Extinction of electromagnetic waves by a small gyrotropic sphere

R. P. Devaty
Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260
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The general solution of Ford and Werner for the scattering and absorption of a plane electromagnetic wave by a gyrotropic sphere is analyzed in the long-wavelength limit. An expression is obtained for the extinction cross section which includes the electric dipole term as well as the magnetic dipole and electric quadrupole terms, which are coupled. The results agree with the Mie solution in the limit of vanishing gyrotropy. The expressions are applied to a lossless single-component plasma sphere. Agreement is obtained with recent work based on a quasistatic approximation.

I. INTRODUCTION

Recently Ford and Werner[1] presented a solution to the problem of the scattering and absorption of a plane electromagnetic wave by a gyrotropic sphere made up of material having a frequency-dependent complex dielectric tensor of the form

$$\overleftrightarrow{\epsilon}(\omega)=\begin{bmatrix}\epsilon_{xx} & \epsilon_{xy} & 0\\ -\epsilon_{xy} & \epsilon_{xx} & 0\\ 0 & 0 & \epsilon_{zz}\end{bmatrix}. \tag{1}$$

Gyrotropy is frequently introduced via a static uniform applied magnetic field **H**. Although not completely general, this tensor applies to microwave and far-infrared measurements on narrow-gap semiconductors[2-9] as well as electron-hole droplets in Ge (Refs. 10–23) for certain orientations of the crystal with respect to the applied magnetic field.

Approximate solutions which apply to particles that are much smaller than the wavelength of the electromagnetic radiation are important because they frequently apply to experiments and because the expressions obtained in this limit are relatively simple and accessible to physical interpretation. Consider the Mie-Debye problem[24,25] of scattering and absorption by an isolated homogeneous isotropic sphere ($\mathbf{H}=\mathbf{0}$) having radius a and complex dielectric function $\epsilon(\omega)$ embedded in a nonabsorbing host with dielectric constant ϵ_2. Define the dimensionless parameter

$$x\equiv ka=\epsilon_2^{1/2}\omega a/c \tag{2}$$

where ω is the frequency of the wave and k is the wave number. In the limit $x\ll 1$ the pair of infinite independent multipole series reduces to three terms, to order x^3. For example, the extinction cross section is[26,27]

$$\sigma_{\rm ext}=\sigma_1^{(e)}+\sigma_1^{(m)}+\sigma_2^{(e)} \tag{3}$$

where

$$\sigma_1^{(e)}=4\pi a^2 x\left\{\mathrm{Im}\left[\frac{\epsilon-\epsilon_2}{\epsilon+2\epsilon_2}\right]+\tfrac{3}{5}x^2\,\mathrm{Im}\left[\frac{(\epsilon-2\epsilon_2)(\epsilon-\epsilon_2)}{(\epsilon+2\epsilon_2)^2}\right]+\tfrac{2}{3}x^3\,\mathrm{Re}\left[\left[\frac{\epsilon-\epsilon_2}{\epsilon+2\epsilon_2}\right]^2\right]+O(x^4)\right\}, \tag{4a}$$

$$\sigma_1^{(m)}=\frac{2\pi}{15}a^2x^3\left[\mathrm{Im}\left[\frac{\epsilon}{\epsilon_2}-1\right]+O(x^2)\right], \tag{4b}$$

$$\sigma_2^{(e)}=\frac{2\pi}{3}a^2x^3\left[\mathrm{Im}\left[\frac{\epsilon-\epsilon_2}{2\epsilon+3\epsilon_2}\right]+O(x^2)\right] \tag{4c}$$

are the electric dipole, magnetic dipole, and electric quadrupole terms, respectively. For most situations the electric dipole term is the most important, and the magnetic dipole and electric quadrupole terms are the leading corrections.

In the long-wavelength limit, it is also possible to obtain expressions for the extinction coefficient and related quantities based on a quasistatic approximation. One can distinguish the condition $x\ll 1$, i.e., the wavelength of the incident wave in the host is much larger than the particle size, from the usually more restrictive condition that the wavelength inside the particle also be much greater than the size (the Rayleigh limit). The electromagnetic wave is approximated by uniform oscillating electric and magnetic fields. Retardation effects are ignored. The responses of the particle to the electric and magnetic fields are treated separately. The results agree with the electric and magnetic dipole terms of the Mie series to leading order in x. It is straightforward to generalize the quasistatic approximation for the electric dipole term to particles having ellipsoidal shape[28] and/or anisotropic dielectric tensor.[29] The quasistatic approximation also underlies the arguments leading to effective-medium theories[30,31] for composite materials.

Ford and Werner[1] obtained expressions for the electric and magnetic dipole moments for a gyrotropic sphere in the Rayleigh limit. Their expressions agree with results derived using a quasistatic approximation,[32-34] for which the electric and magnetic fields are assumed to decouple.

Furdyna *et al.*[35] and Goettig and Trzeciakowski[36] recently calculated the electromagnetic modes of oscillation

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of a small, lossless, single-carrier, conducting sphere in a uniform static magnetic field. Their quasistatic approximation allowed for coupling of the electric and magnetic modes through displacements of the plasma with respect to the uniform compensating background. The modes possess nonvanishing surface and, in some cases, volume charge densities. The interaction of external electromagnetic radiation with these modes and the absorbed power were determined using a quantization scheme originally developed for the zero-field case.[37,38] The frequencies and power absorption for the electric dipole resonances are in agreement with earlier work,[32–34] but not for the magnetic dipole modes. The origin of the discrepancy is that the electric and magnetic fields do not decouple, contrary to the assumptions of the earlier work. In the long-wavelength limit, the magnetic dipole and electric quadrupole terms, which are of the same order in x, mix. This mixing was also recognized in some of the earlier work.[1,7]

The question remains as to why the long-wavelength limit of the Ford-Werner (FW) solution agrees with the older incorrect quasistatic approximation rather than the recent results of Furdyna *et al.*[35] and Goettig and Trzeciakowski.[36] In this paper, we obtain the correct expressions from the FW theory for the magnetic dipole-electric quadrupole extinction cross sections in the long-wavelength limit. In effect we are generalizing Eqs. (4b) and (4c) to the gyrotropic case. We do not treat corrections to the electric dipole absorption that are of the same order as the magnetic-dipole–electric-quadrupole terms. In addition to resolving the discrepancy, this work generalizes the results of Furdyna *et al.*[35] and Goettig and Trzeciakowski[36] to any dielectric tensor given by Eq. (1). Multicomponent plasmas, phenomenological damping, interband absorption, etc. may be included in the treatment. The restrictions are those of the long-wavelength limit. In cases of doubt one should make use of the full FW theory.

The paper is organized as follows. Section II presents the derivation of the extinction coefficient from the FW theory in the long-wavelength limit. Although it is well understood, the electric dipole term is included for completeness. Section III discusses the results in the context of previous work on the single-component Drude plasma. Section IV summarizes the conclusions.

II. EXTINCTION COEFFICIENT IN THE LONG-WAVELENGTH LIMIT

Ford and Werner[1] write the extinction cross section as [Eq. (2.59) in their paper]

$$\sigma_{\text{ext}}=\frac{4\pi}{k}\operatorname{Im}[\mathbf{F}(\hat{\mathbf{k}},\hat{\mathbf{k}})\cdot\mathbf{E}_1^*/|\mathbf{E}_1|^2] \tag{5}$$

where $\mathbf{F}(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2)$ is the vector scattering amplitude [Eq. (FW3.49)], $\hat{\mathbf{k}}$ is a unit vector pointing in the direction of propagation of the electromagnetic wave, and $\mathbf{E}_1$ is the electric field vector of the incident wave. The asterisk denotes complex conjugation. Equations from the paper by Ford and Werner[1] are indicated by the prefix FW. The quantity of greatest interest here is $Z_{ll'}^{m\sigma}$, the ratio of $N\times N$ determinants that appears in $\mathbf{F}(\hat{\mathbf{k}},\hat{\mathbf{k}})$. For example [from Eq. (FW3.47)],

$$Z_{l1}=\begin{vmatrix} Y_{l1} & Y_{l2} & Y_{l3} & \cdots \\ X_{21} & X_{22} & X_{23} & \cdots \\ X_{31} & X_{32} & X_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix} \Bigg/ \begin{vmatrix} X_{11} & X_{12} & X_{13} & \cdots \\ X_{21} & X_{22} & X_{23} & \cdots \\ X_{31} & X_{32} & X_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix} \tag{6}$$

where the expressions for $X_{lk}^{m\sigma}$ and $Y_{lk}^{m\sigma}$ are given by Eqs. (FW3.35) and (FW3.39). The indices m and σ are suppressed in Eq. (6). For details on the FW theory, definitions, and notation the reader is referred to the original paper.[1]

Ford and Werner[1] calculated the electric and magnetic dipole moments of the gyrotropic sphere in the long-wavelength limit. Their results can be used to compute the extinction cross section and related quantities. Their expression for the electric dipole term is correct, but further examination of the magnetic dipole term is required. (1) Due to mixing with the electric quadrupole term, additional terms of the same order must be considered. (2) The assumption $x \ll y$, where $y \equiv qa$ with q a parameter analogous to the wave vector inside the sphere [Eq. (FW3.12)], is not required and in fact is the source of the disagreement of computed resonance frequencies with Furdyna *et al.*[35] and Goettig and Trzeciakowski.[36] We obtain the extinction cross section from Eq. (FW3.49).

Consider the eight terms in Eq. (FW3.49) with $l,l' \le 2$. Represent the indices l, l', m, and σ by (l,l',m,σ). Then $(1,1,m,+)$ denotes the electric dipole term. It is the lowest-order term in x and thus is expected to produce the largest extinction. The next term, $(1,1,m,-)$, is the magnetic dipole term. Although at first glance it appears to be of the same order as the electric dipole term, it is in fact of order x^3 because Z_{11}^{m-} is of order x^2. The term $(2,2,m,-)$, the electric quadrupole term, is also of order x^3. The terms $(1,2,m,-)$ and $(2,1,m,-)$ are of order x^3 and represent the interaction between the electric quadrupole and magnetic dipole terms. Therefore, all four terms should be considered together. In the limit of vanishing gyrotropy (e.g., isotropic material with no dc applied magnetic field), the cross terms vanish. The terms $(1,2,m,+)$, $(2,1,m,+)$, and $(2,2,m,+)$ are of higher order in x and will be neglected. They represent the magnetic quadrupole term and its coupling to the higher-order corrections to the electric dipole term in the long-wavelength limit.

A. Electric dipole term

The extinction cross section for the electric dipole term, or equivalent expressions such as the electric dipole moment, power absorption, and absorption coefficient, have been given correctly to lowest order by a number of authors.[1,32–34,39,40] We give the expressions here for completeness, but do not present the derivation, which is similar to that presented below for the magnetic-dipole–electric-quadrupole terms. The approach followed by FW is also correct.

We find that

$$Z_{11}^{m+}=\begin{cases}-2\dfrac{\epsilon_{zz}-\epsilon_2}{\epsilon_{zz}+2\epsilon_2}, & m=0\\[2ex] -2\dfrac{\epsilon_\pm-\epsilon_2}{\epsilon_\pm+2\epsilon_2}, & m=\pm 1\end{cases} \tag{7}$$

where $\epsilon_\pm \equiv \epsilon_{xx} \pm i\epsilon_{xy}$. Since

$$[\mathbf{Y}_{11}^m(\hat{\mathbf{k}})^* \times \hat{\mathbf{k}}\cdot\mathbf{E}_1][\mathbf{Y}_{11}^m(\hat{\mathbf{k}}) \times \hat{\mathbf{k}}\cdot\mathbf{E}_1^* / |\mathbf{E}_1|^2] = \frac{3}{8\pi}\,|\hat{\mathbf{e}}_m^* \cdot \hat{\mathbf{E}}_1|^2 \tag{8}$$

where $\hat{\mathbf{E}}_1 \equiv \mathbf{E}_1/|\mathbf{E}_1|$,

$$\sigma_1^{(e)} = 4\pi a^2 x \,\mathrm{Im}\begin{cases}\dfrac{\epsilon_{zz}-\epsilon_2}{\epsilon_{zz}+2\epsilon_2}, & m=0\\[2ex] \dfrac{\epsilon_\pm-\epsilon_2}{\epsilon_\pm+2\epsilon_2}, & m=\pm 1\,.\end{cases} \tag{9}$$

B. Magnetic-dipole–electric-quadrupole terms

We present a detailed derivation for the long-wavelength limit of the magnetic dipole term. The derivations for the other terms of the same order are similar. The results are summarized in Table I.

From Eqs. (5) and (FW3.49) the contribution to the extinction cross section due to the magnetic dipole term $(1,1,m,-)$ is

$$\sigma_1^{(m)} = \frac{16\pi^2}{3} a^2 x \,\mathrm{Im}\left[\sum_{m=-1}^{1} Z_{11}^{m-} [\mathbf{Y}_{11}^m(\hat{\mathbf{k}})^* \times \hat{\mathbf{k}}\cdot\mathbf{B}_1] \times [\mathbf{Y}_{11}^m(\hat{\mathbf{k}}) \times \hat{\mathbf{k}}\cdot\mathbf{B}_1^*/|\mathbf{B}_1|^2]\right], \tag{10}$$

where the $\mathbf{Y}_{11}^m(\hat{\mathbf{k}})$ are vector spherical harmonics.[34,41–43] Since

$$\mathbf{Y}_{11}^m(\hat{\mathbf{k}}) = -i\left[\frac{3}{8\pi}\right]^{1/2} \hat{\mathbf{k}} \times \hat{\mathbf{e}}_m \tag{11}$$

and $\hat{\mathbf{k}}\cdot\mathbf{B}_1 = 0$,

$$[\mathbf{Y}_{11}^m(\hat{\mathbf{k}})^* \times \hat{\mathbf{k}}\cdot\mathbf{B}_1][\mathbf{Y}_{11}^m(\hat{\mathbf{k}}) \times \hat{\mathbf{k}}\cdot\mathbf{B}_1^*/|\mathbf{B}_1|^2] = \frac{3}{8\pi}\,|\hat{\mathbf{e}}_m^*\cdot\hat{\mathbf{B}}_1|^2\,. \tag{12}$$

Now consider Z_{11}^{m-}, which is defined by Eq. (6). Since both $x \ll 1$ and $y \ll 1$, we shall expand in these two small parameters. Unlike FW we shall not impose $x \ll y$. We require the expansions of the spherical functions:[44]

$$j_1(x) \simeq \frac{x}{3}\left[1-\frac{x^2}{10}+\cdots\right], \tag{13a}$$

$$j_2(x) \simeq \frac{x^2}{15}\left[1-\frac{x^2}{14}+\cdots\right], \tag{13b}$$

TABLE I. $Z_{ll'}^{m\sigma}$, long-wavelength limit.

(l,l',σ)	$m=0$	$m=\pm 1$	$m=\pm 2$
$(1,1,+)$	$-2\dfrac{\epsilon_{zz}-\epsilon_2}{\epsilon_{zz}+2\epsilon_2}$	$-2\dfrac{\epsilon_\pm-\epsilon_2}{\epsilon_\pm+2\epsilon_2}$	
$(1,1,-)$	$\dfrac{1}{15}\left(\dfrac{\omega a}{c}\right)^2 \dfrac{\epsilon_2[2\epsilon_+\epsilon_- - \epsilon_{xx}\epsilon_{zz} - \epsilon_2(2\epsilon_{xx}+\epsilon_{zz})]+2\epsilon_+\epsilon_-\epsilon_{zz}}{\epsilon_2(2\epsilon_{xx}+\epsilon_{zz})+2\epsilon_{xx}\epsilon_{zz}}$	$\dfrac{1}{30}\left(\dfrac{\omega a}{c}\right)^2 \dfrac{\epsilon_2(\epsilon_\pm+\epsilon_{zz}-6\epsilon_2)+4\epsilon_\pm\epsilon_{zz}}{3\epsilon_2+\epsilon_\pm+\epsilon_{zz}}$	
$(1,2,-)$	$\dfrac{1}{2\sqrt{5}}\left(\dfrac{\omega a}{c}\right) \dfrac{\epsilon_2(\epsilon_+-\epsilon_-)\epsilon_{zz}}{\epsilon_2(2\epsilon_{xx}+\epsilon_{zz})+2\epsilon_{xx}\epsilon_{zz}}$	$\pm\left(\dfrac{3}{20}\right)^{1/2}\left(\dfrac{\omega a}{c}\right) \dfrac{\epsilon_2(\epsilon_\pm-\epsilon_{zz})}{3\epsilon_2+\epsilon_\pm+\epsilon_{zz}}$	
$(2,1,-)$	$-\dfrac{\sqrt{5}}{6}\left(\dfrac{\omega a}{c}\right) \dfrac{(\epsilon_+-\epsilon_-)\epsilon_{zz}}{\epsilon_2(2\epsilon_{xx}+\epsilon_{zz})+2\epsilon_{xx}\epsilon_{zz}}$	$\mp\left(\dfrac{5}{12}\right)^{1/2}\left(\dfrac{\omega a}{c}\right) \dfrac{\epsilon_\pm-\epsilon_{zz}}{3\epsilon_2+\epsilon_\pm+\epsilon_{zz}}$	
$(2,2,-)$	$\dfrac{\epsilon_2(2\epsilon_{xx}+\epsilon_{zz})-3\epsilon_{xx}\epsilon_{zz}}{\epsilon_2(2\epsilon_{xx}+\epsilon_{zz})+2\epsilon_{xx}\epsilon_{zz}}$	$\dfrac{3}{2}\,\dfrac{2\epsilon_2-\epsilon_\pm-\epsilon_{zz}}{3\epsilon_2+\epsilon_\pm+\epsilon_{zz}}$	$3\,\dfrac{\epsilon_2-\epsilon_\pm}{3\epsilon_2+2\epsilon_\pm}$

$$h_1^{(1)}(x) \simeq -\frac{i}{x^2}\left[1+\frac{x^2}{2}+\cdots\right], \tag{13c}$$

$$h_2^{(1)}(x) \simeq -\frac{3i}{x^3}\left[1+\frac{x^2}{6}+\cdots\right], \tag{13d}$$

as well as the expansions for $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_1^{(1)}(x)$, and $\alpha_2^{(1)}(x)$, which are defined by Eq. (FW2.38). The procedure is to expand the matrices for Z_{11}^{m-} in x and y and manipulate the first and second rows of both the numerator and denominator matrices. The two matrices can be made identical to within factors of λ_k (i.e., y) and thus cancel, leaving a relatively simple result. The expansions of the row elements are

$$X_1^{m-}(\lambda) \simeq \left[1-\frac{x^2}{3}-\frac{y^2}{6}\right] d_{1m}^-(\lambda), \tag{14a}$$

$$Y_1^{m-}(\lambda) \simeq -\tfrac{1}{15}(x^2-y^2) d_{1m}^-(\lambda), \tag{14b}$$

$$X_2^{m-}(\lambda) \simeq \left[\frac{\omega a}{c}\right]\left[\frac{y^2}{15x^2}\left[1-\frac{y^2}{14}\right]\times\left[2d_{2m}^-(\lambda)-\frac{\epsilon_2}{\tilde{\epsilon}}\Delta_{2m}^-(\lambda)\right]+\frac{1}{5}\left[1-\frac{29y^2}{126}\right]d_{2m}^-(\lambda)\right]. \tag{14c}$$

Further manipulation is required for Eq. (14c). From the eigenvalue equation (FW3.15) we obtain

$$d_{2m}^-(\lambda) = -\left[\frac{5}{4-m^2}\right]^{1/2}\times\left[\frac{-m^2\tilde{\gamma}+im\tilde{W}-2i\lambda\tilde{W}}{m\tilde{\gamma}+i\tilde{W}}\right]d_{1m}^-(\lambda). \tag{15}$$

From the definition of $\Delta_{lm}^{\sigma}(\lambda)$ [Eq. (FW3.21)],

$$\Delta_{2m}^-(\lambda) = \tfrac{1}{2}[5(4-m^2)]^{1/2}(\tilde{\gamma}m+i\tilde{W})d_{1m}^-(\lambda) + 3\left[\tilde{\gamma}\left[\frac{m^2-4}{6}\right]+\tfrac{1}{2}im\tilde{W}-i\lambda\tilde{W}\right]d_{2m}^-(\lambda). \tag{16}$$

From the definition of λ [Eq. (FW3.12)],

$$(1-i\lambda\tilde{W}) = \frac{\tilde{\epsilon}}{y^2}\left[\frac{\omega a}{c}\right]^2. \tag{17}$$

$\tilde{W}$, $\tilde{\gamma}$, and $\tilde{\epsilon}$ are defined by Eq. (FW3.2). After substitution and algebraic manipulation,

$$X_2^{m-}(\lambda) \simeq -\frac{2}{15}\frac{c}{\epsilon_2\omega a}\left[\frac{5}{4-m^2}\right]^{1/2}\times\frac{1}{m\tilde{\gamma}+i\tilde{W}}(C_m x^2 - A_m y^2)d_{1m}^-(\lambda) \tag{18}$$

where

$$C_m \equiv 2\frac{\tilde{\epsilon}}{\epsilon_2} - \left[\frac{m^2-4}{2}\right]\tilde{\gamma} - \tfrac{3}{2}im\tilde{W}+3 \tag{19}$$

and

$$A_m \equiv (2+m^2\tilde{\gamma}-im\tilde{W}) + \frac{3}{4}\frac{\epsilon_2}{\tilde{\epsilon}}\left\{\left[2+\left[\frac{4-m^2}{3}\right]\tilde{\gamma}-im\tilde{W}\right]\times(2+m^2\tilde{\gamma}-im\tilde{W}) - \tfrac{1}{3}(4-m^2)(m\tilde{\gamma}+i\tilde{W})^2\right\}. \tag{20}$$

Next, we rearrange the numerator determinant by modifying the first row according to

$$Y_1^{m-} \rightarrow Y_1^{m-} - \frac{\epsilon_2\omega a}{2c}\left[\frac{4-m^2}{5}\right]^{1/2}(m\tilde{\gamma}+i\tilde{W})C_m^{-1}X_2^{m-} \tag{21}$$

to obtain

$$Y_1^{m-} \simeq \frac{y^2}{15}(1-A_m/C_m)d_{1m}^-. \tag{22}$$

Similarly, modify the first row of the denominator determinant according to

$$X_1^{m-} \rightarrow X_1^{m-} + \frac{15}{2}\left[\frac{c}{\omega a}\right]\left[\frac{4-m^2}{5}\right]^{1/2}(m\tilde{\gamma}+i\tilde{W})\times C_m^{-1}\left[1-\frac{x^2}{3}\right]X_2^{m-} \tag{23}$$

to obtain

$$X_1^{m-} \simeq \frac{A_m}{C_m}\frac{y^2}{x^2}d_{1m}^-(\lambda) \tag{24}$$

to lowest order.

Factors independent of λ can be factored from the first row of each determinant, leaving the numerator and denominator determinants equal, so that they cancel. Thus, in the long-wavelength limit,

$$Z_{11}^{m-} = \frac{\epsilon_2}{15}\left[\frac{\omega a}{c}\right]^2\left[\frac{C_m}{A_m}-1\right]. \tag{25}$$

Substitution for A_m and C_m leads to the result

$$Z_{11}^{m-}=\begin{cases}\dfrac{1}{15}\left[\dfrac{\omega a}{c}\right]^2\dfrac{\epsilon_2[2\epsilon_+\epsilon_- - \epsilon_{xx}\epsilon_{zz} - \epsilon_2(2\epsilon_{xx}+\epsilon_{zz})]+2\epsilon_+\epsilon_-\epsilon_{zz}}{\epsilon_2(2\epsilon_{xx}+\epsilon_{zz})+2\epsilon_{xx}\epsilon_{zz}}, & m=0\\ \dfrac{1}{30}\left[\dfrac{\omega a}{c}\right]^2\dfrac{\epsilon_2(\epsilon_\pm+\epsilon_{zz}-6\epsilon_2)+4\epsilon_\pm\epsilon_{zz}}{3\epsilon_2+\epsilon_\pm+\epsilon_{zz}}, & m=\pm 1\ .\end{cases} \tag{26}$$

We now summarize the results. The extinction cross section which we have calculated in the long-wavelength limit is

$$\sigma_{\rm ext}=\frac{(4\pi a)^2}{x}\,{\rm Im}\left[\sum_{l,l'}\sum_{m,\sigma}\frac{(-ix)^l}{(2l+1)!!}\frac{(ix)^{l'}}{(2l'-1)!!}Z_{ll'}^{m\sigma}\Phi_{ll'}^{m\sigma}(\hat{\mathbf{k}})\right] \tag{27}$$

where the summations are over the values of l, l', m, and σ discussed previously, and

$$\Phi_{ll'}^{m\sigma}(\hat{\mathbf{k}})\equiv\begin{Bmatrix}\mathbf{Y}_{l'l'}^{m}(\hat{\mathbf{k}})^*\times\hat{\mathbf{k}}\cdot\mathbf{B}_1, & \sigma=(-1)^{l'}\\ -\mathbf{Y}_{l'l'}^{m}(\hat{\mathbf{k}})^*\times\hat{\mathbf{k}}\cdot\mathbf{E}_1, & \sigma=(-1)^{l'+1}\end{Bmatrix}\times\begin{Bmatrix}\mathbf{Y}_{ll}^{m}(\hat{\mathbf{k}})\times\hat{\mathbf{k}}\cdot\mathbf{E}_1^*/|\mathbf{E}_1|^2, & \sigma=(-1)^{l+1}\\ \mathbf{Y}_{ll}^{m}(\hat{\mathbf{k}})\times\hat{\mathbf{k}}\cdot\mathbf{B}_1^*/|\mathbf{B}_1|^2, & \sigma=(-1)^{l}\end{Bmatrix}. \tag{28}$$

Also, $l!!\equiv 1(1-2)(1-4)\cdots$.

Table I lists the expressions for $Z_{ll'}^{m\sigma}$. The $\Phi_{ll'}^{m\sigma}(\hat{\mathbf{k}})$ can be evaluated using

$$\mathbf{Y}_{11}^{m*}(\hat{\mathbf{k}})\times\hat{\mathbf{k}}\cdot\mathbf{E}_1=i\left[\frac{3}{8\pi}\right]^{1/2}(\hat{\mathbf{e}}_m^*\cdot\mathbf{E}_1)\ , \tag{29a}$$

$$\mathbf{Y}_{22}^{m*}(\hat{\mathbf{k}})\times\mathbf{k}\cdot\mathbf{E}_1=\begin{cases}i\left[\dfrac{15}{8\pi}\right]^{1/2}k_0E_0, & m=0\\ i\left[\dfrac{5}{8\pi}\right]^{1/2}(k_\pm E_0+k_0E_\pm), & m=\pm1\\ i\left[\dfrac{5}{4\pi}\right]^{1/2}k_\pm E_\pm, & m=\pm2\end{cases} \tag{29b}$$

where

$$k_\pm\equiv\mp 2^{-1/2}(k_x\mp ik_y)\ ,\qquad k_0\equiv k_z\ . \tag{30}$$

III. DISCUSSION

The expressions which we have derived correctly reduce to known results under the appropriate conditions. The extinction cross section reduces to the prediction of Mie theory [Eqs. (3) and (4)] when the source of gyrotropy is removed and the complex dielectric function of the sphere is assumed isotropic. In particular, we set $\epsilon_{xx}=\epsilon_{zz}\equiv\epsilon$, $\epsilon_{xy}=0$. The cross terms $(1,2,m,-)$ and $(2,1,m,-)$ vanish.

The frequencies of resonances induced by gyrotropy are obtained from the zeroes of the real parts of the denominators of the expressions in Table I. For a given value of m, the denominators for the magnetic dipole, electric quadrupole, and cross terms are the same. Therefore, only one of these terms need be considered for the evaluation of the resonance frequencies, but all are required to determine the strength of the resonances.

To proceed further it is convenient to specify a model for the complex dielectric function of the sphere. The simplest model consistent with Eq. (1) is a single-component Drude plasma in a uniform static applied magnetic field $\mathbf{H}=H\hat{\mathbf{z}}$. The diagonalized dielectric tensor is given by

$$\epsilon_\pm(\omega)=\epsilon_1\left[1-\frac{\omega_p^2}{\omega(\bar{\omega}\pm\omega_c)}\right], \tag{31a}$$

$$\epsilon_{zz}(\omega)=\epsilon_1\left[1-\frac{\omega_p^2}{\omega\bar{\omega}}\right] \tag{31b}$$

where

$$\omega_p\equiv\left[\frac{4\pi ne^2}{\epsilon_1 m^*}\right]^{1/2} \tag{32}$$

is the plasma frequency, n is the carrier density, e is the charge, m^* is the effective mass, and ϵ_1 the dielectric constant due to core polarizability. Also, $\omega_c\equiv eH/(m^*c)$ is the cyclotron frequency and $\bar{\omega}\equiv\omega+i/\tau$ where τ is the electronic scattering time. For a lossless Drude plasma, $\tau\to\infty$ and $\bar{\omega}\to\omega$. This case was studied by Furdyna *et al.*[35] and Goettig and Trzeciakowski.[36]

The electric dipole resonances are given by

$${\rm Re}[\epsilon_{zz}(\omega)+2\epsilon_2]=0 \tag{33}$$

for $m=0$, which leads to

$$\omega_{101}=\omega_p\left[\frac{\epsilon_1}{2\epsilon_2+\epsilon_1}\right]^{1/2}\equiv\omega_1 \tag{34}$$

for the resonance frequency. This frequency, which is independent of the applied magnetic field, is the resonance frequency for electric dipole plasma motion in zero field. The resonances are labeled according to the notation of Furdyna *et al.*[35] A resonance is labeled $\omega_{lm\tau}$, where $\tau=1$ if the frequency tends to ω_l, the lth zero-field mode of oscillation, in the low-field limit and $\tau=2$ if the frequency tends to zero. For $m=\pm1$, we set

$${\rm Re}[\epsilon_\pm(\omega)+2\epsilon_2]=0 \tag{35}$$

to obtain

$$\omega_{1,\pm 1,1} = \mp \frac{\omega_c}{2} + \left[\left[\frac{\omega_c}{2} \right]^2 + \omega_1^2 \right]^{1/2} . \tag{36}$$

This result is the well-known plasma-shifted cyclotron resonance,[32,40,45,46] which describes the interaction between the cyclotron resonance and plasmon modes.

The magnetic-dipole–electric-quadrupole terms are considered together. For $m=0$, resonances are located from

$$\mathrm{Re}[\epsilon_2(2\epsilon_{xx}+\epsilon_{zz})+2\epsilon_{xx}\epsilon_{zz}]=0 . \tag{37}$$

The resonance frequencies are the two real positive solutions of

$$\omega^4-(\omega_c^2+\omega_p^2+\omega_2^2)\omega^2+[(1+\epsilon_2/2\epsilon_1)\omega_c^2+\omega_p^2]\omega_2^2=0 \tag{38}$$

where

$$\omega_2^2=2\epsilon_1\omega_p^2/(3\epsilon_2+2\epsilon_1) . \tag{39}$$

One of these modes, which is written to order $(\omega_c/\omega_p)^2$ as

$$\omega_{201} \simeq \omega_2+\tfrac{1}{6}\omega_c^2/\omega_2 , \tag{40}$$

is the ω_{201} mode of Furdyna *et al.*[35] The other mode,

$$\omega \simeq \omega_p+\tfrac{1}{3}\omega_c^2/\omega_p , \tag{41}$$

is the bulk magnetoplasmon of Goettig and Trzeciakowski[36] (GT). According to GT only the component of the oscillating magnetic field parallel to the dc applied magnetic field interacts with this mode, which arises from the $m=0$ magnetic dipole term. However, we find that this mode appears in the electric quadrupole and cross terms as well. To lowest order in ω_c/ω_p, the power absorbed due to the magnetic dipole term agrees with Eq. (57) of GT and the contribution of the remaining three terms is of higher order, which resolves the discrepancy.

For $m=\pm 1$, the condition for resonances is

$$\mathrm{Re}(3\epsilon_2+\epsilon_\pm+\epsilon_{zz})=0 . \tag{42}$$

For the lossless Drude plasma, the resonance frequencies are given by the real positive roots of

$$\omega^3 \pm \omega_c\omega^2-\omega_2^2\left[\omega \pm \frac{\omega_c}{2}\right]=0 . \tag{43}$$

Note that the cyclotron frequency can be positive or negative depending on the sign of the carriers. This equation agrees with Eq. (23) of Furdyna *et al.*[35] for $l=2$. The expressions for the roots are quite complicated. For the low-field limit,

$$\omega_{2,\pm 1,1} \simeq \omega_2 \mp \frac{\omega_c}{4}+\frac{5}{32}\frac{\omega_c^2}{\omega_2} \tag{44}$$

and

$$\omega_{2,-\mathrm{sgn}q,2} \simeq \frac{|\omega_c|}{2}\left[1-\frac{1}{4}\left[\frac{\omega_c}{\omega_2}\right]^2\right] \tag{45}$$

in agreement with Furdyna *et al.*[35] sgn$q$ denotes the sign of the carriers.

For $m=\pm 2$, resonances are located using

$$\mathrm{Re}(3\epsilon_2+2\epsilon_\pm)=0 . \tag{46}$$

For the lossless Drude plasma, this condition becomes

$$\omega^2 \pm \omega_c\omega-\omega_2^2=0 . \tag{47}$$

The frequencies are

$$\omega = \mp \frac{\omega_c}{2}+\left[\left[\frac{\omega_c}{2}\right]^2+\omega_2^2\right]^{1/2} . \tag{48}$$

The power absorbed can also be calculated for the "lossless" plasma. It is related to the extinction cross section by

$$P=\sigma_{\mathrm{ext}}S \tag{49}$$

where S is the magnitude of the Poynting vector of the incident wave. Although our results do not appear similar at first glance they are identical to those obtained by Furdyna *et al.*,[35] who followed a quantum-mechanical approach: (1) Compute the modes of the system. (2) Write down the Hamiltonian. (3) Quantize it. (4) Use it to compute the power absorption by treating the interaction of the modes with a perturbing electromagnetic field. Our approach is classical. We take our results for the extinction cross section, plug in the dielectric tensor for the single-component Drude plasma, and take the lossless limit by letting the relaxation time become infinite. Use is made of the relation

$$\lim_{a\to 0}\frac{a}{x^2+a^2}=\pi\delta(x) . \tag{50}$$

Our results not only confirm the work of Furdyna *et al.*,[35] but are considerably more general because we can treat a sphere having any dielectric tensor consistent with Eq. (1). We may include phenomenological damping, multicomponent systems (which allow for many resonances, including types not seen for the simple example), optical phonons, and anisotropic carrier effective-mass tensors with the proper symmetry. The generality offers the possibility of comparison with data on real systems including powdered semiconductors (e.g., n-type InSb, n-type InAs),[2–9] electron-hole droplets (for specific orientations only),[10–22] and possibly the characterization of the new semiconductor "microdots."[47–51] Cyclotron resonance and magnetospectroscopy in general has been of use in the study of bulk and thin-film semiconductors as well as quantum wells, heterojunctions, and the two-dimensional electron gas.[52] If quantum dot structures take on some importance for new device structures, our expressions should be of use in the interpretation of magnetooptical data, particularly if the dots are spheres.

The behavior of the electric dipole extinction in an applied magnetic field in the long-wavelength limit is well known. What is new is the expression for the magnetic-dipole–electric-quadrupole extinction cross section. Experimentally, resonances have been observed for electric dipole excitations in several systems,[2–22] but there have been few reports of magnetic dipole excitations.[20] The magnetic-dipole–electric-quadrupole term is relatively

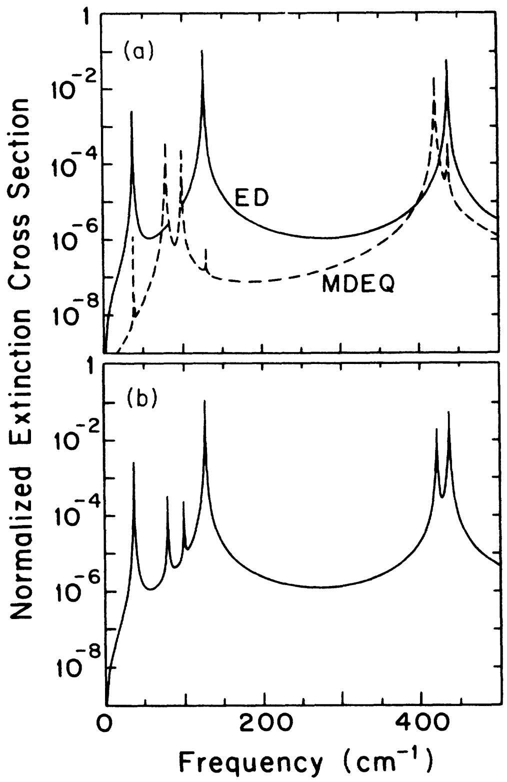

FIG. 1. Normalized extinction cross section for a small gyrotropic sphere. Parameters for a single-component Drude plasma are chosen to model a 1-μm-radius n-type InSb sphere in vacuum: $n=5\times10^{16}$ cm^{-3}, $m^*/m=0.014$, $1/\tau=0.5$ cm^{-1}, $\epsilon_1=18$, $\epsilon_2=1$. A 6-T magnetic field is applied in the z direction. The incident wave propagates in the x direction. (a) Electric dipole (ED), solid line, and magnetic-dipole–electric-quadrupole (MDEQ), dashed line, contributions. (b) Total extinction cross section in the long-wavelength limit. The ED term dominates, especially for the low-frequency (<150 cm^{-1}) group of resonances.

small and should be more difficult to observe, but the effect should be observable. Figure 1 shows the normalized extinction cross section $\sigma_{\text{ext}}/(\pi a^2)$, with parameters for the single-component Drude plasma chosen to model a 1-μm-radius sphere of n-type InSb in vacuum. Figure 1(a) shows the contributions of the electric dipole (ED) and magnetic-dipole–electric-quadrupole (MDEQ) terms separately. Figure 1(b) shows the total extinction cross section in the long-wavelength limit. The 6-T magnetic field is applied in the z direction. The incident electromagnetic wave propagates in the x direction. The MDEQ resonances are weaker by 2–3 orders of magnitude in the low-frequency multiplet below 150 cm^{-1}, and some of them overlap the ED resonances. The ED and MDEQ resonances are closer in magnitude for the group at high frequencies (>400 cm^{-1}). The strength of the MDEQ resonances relative to the ED resonances increases with particle size, but the applicability of the long-wavelength approximation becomes more questionable as well. The scattering rate was assigned a low value $[1/\tau(\text{cm}^{-1})]=0.5$ cm$^{-1}=1/[2\pi c\tau(\text{s})]$ to produce sharp resonances on the figure. If the resonances in actual materials are broader, MDEQ resonances will be more difficult to resolve and identify. For an experiment, it is important to have well-dispersed spherical particles with a controlled sharp size distribution. If the effective-mass tensors of the carriers are anisotropic, it is desirable that the particles be aligned.

IV. CONCLUSIONS

We have obtained the long-wavelength limit of the extinction cross section for a gyrotropic sphere from the general solution of Ford and Werner.[1] We applied the results to a lossless single-component Drude plasma sphere and obtained agreement with recent calculations by Furdyna *et al.*[35] and GT,[36] which were performed using a quasistatic approximation. Thus, our results resolve the discrepancy between these recent studies and the earlier, incorrect long-wavelength expressions. In addition, their generality permits potential applications to a number of actual materials.

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