$$

where $R_d = \pi^{7/2}/45\zeta(\frac{3}{2})^{8/3}$ in $d = 3$. This demonstrates once again the discontinuous change in exponents when interactions are included. These results agree with (1.5) and (1.8) in the Introduction.

In Appendix B it is shown that the result (4.26) for ρ_n, which was obtained from (4.22) by fixing the density, is in fact the same as that which is obtained using the canonical (i.e., fixed density) free energy in (4.6) and (4.23). Therefore, the superfluid density is ensemble independent—a result which has always been implicitly assumed.

V. FURTHER RESULTS AND CONCLUSIONS

A. Perturbation theory

In order to progress beyond the Bogoliubov approximation, some form of higher-order perturbative approach

must be developed. The standard method, which seems to simplify formal manipulations, begins at the level of the ideal gas [represented by the first term in (2.1)] and treats all remaining terms within a diagrammatic perturbation theory.[8,9,19] Since the chemical potential μ is positive in most cases of interest (for example, in the hard-core Bose gas), this method makes only formal sense, and meaningful calculations must begin with some kind of diagrammatic resummation. To lowest order in the interactions, such a resummation leads directly to the Bogoliubov model.[8,9] This illustrates how, within a more controlled calculation, the Bogoliubov approximation arises naturally, and why it gives the correct physics at low temperatures.

The essential piece of physics that emerges from the Bogoliubov model is the linear form (1.3) of the excitation spectrum at long wavelengths. All of the low-temperature thermodynamics follows from this form. Gavoret and Nozières[19] were first to establish (1.3) (at $T=0$) to all orders in perturbation theory (modulo a few technical problems, see below) showing, in addition, that the quasiparticle speed of sound is precisely the hydrodynamic compressional speed of sound given by the bulk modulus. Hohenberg and Martin[20] rederived this result by a shorter route using functional calculus.

Götze and Wagner[21] used this result to establish the T^3 law, (1.4), for the specific heat. Kehr[22,23] used the same techniques to establish the T^4 law, (1.5), for the superfluid density,[22] showing also that ρ_s coincides with the density at zero temperature. Kehr also established the T^2 law[23] for the condensate fraction

$$n_0(T)/n_0(0)\approx 1-m(k_BT)^2/12n\hbar^3c\ , \quad T\to 0 (d=3) \tag{5.1}$$

which generalizes (3.26) to arbitrary densities.

These results are exact to all orders in perturbation theory, but involve the unknown speed of sound, c. Actual calculation of this, and other physical quantities, must be carried out within perturbation theory—the Bogoliubov model giving the correct lowest order results. In order to go beyond Bogoliubov it is natural to take the Bogoliubov Hamiltonian, $H_0[\bar{\psi}_0]$, as the unperturbed model,[8,9] treating the remainder

$$H-H_0[\bar{\psi}_0]=H_1+H_3+H_4+\delta H_2+\delta C \tag{5.2}$$

as the perturbation. Here H_1, H_3, and H_4 are given by (2.12), (2.14), and (2.15), $\delta C=C[\psi_0]-C[\bar{\psi}_0]$, while

$$\delta H_2=\delta n_0\sum_{\mathbf{k}}[(\epsilon_0+\epsilon_{\mathbf{k}})a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}} + \tfrac{1}{2}\nu_{\mathbf{k}}(e^{-2i\phi}a_{\mathbf{k}}a_{-\mathbf{k}}+e^{2i\phi}a_{\mathbf{k}}^{\dagger}a_{-\mathbf{k}}^{\dagger})] \tag{5.3}$$

with $\delta n_0=|\psi_0|^2-|\bar{\psi}_0|^2$ and $\psi_0=|\psi_0|e^{i\phi}$.

In developing the diagrammatic formalism, the important quantities are the Matsubara Green's functions. These are defined by

$$G_{ij}(\mathbf{x}_1-\mathbf{x}_2,\tau_1-\tau_2)=-\langle T_\tau\Phi^i(\mathbf{x}_1,\tau_1)\Phi^j(\mathbf{x}_2,\tau_2)\rangle\ , \quad i,j=1,2\ , \tag{5.4}$$

where T_τ is the time-ordering operator and the notation

$$\Phi^i(1)\Phi^j(2)=\begin{cases}\varphi(1)\varphi^{\dagger}(2), & i=1,j=1\ ,\\ \varphi^{\dagger}(1)\varphi(2), & i=2,j=2\ ,\\ \varphi^{\dagger}(1)\varphi^{\dagger}(2), & i=2,j=1\ ,\\ \varphi(1)\varphi(2), & i=1,j=2\ ,\end{cases} \tag{5.5}$$

has been used. In the normal phase G_{ij} is diagonal. The superfluid phase is characterized by the appearance of nonzero off-diagonal elements of G_{ij} which are termed "anomalous." Thus, for example, the Bogoliubov Hamiltonian yields the Fourier transformed Green's functions[24]

$$\begin{aligned}G_{11}^B(\mathbf{k},ik_n)&=G_{22}^B(-\mathbf{k},-ik_n)\\&=\frac{|u_{\mathbf{k}}|^2}{ik_n-E_{\mathbf{k}}}-\frac{|v_{\mathbf{k}}|^2}{ik_n+E_{\mathbf{k}}}\ ,\\ G_{21}^B(\mathbf{k},ik_n)&=G_{12}^B(-\mathbf{k},-ik_n)^*\\&=-u_{\mathbf{k}}v_{\mathbf{k}}\left[\frac{1}{ik_n-E_{\mathbf{k}}}-\frac{1}{ik_n+E_{\mathbf{k}}}\right]\ ,\end{aligned} \tag{5.6}$$

where $k_n=2\pi nk_BT$, n an integer, are the Matsubara frequencies.

Feynman diagrams, and their associated analytic expressions, are generated by associating a vertex with each term in (5.2). These vertices are shown in Fig. 2. Each vertex has one or more legs, and each leg has an associated direction, represented by an arrow. An arrow pointing away from the vertex represents a creation operator, an arrow pointing into the vertex represents an annihilation operator. The diagrams are formed by connecting legs pairwise in all possible ways. Such a connection results in a bond, or propagator, with two associated ar-

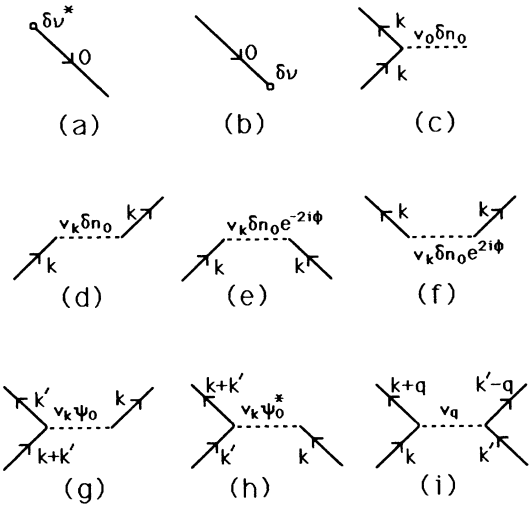

FIG. 2. Diagrammatic vertices for perturbation theory in the ordered phase, of which only the last is present in the disordered phase ($\nu=0, T>T_c$). Vertices (a) and (b) arise from H_1; here $\delta\nu=(-\mu+\epsilon_0|\psi_0|^2)\psi_0-\frac{1}{2}\nu$. Vertices (c)–(f) arise from δH_2, (g) and (h) from H_3, and (i) from H_4. An arrow pointing away from a vertex represents a creation operator $a_{\mathbf{k}}^{\dagger}$, an arrow pointing into a vertex represents a destruction operator $a_{\mathbf{k}}$.

rows. Each of the four combinations of arrow directions represents one of the four Bogoliubov Green's functions (5.6), as shown in Fig. 3. The analytic expression associated with each diagram so constructed is obtained in the usual manner by multiplying together the propagators from each bond and the factors of $\nu_{\mathbf{k}}$ and ψ_0 which weight each vertex (see Fig. 2), then summing over all internal frequencies and momenta, subject to the usual conservation rules (which have already been taken into account in Fig. 2). In the normal phase, the vanishing of the anomalous propagators implies that momentum and frequency can only flow *along* the arrow directions—reducing the set of diagrams considerably. This is also the case if the ideal gas is taken as the unperturbed model and in this sense the set of diagrams is easier to describe. However, as stated earlier, an infinite class of them must be resummed to give a single diagram in the present perturbation theory. This greatly complicates actual calculations.

As mentioned, Gavoret and Norières[19] encountered some technical problems in their analysis of the zero-temperature perturbation series. These are closely related to the difficulties found in Sec. III involving the mixture of mean-field and fluctuation corrected results, and, in particular, to the proper treatment of the spin waves in the broken symmetry state. To understand how these difficulties arise, consider once again the longitudinal susceptibility, defined by

$$\begin{aligned}\chi_\nu &= \frac{\partial|\psi_0|}{\partial|\nu|} \\ &= \tfrac{1}{4}\int_0^\beta d\tau \langle T_\tau[a_0(\tau)+a_0^\dagger(\tau)][a_0(0)+a_0^\dagger(0)]\rangle \\ &= -\lim_{q,iq_n\to 0} G_L(\mathbf{q},iq_n)\ , \end{aligned} \tag{5.7}$$

where

$$G_L(\mathbf{q},iq_n)=\tfrac{1}{4}\sum_{i,j} G_{ij}(\mathbf{q},iq_n) \tag{5.8}$$

is the longitudinal Green's function, and ψ_0 has been chosen real for convenience. Using (5.6), the Bogoliubov result is

$$\chi_\nu^{(B)} = -\tfrac{1}{2}\lim_{\mathbf{k}\to 0}\left[\frac{2u_{\mathbf{k}}v_{\mathbf{k}}}{E_{\mathbf{k}}} - \frac{|u_{\mathbf{k}}|^2+|v_{\mathbf{k}}|^2}{E_{\mathbf{k}}}\right] = \frac{1}{4\mu}\ . \tag{5.9}$$

This represents the mean-field result, which can also be obtained from (2.36), and disagrees with the spin-wave result, $\chi_\nu \equiv \infty$ [see (3.8)]. The origin of the problem lies in the fact that the Bogoliubov model does not respect the full phase symmetry of the order parameter. Rather, a harmonic approximation is made in which only quadratic fluctuations of the field $\psi(\mathbf{x})$ about its mean value, ψ_0, are kept. This has the effect of treating the longitudinal fluctuations, $\frac{1}{2}[\varphi(x)+\varphi^\dagger(x)]$, and the transverse fluctuations, $(1/2i)[\varphi(x)-\varphi^\dagger(x)]$, as independent. Geometrically, one is replacing the familiar sombrero-shaped Landau free-energy surface by a linear trough (Fig. 4). Therefore, the circular phase symmetry is replaced by a linear translational symmetry. This is the standard, noninteracting, spin-wave approximation. As long as one is not interested in topological excitations, such as vortices (which are unimportant at low temperatures), this approximation gives the correct, long wavelength behavior for the transverse (i.e., phase) fluctuations. However, in an applied field the geometry becomes important. In the trough approximation a uniform longitudinal field does not break the transverse symmetry, and the response of the order parameter to such a field is completely decoupled from the transverse fluctuations. For this reason the resulting susceptibility is finite. Conversely the Landau sombrero becomes tilted in a uniform field, and a unique minimum is selected. The field therefore couples to the phase fluctuations as well as the longitudinal fluctuations, and for this reason the response, as measured by the susceptibility, is singular.

These qualitative conclusions can be extracted quantitatively from the Bose gas Hamiltonian. The cubic and quartic terms, H_3 and H_4, in (5.2) contain the nonlinear geometric effects that connect the longitudinal and phase fluctuations. In Appendix C the nature of these two terms is clarified in the context of the conceptually more simple continuous spin s^4 model. In Fig. 5 the relevant higher-order Feynman diagrams involving these two terms are shown. These yield the second order result

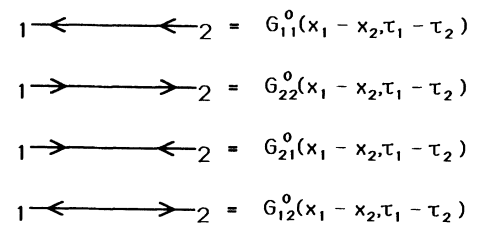

FIG. 3. The four possible propagators in the ordered phase, of which only the first two are nonvanishing in the disordered phase. The propagators are used to connect the various vertices, shown in Fig. 2, to form a diagram. Each arrow is associated with the vertex closest to it. In Fourier space each propagator is labeled by a momentum $\mathbf{k}$ and frequency k_n rather than the two endpoints (x_1,τ_1) and (x_2,τ_2).

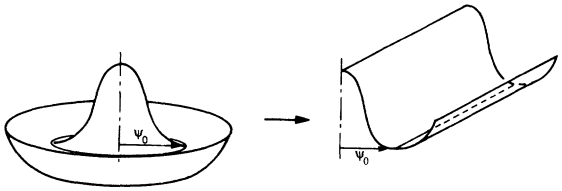

FIG. 4. Landau free-energy surface in the harmonic approximation.

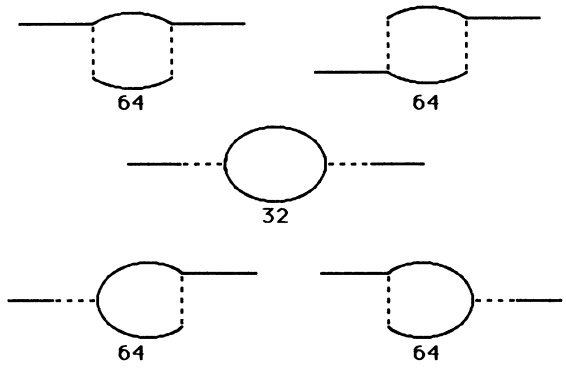

FIG. 5. Important one-loop diagrams contributing to the longitudinal susceptibility. The arrows on the propagators have been suppressed. The number beneath each diagram represents the number of distinct diagrams with the same topology obtained by restoring the arrows in all permitted ways. There are 288 diagrams in all.

$$\chi_\nu^{(1)}=\frac{\bar{n}_0}{(3\bar{n}_0\omega_0-\mu)^2}\times\int_{\mathbf{k}}\left[\frac{-n_B'(E_{\mathbf{k}})}{E_{\mathbf{k}}^2}[\xi_{\mathbf{k}}(\omega_0+\omega_{\mathbf{k}})-\bar{n}_0\omega_{\mathbf{k}}^2]^2+\frac{2n_B(E_{\mathbf{k}})+1}{2E_{\mathbf{k}}^3}\omega_{\mathbf{k}}^2(\varepsilon_{\mathbf{k}}-\mu)^2\right], \tag{5.10}$$

where $\bar{n}_0=|\bar{\psi}_0|^2$ and $\bar{\psi}_0(\mu,\nu)$ was defined as the solution to (2.36). It is straightforward to verify that if the remaining, less singular contributions to χ_ν at this order, represented by the graphs shown in Fig. 6, are included, namely

$$\delta\chi_\nu^{(1)}=\frac{3\bar{n}_0\omega_0+\mu}{2(3\bar{n}_0\omega_0-\mu)^3}I[\mu,\nu;\bar{\psi}_0], \tag{5.11}$$

where $I[\mu,\nu;\psi_0]$ was defined in (3.4), then the result $\chi_\nu^{(B)}+\chi_\nu^{(1)}+\delta\chi_\nu^{(1)}$ is precisely $\partial\psi_0/\partial|\nu|$ from (3.7). The requisite spin-wave results, (3.8) and (3.9), therefore follow immediately from this approach as well. In addition, this approach yields the corrections, at small momentum and frequency, to the longitudinal part of the Green's function. At lowest order one has (setting $\nu=0$)

$$G_L^B(\mathbf{k},ik_n)=-\frac{1}{2}\frac{\varepsilon_{\mathbf{k}}}{k_n^2+E_{\mathbf{k}}^2}, \tag{5.12}$$

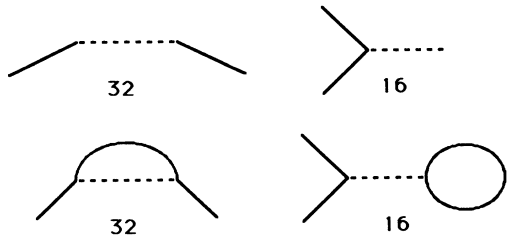

FIG. 6. Additional graphs contributing to the susceptibility at one-loop order. As in Fig. 4, the number beneath each graph represents the number of distinct ways of labeling the propagators. There are 96 diagrams in all.

which yields (5.9) when $k_n,\mathbf{k}\to 0$. The important geometrical corrections to this expressions arise from the first diagram in Fig. 5. [This diagram yields the divergent part of the susceptibility, (3.8) and (3.9), when evaluated at zero external frequency and momentum.] At finite temperatures one finds for $k\ll(\beta\hbar c)^{-1}$ and $2<d<4$

$$G_L(\mathbf{k},0)\approx\frac{m^2}{\beta n\hbar^4 2^{2d}\pi^{(d-1)/2}}\times\frac{\Gamma(\frac{1}{2}(d-2))\Gamma(\frac{1}{2}(4-d))}{\Gamma(\frac{1}{2}(d-1))}\frac{1}{|\mathbf{k}|^{4-d}}. \tag{5.13}$$

The other Matsubara frequencies give nonsingular results in this limit. The $|\mathbf{k}|^{-\epsilon}$ divergence is characteristic of ordered phases with a continuous broken symmetry. In Appendix C it is shown to emerge from the classical spin-wave calculation as well. Quantum mechanics (as embodied in the operator character of the fields) is apparently not important at finite temperatures. Conversely, at zero-temperature quantum mechanics provides the only dynamics, and should therefore be important. This manifests itself in the disappearance of the gap between Matsubara frequencies at zero temperature, so that k_n can be treated as an extra, continuous dimension. At small $|\mathbf{k}|$, the denominator of (5.12) is $k_n^2+(\hbar c)^2\mathbf{k}^2$; therefore, this extra temporal dimension enters in the same quadratic fashion as the spatial dimensions. Hence, one expects an exponent characteristic of one higher dimension in the zero-temperature analog of (5.13) [this should be contrasted with the behavior *at* criticality where $d\to d+2$ (Ref. 1). Indeed one finds

$$G_L(\mathbf{k},ik_n)\approx\begin{cases}G_L^B(\mathbf{k},ik_n)^2\dfrac{8(mc^2)^4}{(\hbar c)^d(16\pi)^{d/2}n}\dfrac{\Gamma[\frac{1}{2}(d-1)]\Gamma[\frac{1}{2}(3-d)]}{\Gamma(d/2)}\dfrac{1}{\kappa^{3-d}} & 2(<d<3)\\ -G_L^B(\mathbf{k},ik_n)^2\dfrac{(mc^2)^4}{4(\hbar c)^3\pi^2 n}\ln(\kappa/\kappa_0) & (d=3)\\ \text{const} & (d>3),\end{cases} \tag{5.14}$$

where $\kappa^2 \equiv k_n^2 + \hbar^2 c^2 \mathbf{k}^2$, and $\kappa_0 \sim mc^2$ is the characteristic energy scale. The logarithmic divergence in $d=3$ was noted in Ref. 19, but its physical significance was not then realized.

Associated with the divergence of G_L at small momentum is yet another problem in the diagrammatic formalism. The diagrammatic resummation method mentioned earlier expresses the Green's functions G_{ij} in terms of self-energy functions Σ_{ij} via the matrix equations,[25]

$$G = G^{(0)} + G\Sigma G^{(0)} , \tag{5.15}$$

where $G^{(0)}$ is the ideal-gas Green's function

$$\begin{aligned} G_{11}^{(0)}(\mathbf{k}, ik_n) &= (ik_n - \varepsilon_{\mathbf{k}} + \mu)^{-1} \\ &= G_{22}^{(0)}(-\mathbf{k}, -ik_n) , \\ G_{12}^{0}(\mathbf{k}, ik_n) &= G_{21}^{0}(\mathbf{k}, ik_n) = 0 . \end{aligned} \tag{5.16}$$

The self-energies are expressed in the usual manner as the sum of all one-particle irreducible diagrams, and are generally assumed to have well-behaved expansions about zero momentum. For example, the Bogoliubov results are

$$\begin{aligned} \Sigma_{11}(\mathbf{k}, ik_n) &= \Sigma_{22}(\mathbf{k}, ik_n) = n(v_0 + v_{\mathbf{k}}) , \\ \Sigma_{12}(\mathbf{k}, ik_n) &= \Sigma_{21}(\mathbf{k}, ik_n) = n_0 v_{\mathbf{k}} . \end{aligned} \tag{5.17}$$

In fact, although the transverse, $1/\kappa^2$, sound-mode singularity is removed by this method, the longitudinal singularity is not.[26–28] To see this, use (5.15) to express Σ in terms of the Green's functions as

$$\Sigma(\mathbf{k}, ik_n) = G^{(0)^{-1}}(\mathbf{k}, ik_n) - G^{-1}(\mathbf{k}, ik_n) , \tag{5.18}$$

in particular,

$$\Sigma_{12} = G_{12}/(G_{11}G_{22} - G_{12}G_{21}) . \tag{5.19}$$

In the limit of small $|\mathbf{k}|, ik_n$ the transverse singularity cancels and one is left with[27]

$$\Sigma_{12}(\mathbf{k}, ik_n) \approx \tfrac{1}{2}(G_{11} + G_{12}) \approx \tfrac{1}{4} G_L(\mathbf{k}, ik_n) . \tag{5.20}$$

so that Σ_{12} *vanishes*, either as κ^{3-d} [or $1/\ln(\kappa)$ in $d=3$] if $T=0$, or as $|\mathbf{k}|^{4-d}$ if $T>0$. The Bogoliubov result (5.17) is therefore somewhat misleading. It seems clear from (5.20) that the origin of this singularity is again geometrical, as in the case of the susceptibility. This vanishing of Σ_{12} causes infrared (small $\mathbf{k}, ik_n$) divergences in the perturbation theory for Σ_{12} and Σ_{11} (although, as noted in Ref. 19, and elsewhere, these divergences appear to cancel out in all physical quantities). In Ref. 26 it is shown how to eliminate these divergences, order-by-order in the perturbation expansion.

In order to decouple the longitudinal and transverse fluctuations completely from the outset, one might consider transforming the fields to polar coordinates. Such an approach has been developed by Popov.[8] Within a functional integral formalism,[8] his approach begins with the derivation of an effective hydrodynamical Hamiltonian. This is accomplished by integrating out field fluctuations with wave numbers greater than some small value, k_0. The remaining, slowly varying, part of the field, $\psi_0(\mathbf{x}, \tau)$, is written then in the form

$$\psi_0(\mathbf{x}, \tau) = \rho_0(\mathbf{x}, \tau)^{1/2} e^{i\phi(\mathbf{x}, \tau)} . \tag{5.21}$$

The broken symmetry of the superfluid state is incorporated by writing

$$\rho_0(\mathbf{x}, \tau) = \rho_0(k_0) + \pi(\mathbf{x}, \tau), \quad \langle \pi(\mathbf{x}, \tau) \rangle = 0 \tag{5.22}$$

where $\rho_0(k_0)$ is the density of particles with momenta less than k_0, and is essentially equal to the condensate density for k_0 sufficiently small. The Hamiltonian is an expansion in powers of the fields π and ϕ (and their derivatives with respect to $\mathbf{x}$ and τ), with coefficients given by appropriate thermodynamic derivatives[8] (hence the term "hydrodynamical"). For example, the coefficient of $|\nabla\phi|^2$ is proportional to the superfluid density ρ_s, corresponding to the fact that gradients in the phase give rise to an additional free-energy density $\frac{1}{2}\rho_s v_s^2$ with $v_s = (\hbar/m)\nabla\phi$.

As expected, correlations in the field π are nonsingular whereas correlations in ϕ contain the transverse singularity. The longitudinal singularity in the standard Green's function (5.4) arises from the higher-order mixing of the fields π and ϕ when (5.21) is expanded.[27] In particular, $G_L(x-y) \sim \langle \phi(x)\phi(y) \rangle^2$ as expected.[27]

The hydrodynamical Hamiltonian approach also allows one to recover other results, such as the damping of collisionless sound (i.e., the imaginary part of the Bogoliubov quasiparticle energies, $E_{\mathbf{k}}$). At finite temperature the damping is proportional to $T^4|\mathbf{k}|$; at zero temperature it is proportional to $|\mathbf{k}|^5$.[8,25] Via a complicated diagrammatic resummation technique, the approach also can be used to derive the standard superfluid hydrodynamic results[29] in the collision dominated regime. In particular, both first (density wave) and second (temperature wave) sound poles appear in the Green's function. For the weakly interacting Bose gas, the second sound pole has an amplitude proportional to ρ_n and a speed $c/\sqrt{3}$ where c is the speed of first sound, (1.12).[8,28]

B. Unified scaling picture

In this final subsection the scaling properties of a dilute, weakly interacting superfluid are summarized within a more unified picture. To begin, observe that zero temperature *and* zero density (or chemical potential) can be considered as a type of *onset* critical point. Thus, near this point, a given thermodynamic quantity will "turn on" with a characteristic power of the density (or chemical potential). For example, one has for $\mu, \rho \to 0$

$$\begin{aligned} \rho_s(T=0) &= \rho \approx \mu/v_0 , \\ \psi_0(T=0) &\approx \rho^{1/2} \approx (\mu/v_0)^{1/2} , \end{aligned} \tag{5.23}$$

so that the superfluid density and order parameter onset with an exponent of unity and $\frac{1}{2}$, respectively. Note, if ρ is used as the independent variable, the above coincide with the ideal-gas results.

The next step involves the introduction of the temperature. At any finite temperature, the superfluid transition must occur, but now at a finite value, $\rho_c(T)$, of the densi-

ty, or, $\mu_c(T)$, of the chemical potential. This transition is determined essentially by the condition that the interparticle spacing be of order the thermal de Broglie wavelength [see (1.1)]. The onset of ρ_s and ψ_0 now follow from the usual critical exponent definitions as

$$\rho_s(T;\mu)\sim(\mu-\mu_c)^{\nu}\sim\Delta\rho^{\nu/(1-\alpha)}\ ,$$
$$\psi_0(T;\mu)\sim(\mu-\mu_c)^{\beta}\sim\Delta\rho^{\beta/(1-\alpha)}\ , \tag{5.24}$$
$$\Delta\rho(T;\mu)\equiv\rho(T;\mu)-\rho_c(T)\sim(\mu-\mu_c)^{(1-\alpha)}\ ,$$

for $T>0$, $\alpha>0$, where α is the specific-heat exponent. Only the definitions in terms of ρ make sense for the ideal gas [if $\alpha<0$ one has $\Delta\rho\sim\mu-\mu_c$ with corresponding changes in the two lines above].

An attempt is now made to include both (5.23) and (5.24) into a single scaling expression, the temperature being treated as a relevant variable. Thus, one first attempts to write ρ_s in the form

$$\rho_s(T;\rho)\approx\rho Y^{(0)}(D^{(0)}T/\rho^{\phi^{(0)}}),\quad T,\rho\ll 1 \tag{5.25}$$

with $Y^{(0)}(0)=1$, and $\phi^{(0)}$ to be determined below. The major shortcoming of this form is that it predicts that all superfluid density profiles have precisely the same shape: the finite-temperature transition must take place at some critical value, y_c, of the argument of $Y^{(0)}$. This implies

$$T_c(\rho)=y_c\rho^{\phi^{(0)}}/D^{(0)}\ , \tag{5.26}$$

and hence,

$$\rho_s(T;\rho)/\rho\approx Y^{(0)}(y_cT/T_c)\ , \tag{5.27}$$

so that ρ_s/ρ as a function of T/T_c yields a single universal curve. This completely contradicts Fig. 1 which dictates the existence of two intruding crossover regions, one near $T=0$ and the other near $T=T_c$, where such a simple scale invariance breaks down. If these two regions were to be ignored (they disappear in the zero density limit), the function $Y^{(0)}$ and the exponent $\phi^{(0)}$ follow immediately from the ideal-gas results as

$$Y^{(0)}(y)=1-(y/y_c)^{d/2},\quad \phi^{(0)}=2/d\ . \tag{5.28}$$

This naive scaling therefore is capable only of describing the ideal gas, and completely misses the effects of interactions.[30]

In order to properly include interactions, two new variables must be added to the scaling form. These are simply the scaling combinations appropriate to the zero temperature and critical regimes discussed in the Introduction and body of this paper. Thus write,

$$\rho_s(t;\rho)/\rho=Y\left[\frac{D^{(0)}T}{\rho^{\phi^{(0)}}};\frac{D_0\rho}{(T/T_c)^{\psi_0}},\frac{D\rho}{[(T_c-T)/T_c]^{\psi}}\right]\ , \tag{5.29}$$

where the new crossover exponents are

$$\psi_0=\frac{d}{d-2}\phi_0=\frac{d}{d-2},\quad \psi=\frac{d}{d-2}\phi=\frac{d(4-d)}{(d-2)^2}\ , \tag{5.30}$$

where ϕ and ϕ_0 were defined in (1.9) and (1.13), respectively. This can be rewritten in terms of the variable $y=D^{(0)}T/\rho^{\phi^{(0)}}$ as

$$\rho_s(y;\rho)/\rho\approx Y\left[y;\frac{D_0\rho}{(y/y_c)^{\psi_0}},\frac{D\rho}{[(y_c-y)/y_c]^{\psi}}\right]\ . \tag{5.31}$$

If y is held fixed and ρ is allowed to vanish one must recover

$$Y(y;0,0)=Y^{(0)}(y)\ . \tag{5.32}$$

It is therefore natural to write

$$Y(y;y_0,y_T)=Y^{(0)}(y)Y_{0,T}(y_0,y_T)\ , \tag{5.33}$$

where $Y_{0,T}$ represents the effects of interactions and has the limiting forms

$$Y_{0,T}(y_0,y_T)\approx\begin{cases}Y_T(y_T), & y_T\to\infty\quad(y\to y_c)\ ,\\ Y_0(y_0), & y_0\to\infty\quad(y\to 0)\ ,\end{cases} \tag{5.34}$$

where the functions Y_T and Y_0 are the critical and zero-temperature crossover scaling functions discussed in Ref. 1 and Sec. IV (with appropriate allowance for the fact that the scaling variables y_0 and y_T take slightly different forms from those used previously).

Scaling forms generally imply some sort of underlying fixed-point renormalization-group flow structure. In Fig. 7 such a flow structure for the Bose fluid is shown. This figure is highly schematic and is meant only to illustrate

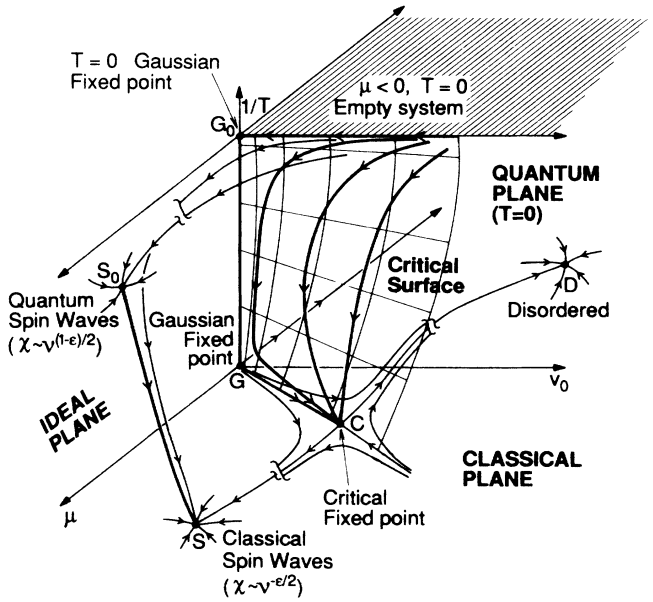

FIG. 7. Schematic fixed point diagram for the pure interacting Bose gas. At finite temperature the flows eventually collapse onto the classical plane. Critical scaling involves the crossover of flows passing from the vicinity of G_0 to that of G, then finally to that of C. Zero-temperature scaling involves crossover from the quantum to classical spin-wave regimes. The half-plane $\nu_0=0$, $\mu>0$ is singular as it corresponds to infinite density ($\rho=\infty$).

the qualitive relationships between the various fixed points. Primarily, this is due to the fact that different forms of renormalization are required in different regions of the diagram; therefore, all the flows cannot actually be shown simultaneously.

The basic structure of the flows is as follows: All flows originating at finite temperature inevitably collapse onto the classical plane in which only the zeroth Matsubara frequency survives. This property reflects the unimportance of quantum mechanics at finite temperature. The structure within this plane is well known.[31,32] The critical properties are described by the Gaussian fixed point, G, and the critical fixed point, C, characterized by the exponents (5.24). Flows beginning in the disordered (normal fluid) phase are attracted to an "infinite-temperature" sink, D. Flows beginning in the ordered (superfluid) phase are attracted to a sink, S, which describes the spin-wave properties. Flows in the plane separating the ordered and disordered phases describe the system at criticality, and are eventually attracted to C.

The crossover to ideal-gas behavior, when the density and transition temperature are reduced, is described by flows that begin near the $T=0$ plane. The detailed analysis of the flows describing the critical crossover was carried out in Ref. 1(b) (see especially, Sec. VII). These flows must begin near the critical surface, $\mu_c(T,\nu_0)$, and are first attracted toward the zero-temperature Gaussian fixed point, G_0, before contracting onto the classical plane. If the starting density is sufficiently small, the flows will remain close to the $\nu_0=0$, $\mu=0$ axis and pass near G (it is this property of the flows which in fact determines what is meant by low density[1(b)]) before crossing over to C, and finally, S. The simple nature of the flows from the starting point to the neighborhood of G allows one to map the starting point directly onto the flows crossing over from G to C.[1(b)] This is the essence of the correspondence between the dilute Bose gas near criticality and the classical continuous spin s^4 model.[1]

Conversely, if the flows do not begin sufficiently near the critical plane, they will be attracted first toward the ideal plane ($\nu_0=0$) before collapsing onto the classical plane and arriving at S without passing near G. These flows may still pass near G (if the starting point lies in the critical region of Fig. 1, but not the interacting critical region), or they may miss G entirely (if the starting point lies in the intermediate region of Fig. 1).

It is at very low temperatures that the quantum spin-wave fixed point finally comes into play. *At* zero temperature the flows will terminate at S_0, and the behavior will be that of the $(d+1)$-dimensional classical spin-wave system, as noted in Eqs. (5.14) and (3.9). Close to zero temperature, the flows pass very close to S_0 before contracting down onto S. It is precisely this crossover from S_0 to S which results in the scaling behavior that has been the main concern of this paper. Although the analysis in this paper was not carried out within a renormalization-group framework, it seems clear that such a treatment of the Bogoliubov Hamiltonian would present little more difficulty than the ideal Bose gas or Gaussian models (see Appendix D).

This completes the general scaling picture of the dilute Bose fluid. An interesting question that remains to be discussed is how these results are affected by the presence of an external random potential. The work in Ref. 1 was strongly motivated by the experimental results of Reppy and co-workers[33] on helium adsorbed in porous Vycor glass. These experiments were consistent with the assumption that randomness could be ignored, at least in the density and temperature ranges explored. Although heuristic scaling arguments exist that make this plausible,[34] a full renormalization-group picture, analogous to that of the flows about G_0, G, and C in Fig. 7, has yet to be presented. This will be addressed in future publications.[35]

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APPENDIX A: CRITICAL CROSSOVER IN THE ORDERED PHASE

In this appendix, the forms of the critical scaling fields beneath T_c will be derived. The calculations will be performed at fixed chemical potential μ. The imposition of the constant density constraint may be treated separately, as discussed in detail in Ref. 1(b) and in Appendix B.

It is most convenient to work with the condensate wave function given by (3.7). Near the critical point, μ and ψ_0 will be small, and one seeks the leading $\mu\to 0$ singularities in the integral $I[\mu,\nu=0;(\mu/\nu_0)^{1/2}]$. The coefficients and degree of these singularities will be related to the scaling properties.[1]

To begin, the second, zero temperature, term in the integrand varies as

$$I_0 \equiv \tfrac{1}{2}\int_k [(\nu_0+\nu_{\mathbf{k}})(\xi_{\mathbf{k}}/E_{\mathbf{k}}-1)-\mu\nu_{\mathbf{k}}^2/\nu_0 E_{\mathbf{k}}]$$
$$\approx \xi_0^{-d}\nu_0 K_d \int_0^\infty y^{d-1}dy \left[\left[\frac{y^2+1}{y(y^2+2)^{1/2}} - 1 \right] - \frac{\phi^2}{2y(y^2+2\phi)^{1/2}} \right], \qquad \text{(A1)}$$

where ϕ and ξ_0 were defined in Sec. III. Since its integral diverges in $d>2$, unless the cutoff inherent in ϕ is used, the last term in square brackets yields the leading dependence on ξ_0. The result is

$$I_0 \simeq -\tfrac{1}{2}\nu_0 K_d (a_0/\xi_0)^2 a_0^{-d}\int_0^\infty du\, u^{d-3}\phi(u)^2 , \quad \xi_0\to\infty . \qquad \text{(A2)}$$

The analysis of the temperature dependent part of I is more complicated. The exponential cutoff in the Bose factor allows one to set $\phi=1$ to obtain the leading depen-

dence on $\beta\mu$. Via the same substitution used to derive (3.23), one has

$$\Delta I \equiv \int_k \frac{(\epsilon_0+\epsilon_{\mathbf{k}})\xi_{\mathbf{k}}-n_0\epsilon_{\mathbf{k}}^2}{E_{\mathbf{k}}} n_B(E_{\mathbf{k}}) \approx \frac{K_d \xi_0^{-d}\epsilon_0}{2r}\int_0^\infty dx \frac{2(x^2/r^2+1)^{1/2}-1}{(x^2/r^2+1)^{1/2}(e^x-1)} \times[(x^2/r^2+1)^{1/2}-1]^{(d-2)/2} . \quad \text{(A3)}$$

In the limit $r\to 0$ the result is

$$\Delta I(r=0)\approx \int_0^\infty dx \frac{x^{(d-2)/2}}{e^x-1}\frac{2\epsilon_0}{\Gamma(\frac{1}{2}d)\Lambda_T^d}=\frac{2\zeta(\frac{1}{2}d)}{\Lambda_T^d}\epsilon_0 . \quad \text{(A4)}$$

The leading correction to this result is also important, and is extracted by subtracting the $r=0$ result and changing variables to $u=x/r$:

$$\Delta I(r)-\Delta I(0)\simeq \frac{K_d\xi_0^{-d}\epsilon_0}{2r}\int_0^\infty du \left[\frac{2(1+u^2)^{1/2}-1}{(1+u^2)^{1/2}}[(1+u^2)^{1/2}-1]^{(d-2)/2}-2u^{(d-2)/2}\right]\frac{r}{e^{ru}-1} . \quad \text{(A5)}$$

The subtraction ensures the existence of the $r=0$ limit of the integral for $d<4$, so that the leading dependence is obtained by replacing $r(e^{ru}-1)^{-1}$ simply by u^{-1}. The resulting integral can be evaluated using the substitution $s^2=1+u^2$, and the identity[9]

$$\int_1^\infty \frac{(s-1)^{\mu-1}}{(s+1)^\nu}ds = 2^{\mu-\nu}B(\nu-\mu,\mu) \quad \text{(A6)}$$

to give the remarkably simple result

$$\Delta I(r)-\Delta I(0)\simeq \epsilon_0 \frac{3\times 2^{(d-4)/2}}{\Lambda_T^d}\Gamma[\tfrac{1}{2}(2-d)]r^{(d-2)/2} . \quad \text{(A7)}$$

The final result is therefore

$$|\psi_0|\simeq(\mu/\epsilon_0)^{1/2}\left[1-\frac{\epsilon_0}{2\mu}\left[\frac{2\zeta(\frac{1}{2}d)}{\Lambda_T^d}+\frac{3\times 2^{(d-4)/2}}{\Lambda_T^d}\Gamma[\tfrac{1}{2}(2-d)](\beta\mu)^{(d-2)/2}+O(\beta\mu)\right]\right] . \quad \text{(A8)}$$

Note that (A2) does not contribute at the order shown.

In order to make the connection with scaling, one matches (A8) to a scaling form[8]

$$\Lambda_T^{d/2}|\psi_0| \simeq A_1 g_\mu^{\beta_0}\Psi[A_2 g_\epsilon/g_\mu^\phi] \quad \text{(A9)}$$

in which g_μ and g_ϵ are scaling fields[1] and the factor of $\Lambda_T^{d/2}$ on the left makes the equation dimensionless. Since ψ_0 evidently diverges as $\epsilon_0^{-1/2}$ for small ϵ_0 at fixed $\mu>0$, one anticipates, by appropriate choice of the metrical factors A_1 and A_2,

$$\Psi[y]=y^{-\Gamma_0}[1+y+O(y^2)] \quad \text{(A10)}$$

with $\Gamma_0=\frac{1}{2}$. The quantity $g_\epsilon^{1/2}|\psi_0|\Lambda_T^{d/2}$ then has, at least in low orders, an expansion in powers of ϵ_0 given by

$$g_\epsilon^{1/2}|\psi_0|\Lambda_T^{d/2}=g_\mu(0)^{\beta_0+\Gamma_0\phi}A_1A_2^{-\Gamma_0} \times[1+\epsilon_0 A_2 g_\epsilon'(0)/g_\mu(0)^\phi +\epsilon_0(\beta_0+\Gamma_0\phi)g_\mu'(0)/g_\mu(0)+\cdots] , \quad \text{(A11)}$$

where $g_\mu'(0)=(dg_\mu/d\epsilon_0)_{\epsilon_0=0}$, etc. Comparing (A11) with (A8) one identifies [see Ref. 1(b) for details of the matching procedure]:

$$g_\epsilon\simeq\beta\epsilon_0/\Lambda_T^d,\quad g_\mu\simeq\beta\mu-2\zeta(\tfrac{1}{2}d)\beta\epsilon_0/\Lambda_T^d ,$$
$$\phi=\tfrac{1}{2}(4-d),\quad \beta_0=\tfrac{1}{2}-\Gamma_0\phi=(d-2)/4,\quad \Gamma_0=\tfrac{1}{2} ,$$
$$A_2=-3\times 2^{(d-6)/2}\Gamma[\tfrac{1}{2}(2-d)] ,$$
$$A_1=A_2^{\Gamma_0}=3^{1/2}2^{(d-6)/4}\{-\Gamma[\tfrac{1}{2}(2-d)]\}^{1/2} . \quad \text{(A12)}$$

These results are completely consistent with those derived in the disordered phase[1]—the linear scaling fields and crossover exponent being identical in the two cases.

The same calculation can be carried through for the superfluid density using (3.7), (3.11), and (4.20). The appropriate scaling form here is

$$\Lambda_T^d\rho_s/m\simeq B_1 g_\mu^{\nu_0}R_s[B_2 g_\epsilon/g_\mu^\phi] \quad \text{(A13)}$$

with

$$R_s[y_s]=y_s^{-\Delta_0}[1+y_s+O(y_s^2)] . \quad \text{(A14)}$$

As expected, the results for g_μ, g_ϵ, and ϕ are the same as (A12). The other, property *dependent*, parameters are given by

$$\nu_0=\tfrac{1}{2}(d-2),\quad \Delta_0=1 ,$$
$$B_1=B_2=2^{(d-2)/2}[(d+2)/d]\{-\Gamma[\tfrac{1}{2}(2-d)]\} . \quad \text{(A15)}$$

This result also demonstrates that the scaling forms for ρ_s/m and $n_0=|\psi_0|^2$ differ: Although $\nu_0=2\beta_0$ and

$\Delta_0=2\Gamma_0$, the metrical factors resulting from squaring (A9) and (A10) are

$$\overline{A}_1=\overline{A}_2=3\times 2^{(d-4)/2}[-\Gamma(\tfrac{1}{2}(2-d))] \tag{A16}$$

which are different from B_1 and B_2 so long as $d<4$.

APPENDIX B: ENSEMBLE INDEPENDENCE OF THE SUPERFLUID DENSITY

In Eq. (4.6) the superfluid density was defined as a second derivative of the free energy with respect to an imposed twist. The free energy used was the grand canonical free energy—a function of the variable μ. If, as is appropriate in some experiments, the density is constrained to be fixed, one might be concerned that the corresponding constrained derivatives will yield a different answer for ρ_s. The following observations illustrate that this concern is unfounded.

Let

$$\rho_s[\mu]=\frac{m^2}{\hbar^2}\frac{\partial^2}{\partial k_0^2}F^1[\mu,T,v_0;k_0]_{v_0=0,k_0=0} \tag{B1}$$

and

$$\rho_s[n]=\frac{m^2}{\hbar^2}\frac{\partial^2}{\partial k_0^2}G^1[n,T,v_0;k_0]_{v_0=0,k_0=0}\ , \tag{B2}$$

where

$$G^1=F^1+\mu n \tag{B3}$$

is the canonical free energy. It will be shown that $\rho_s[\mu]=\rho_s[n]$, where $n=-\partial F^1/\partial\mu$. Hence the constraint commutes with the k_0 derivatives.

Begin with the standard identity

$$\left[\frac{\partial h(x,y)}{\partial x}\right]_D=\left[\frac{\partial h}{\partial x}\right]_y+\left[\frac{\partial h}{\partial y}\right]_x\left[\frac{\partial y}{\partial x}\right]_D\ , \tag{B4}$$

where $h(x,y)$ is any function, and the subscript D means that the corresponding derivative is to be taken at fixed $D=D(x,y)$. One has then

$$\begin{aligned}\left[\frac{\partial G^1}{\partial k_0}\right]_n&=\left[\frac{\partial F^1}{\partial k_0}\right]_n+n\left[\frac{\partial\mu}{\partial k_0}\right]_n\\&=\left[\frac{\partial F^1}{\partial k_0}\right]_\mu+\left[\frac{\partial F^1}{\partial\mu}\right]_{k_0}\left[\frac{\partial\mu}{\partial k_0}\right]_n+n\left[\frac{\partial\mu}{\partial k_0}\right]_n\\&=\left[\frac{\partial F^1}{\partial k_0}\right]_\mu\end{aligned} \tag{B5}$$

since $n=-(\partial F^1/\partial\mu)_{k_0}$. Therefore

$$\left[\frac{\partial^2 G^1}{\partial k_0^2}\right]_n=\left[\frac{\partial^2 F^1}{\partial k_0^2}\right]_\mu-\left[\frac{\partial n}{\partial k_0}\right]_\mu\left[\frac{\partial\mu}{\partial k_0}\right]_n\ . \tag{B6}$$

If F^1 is an even function of k_0, as is certainly the case for helium, the second term vanishes when $k_0\to 0$, and the required equality follows immediately.

APPENDIX C: SPIN WAVES IN THE s^4 MODEL

The s^4-model Hamiltonian is given by

$$H_s=\tfrac{1}{2}\int d^dx\,(\,|\nabla\mathbf{s}|^2+r\,|\mathbf{s}|^2+\tfrac{1}{2}u\,|\mathbf{s}|^4)\ , \tag{C1}$$

where $\mathbf{s}$ is an n-dimensional continuous spin, and an underlying lattice spacing, b_0, should be understood. Comparisons with helium are made by taking $n=2$— as is appropriate to the phase symmetry of the field operator. The first two terms in (C1) represent the Gaussian model, which is closely analogous to the ideal gas.[1] In the ordered phase (characterized within mean-field theory by $r<0$) there will be a nonzero magnetization, $\mathbf{M}=\langle\mathbf{s}\rangle$. It is therefore convenient to perform a shift by defining

$$\boldsymbol{\sigma}=\mathbf{s}-\mathbf{M},\quad\langle\boldsymbol{\sigma}\rangle=0 \tag{C2}$$

[cf. Eq. (2.3)]. The Hamiltonian then reads

$$\begin{aligned}H_s=\int d^dx\,[&\tfrac{1}{2}|\nabla\boldsymbol{\sigma}|^2+\tfrac{1}{2}(r+u\mathbf{M}^2)(\mathbf{M}\cdot\boldsymbol{\sigma}+\boldsymbol{\sigma}^2)\\&+u\mathbf{M}\cdot\boldsymbol{\sigma}(\mathbf{M}\cdot\boldsymbol{\sigma}+\boldsymbol{\sigma}^2)+\tfrac{1}{4}u\boldsymbol{\sigma}^4]+F_0V\ ,\end{aligned} \tag{C3}$$

where $F_0=\frac{1}{2}r\mathbf{M}^2+\frac{1}{4}u\mathbf{M}^4$ is the Landau free energy. Treating the deviation, $\boldsymbol{\sigma}$, as small, the initial approximation consists of dropping terms cubic and higher in $\boldsymbol{\sigma}$. Equation (C2) then implies

$$\mathbf{M}^2\simeq\mathbf{M}_0^2=-r/u \tag{C4}$$

[cf. Eq. (2.37)], and the Hamiltonian becomes

$$\begin{aligned}H_s&\approx H_s^0+F_0(\mathbf{M}_0)V\\&\equiv\int d^dx\,[\tfrac{1}{2}|\nabla\boldsymbol{\sigma}|^2+u(\mathbf{M}_0\cdot\boldsymbol{\sigma})^2]-(r^2/4u)V\end{aligned} \tag{C5}$$

[cf. Eq. (2.19)]. Writing $\boldsymbol{\sigma}=(\sigma^\parallel,\boldsymbol{\sigma}^\perp)$, where $\sigma^\parallel=\boldsymbol{\sigma}\cdot\hat{\mathbf{M}}$ and $\boldsymbol{\sigma}^\perp\cdot\hat{\mathbf{M}}=0$ (here $\hat{\mathbf{M}}=\mathbf{M}/|\mathbf{M}|=\mathbf{M}_0/|\mathbf{M}_0|$), one has

$$H_s^0=\int d^dx\,[\tfrac{1}{2}|\nabla\boldsymbol{\sigma}^\perp|^2+\tfrac{1}{2}|\nabla\sigma^\parallel|^2+|r|(\sigma^\parallel)^2]\ . \tag{C6}$$

The transverse fluctuations are therefore "massless"—these represent the Goldstone modes—while the longitudinal fluctuations are not. The momentum space correlation function is given by

$$\begin{aligned}\frac{1}{n}\langle\boldsymbol{\sigma}_{\mathbf{q}}\cdot\boldsymbol{\sigma}_{-\mathbf{q}}\rangle_0&=\frac{1}{n}\langle\,|\boldsymbol{\sigma}_{\mathbf{q}}^\perp|^2+(\sigma_{\mathbf{q}}^\parallel)^2\rangle_0\\&=\frac{1}{\beta n}\left[\frac{n-1}{q^2}+\frac{1}{q^2+2|r|}\right]\\&=\frac{q^2+2(n-1)|r|/n}{\beta q^2(q^2+2|r|)}\ .\end{aligned} \tag{C7}$$

If one takes $n=2$, $\hbar^2/2m=1$, and $|r|$ is identified with μ, this last form corresponds exactly to the zero-frequency propagator, $\beta^{-1}G_{11}^0(\mathbf{q},iq_n=0)$ [cf. Eq. (5.6)]. To be more explicit, the Bogoliubov *canonical* transformation corresponds precisely to the *Jacobian preserving* transformation

$$\hat{\boldsymbol{\sigma}}_{\mathbf{q}}=[(q^2+|r|)^{1/2}q^{-1}\sigma_q^\parallel,q(q^2+|r|)^{-1/2}\boldsymbol{\sigma}_q^\perp]\ . \tag{C8}$$

In terms of this transformed spin, the Hamiltonian be-

comes

$$H_s^0=\tfrac{1}{2}\sum_{\mathbf{q}}[(\mathbf{q}^2+|r|)(\sigma_{\mathbf{q}}^{\parallel})^2+q^2|\sigma_{\mathbf{q}}^{\perp}|^2]$$

$$=\tfrac{1}{2}\sum_{\mathbf{q}}E_{\mathbf{q}}^s|\hat{\sigma}_{\mathbf{q}}|^2, \tag{C9}$$

where $E_{\mathbf{q}}^s=[q^4+2|r|q^2]^{1/2}$ [cf. Eqs. (2.28) and (2.38)]. A cutoff $q_\Lambda\sim\pi/b_0$ on the momentum sums should be understood.

Carrying the analogy further, the analog of the anomalous Green's function is

$$\tfrac{1}{2}\langle(\sigma_q^{\perp})^2-(\sigma_q^{\parallel})^2\rangle_0=\frac{|r|}{\beta(E_{\mathbf{q}}^s)^2} \tag{C10}$$

which corresponds precisely to $\beta^{-1}G_{22}^0(\mathbf{q},iq_n=0)$ [cf. Eq. (5.6); to see this, identify $\sigma_{\mathbf{q}}^{\perp}+i\sigma_{\mathbf{q}}^{\parallel}$ with $a_{\mathbf{q}}$ and $\sigma_{-\mathbf{q}}^{\perp}-i\sigma_{-q}^{\parallel}$ with $a_{\mathbf{q}}^{\dagger}$], and is nonzero simply because the broken symmetry makes the longitudinal and transverse fluctuations inequivalent.

The superfluid density or, in spin language, the helicity modulus, can be obtained by adding the term analogous to (4.2), namely

$$\tfrac{1}{2}\int d^dx\,\mathbf{k}_0\cdot(\sigma^{\perp}\nabla\sigma^{\parallel}-\sigma^{\parallel}\nabla\sigma^{\perp})$$

$$=\tfrac{1}{2}\sum_{\mathbf{q}}\mathbf{k}_0\cdot\mathbf{q}(\sigma_{\mathbf{q}}^{\perp}\sigma_{-\mathbf{q}}^{\parallel}-\sigma_{\mathbf{q}}^{\parallel}\sigma_{-\mathbf{q}}^{\perp}) \tag{C11}$$

to H_s, and taking two derivatives with respect to k_0. The result is

$$\rho_n^s=\frac{\beta}{V}\sum_{q}\frac{2}{d}q^2\langle\sigma_{\mathbf{q}}^{\perp}\sigma_{-\mathbf{q}}^{\perp}\rangle_0\langle\sigma_{\mathbf{q}}^{\parallel}\sigma_{-\mathbf{q}}^{\parallel}\rangle_0$$

$$=\frac{1}{V}\sum_{\mathbf{q}}\frac{2}{d}q^2\beta(E_{\mathbf{q}}^s)^2 \tag{C12}$$

which should be compared to the small momentum form of (4.21).

Finally, to obtain the singularity in the actual longitudinal correlation function, the higher-order terms left out of H_s^0 must be taken into account, in particular the cubic term

$$u\mathbf{M}\cdot\boldsymbol{\sigma}\sigma^2=uM\sigma^{\parallel}[(\sigma^{\parallel})^2+(\sigma^{\perp})^2] \tag{C13}$$

which connects $\sigma^{\parallel}$ to $(\sigma^{\perp})^2$. The origin of this term is clearer if one considers the model more commonly encountered when discussing spin waves—namely the fixed spin ferromagnet

$$H_s'=\int d^dx\,|\nabla\mathbf{s}|^2,\quad |\mathbf{s}|\equiv 1. \tag{C14}$$

If one considers small fluctuations $\boldsymbol{\sigma}$ about unit magnetization, then one has precisely

$$\sigma^{\parallel}=[1-(\sigma^{\perp})^2]^{1/2}-1\simeq-\tfrac{1}{2}(\sigma^{\perp})^2, \tag{C15}$$

i.e., the fixed length constraint implies that longitudinal fluctuations are directly related to transverse fluctuations. For soft spins the relationship is not strict, but $\boldsymbol{\sigma}$ still will be dominated by the easy transverse fluctuations, and these will drive the longitudinal fluctuations. The term $M\sigma^{\parallel}(\sigma^{\perp})^2$ in (C13) represents just this effect.

To see the effect of (C13) on the longitudinal correlations, consider the following second-order contribution to $G_{\parallel}(x-y)=\langle\sigma^{\parallel}(x)\sigma^{\parallel}(y)\rangle$

$$\delta G_{\parallel}(x-y)=\tfrac{1}{2}\beta^2u^2M^2\left\langle\sigma^{\parallel}(x)\sigma^{\parallel}(y)\left[\int\sigma^{\parallel}(z)\sigma^{\perp}(z)^2d^dz\right]^2\right\rangle_{0,c}, \tag{C16}$$

where the subscript c indicates the connected part. This yields

$$\delta G_{\parallel}(x-y)=2\beta^2u^2M^2\int d^dzd^dw\,G_{\parallel}^0(x-z)$$
$$\times G_{\perp}^0(w-z)^2G_{\parallel}^0(y-z)$$
$$\simeq 2\beta^2u^2M^2G_{\parallel}^0(q=0)^2G_{\perp}^0(x-y)^2$$
$$=(u^2M^2/2|r|^2)G_{\perp}^0(x-y)^2, \tag{C17}$$

where the last two expressions follow in the limit of large $|x-y|$ by virtue of the short-ranged character of

$$G_{\parallel}^0(z)\sim\exp[-|z|/(2|r|)^{1/2}]$$

[as follows from (C7)]. Also from (C7) one has $G_{\perp}^0(z)\sim\beta^{-1}|z|^{2-d}$, when $|z|$ is large, so that (C17) yields the Fourier transform

$$\delta G_{\parallel}(q)\sim(u^2M^2/2|r|^2)(k_BT)^2q^{-\epsilon},\quad q\to 0. \tag{C18}$$

This dominates $G_{\parallel}^0(q)$ at small q, and leads to the requisite divergence of the longitudinal susceptibility $\chi_L=G_{\parallel}(q=0)$ [cf. Eq. (3.8)].

As a final observation, note that if the momentum cutoff q_Λ is taken to be temperature dependent in such a way as to mimic the effective energy cutoff $\sim k_BT$ inherent in the Bose factor arising in the helium diagrammatics, then the various temperature laws for $n_0(0)-n_0(T)$, $\rho_n(T)$, etc., hold precisely in the s^4 case as well. Without such a cutoff the temperature dependence would be trivial, governed only by the temperature prefactors such as that which appears in (C12). Thus, for example, ρ_n would vary strictly linearly with temperature at low temperatures, independent of dimensionality. Note, however, that the actual temperature dependence of q_Λ will not be simple because it would entail inverting an equation such as $\beta E_{q_\Lambda}^s\sim 1$. In contrast, near T_c it was found[1] that one needed only to invert $\beta q_\Lambda^2\sim 1$. This is not surprising as in this regime one has $\beta mc^2\ll 1$.

APPENDIX D: RENORMALIZATION-GROUP ANALYSIS OF THE BOGOLIUBOV MODEL

In this appendix, the path-integral formulation of the Bose fluid is used to derive renormalization-group flows for the Bogoliubov model. In this approach, the Hamiltonian is replaced by a Lagrangian which is a functional of a classical complex field[8,1(b)] $\varphi(\mathbf{x},\tau)$, with Fourier transform $a(\mathbf{k},k_n)$. The Bogoliubov Lagrangian reads [cf. Eq. (2.13)]

$$L_0=\sum_{\mathbf{k},ik_n}\{[ik_n-\varepsilon_{\mathbf{k}}^0+\mu-n_0(v_0+v_{\mathbf{k}})]a^*(\mathbf{k},ik_n)a(\mathbf{k},ik_n)-\tfrac{1}{2}n_0v_{\mathbf{k}}[a(\mathbf{k},ik_n)a(-\mathbf{k},-ik_n)+a^*(\mathbf{k},ik_n)a^*(-\mathbf{k},-ik_n)]\}\ . \tag{D1}$$

The trivial constant term (2.10) has been dropped for simplicity. Equation (2.36) still is taken as the equation of state determining $n_0(\mu,v)$. As in Sec. II, $a(\mathbf{k},ik_n)$ is divided into longitudinal and transverse parts,

$$a(\mathbf{k},ik_n)=a^{\parallel}(\mathbf{k},ik_n)+ia^{\perp}(\mathbf{k},ik_n)\ .$$

This yields

$$L_0=-\sum_{\mathbf{k},ik_n}\{(\varepsilon_{\mathbf{k}}^0+v/2\psi_0)\,|\,a^{\perp}(\mathbf{k},ik_n)\,|^2+(\varepsilon_{\mathbf{k}}^0+2n_0v_{\mathbf{k}}+v/2\psi_0)\,|\,a^{\parallel}(\mathbf{k},ik_n)\,|^2$$
$$+k_n[a^{\perp}(\mathbf{k},ik_n)a^{\parallel}(-\mathbf{k},-ik_n)-a^{\parallel}(\mathbf{k},ik_n)a^{\perp}(-\mathbf{k},-ik_n)]\}\ . \tag{D2}$$

L_0 is put into a standard form by rescaling $a^{\perp}$ and $a^{\parallel}$ to yield

$$L_0=-\sum_{\bar{\mathbf{k}},i\bar{k}_n}\{(\bar{\mathbf{k}}^2+\bar{v})\,|\,\bar{a}^{\perp}(\bar{\mathbf{k}},i\bar{k}_n)\,|^2+(1+e_2\bar{\mathbf{k}}^2)\,|\,\bar{a}^{\parallel}(\bar{\mathbf{k}},i\bar{k}_n)\,|^2+\bar{k}_n[\bar{a}^{\perp}(\bar{\mathbf{k}},\bar{k}_n)\bar{a}^{\parallel}(-\bar{\mathbf{k}},-\bar{k}_n)-\bar{a}^{\parallel}(\bar{\mathbf{k}},\bar{k}_n)\bar{a}^{\perp}(-\bar{\mathbf{k}},-\bar{k}_n)]\}\ , \tag{D3}$$

where

$$\bar{v}=mva^2/\hbar^2\psi_0\ ,$$
$$\bar{a}^{\perp}=(\hbar^2/2ma^2)^{1/2}a^{\perp}\ ,$$
$$\bar{a}^{\parallel}=(2n_0v_0+v/2\psi_0)^{1/2}a^{\parallel}\ ,$$
$$e_2=\hbar^2/2m(2n_0v_0+v/2\psi_0)a^2\ ,$$
$$\bar{\mathbf{k}}=\mathbf{k}a\ ,$$
$$\bar{k}_n=[2ma^2/\hbar^2(2n_0v_0+v/2\psi_0)]^{1/2}k_n$$

and the unimportant $\mathbf{k}$ dependence of $v_{\mathbf{k}}$ has been dropped.

The renormalization-group transformation involves integrating out all field components with wave numbers in a shell $b^{-1}\bar{k}_\Lambda<|\bar{\mathbf{k}}|<\bar{k}_\Lambda,(b>1)$, where $\bar{k}_\Lambda=O(1)$ is a cutoff [corresponding to a cutoff $k_\Lambda=O(\pi/a)$ on $|\mathbf{k}|$], then rescaling momentum and fields so as to restore the original cutoff, as well as the standard form (D3), but with renormalized parameters $\bar{v}',e_2',T'$. Since L_0 is quadratic, the first operation simply yields a constant which contributes to the free energy, while the rescaling operation yields

$$\bar{v}'=b^2\bar{v},\quad e_2'=b^{-2}e_2,\quad T'=bT\ ,$$
$$\zeta^{\perp}=b^{-1},\quad \zeta^{\parallel}=1\ , \tag{D4}$$

where $\zeta^{\perp}$ and $\zeta^{\parallel}$ are the transverse and longitudinal field rescaling factors, respectively. It is seen immediately that the spin-wave fixed points are located at $\bar{v}=0$, $e_2=0$, and either $T=0$ or $T=\infty$. The $T=0$ fixed point is unstable to both $\bar{v}$ and T, while the $T=\infty$ fixed point is unstable only to $\bar{v}$. This essentially reproduces the structure shown in Fig. 7.

The various exponents near zero temperature follow from the free-energy density which, from (D4), obeys

$$f(\bar{v},T)=b^{-d-1}f(b^2\bar{v},bT)+\delta f\ , \tag{D5}$$

where δf comes from the renormalization constant mentioned above, and is presumed nonsingular. The prefactor b^{-d-1} follows from $f=F/\beta V$, where F is the extensive free energy. Two derivatives with respect to $\bar{v}$ yields

$$\chi(\bar{v},T=0)\sim b^{3-d}\chi(b^2\bar{v},T=0) \tag{D6}$$

which implies $\chi_{T=0}\sim\bar{v}^{d-3}$ [cf. (3.9)]. A single derivative yields for the singular part of ψ_0

$$\psi_{0,\mathrm{sing}}(\bar{v}=0,T)\sim b^{1-d}\psi_{0,\mathrm{sing}}(\bar{v}=0,bT) \tag{D7}$$

which implies $\psi_{0,\mathrm{sing}}\sim T^{d-1}$ [cf. (3.26); since $\alpha>0$ the fact that (3.26) is a constant density result does not affect the exponent here, only the detailed scaling function]. The crossover from ideal-gas behavior manifests in the irrelevance of $e_2\approx\xi_0^2/a^2$. In the dilute limit the initial value of e_2 is very large ($e_2=\infty$, or $v=0$, is the ideal-gas fixed point), and only after many renormalizations can it be neglected, as was done in (D5), (D6), and (D7).

As T renormalizes to infinity, the gap between the frequencies $\bar{k}_n$ in (D3) acts to suppress all nonzero $\bar{k}_n$. To obtain a proper finite limiting free energy when T is large, it is convenient to change the renormalization scheme slightly. Instead of renormalizing the temperature, one fixes it at, say, $T=1$, and absorbs the renormalization into a coefficient Γ^{-1} of k_n in (D3).[6] The result after re-

normalization is

$$\Gamma' = b^{-1}\Gamma \tag{D8}$$

and Γ is therefore irrelevant. The free energy renormalizes as

$$f(\bar{v},\Gamma) = b^{-d} f(b^2\bar{v}, b^{-1}\Gamma) + \delta f \ . \tag{D9}$$

This new procedure amounts to an essentially trivial change: The renormalized free-energy density is simply taken as $F/\beta V'$ rather than $F/\beta' V'$, where $V' = b^{-d}V$ is the renormalized volume—the former has a finite nonzero limit as $\Gamma \to 0$, while the latter diverges to infinity as $\beta \to 0$.

From (D9) the renormalization of the finite-temperature susceptibility follows as

$$\chi(\bar{v},\Gamma=0) \sim b^{4-d}\chi(b^2\bar{v},\Gamma=0) \ , \tag{D10}$$

hence $\chi(\Gamma=0) \sim \bar{v}^{\,d-4}$ [cf. Eq. (3.8)]. The only temperature dependence comes from the factor $1/\beta$ in the above definition for the free energy: All quantities, therefore, vary linearly with T. This is a well-known property of classical continuous spin models. Nontrivial temperature dependence results only if the cutoff k_Λ is made temperature dependent (see Appendix C and Ref. 1).

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