PHYSICAL REVIEW B

CONDENSED MATTER

THIRD SERIES, VOLUME 38, NUMBER 12 15 OCTOBER 1988-II

Resistance fluctuations in a four-probe geometry with infinite leads

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(Received 13 October 1987; revised manuscript received 29 April 1988)

The mean-square variation in the magnetoresistance of small metal wires is calculated in a four-probe geometry with infinite leads. Dephasing scattering is included in a simplified but current-conserving manner. As a result, a region of each lead approximately one dephasing length from the sample contributes to the fluctuations, which, however, then remain finite as the leads are made long, as one would expect on phyiscal grounds. The calculated symmetric and antisymmetric parts of the resistance fluctuations are in semiquantitative agreement with experiment.

That quantum interference is the cause of reproducible aperiodic oscillations in the low-temperature conductance of small metal samples as a function of magnetic field and Fermi energy is now conclusively established.[1–3] In the pioneering theoretical work on this subject,[4,5] these conductance fluctuations were calculated to have a magnitude of e^2/h, in agreement with the early experiments. As samples have become smaller, though, it has become apparent that the variance of the conductance can be larger than the putative universal value.[1,6–8] In particular, for sample lengths much smaller than a dephasing length, the fluctuations in the conductance, as defined by the current injected divided by the voltage measured, diverge. The fluctuations in the resistance, on the other hand, do remain finite, a fact which can be seen by voltage additivity.[1]

The resistance fluctuations do not go to zero because the actual region which a resistance measurement samples cannot be smaller than a dephasing length, even if the lithographical sample can be. There have been a number of attempts to make this simple intuitive idea more precise.[1,7–15] Benoit *et al.*[7] have used voltage additivity and the Onsager relations to show that the antisymmetric part of the voltage fluctuations in a magnetic field is bounded as the length of the sample increases. Buttiker[9] has used his generalization of the Landauer formula,[16] which includes the leads in a multiterminal device, to estimate the voltage fluctuations for a three-terminal geometry. Numerical simulations and analytic calculations using the same multiterminal Landauer formula for four terminal geometries have also been done at the same time as the present work.[12,13,15] The numerical simulations include dephasing through the placement of the electron reservoirs. This does not correspond to present experiments in small metal wires and metal-oxide-semiconductor field-effect transistors (MOSFET's) because there are no perfect leads in these experiments. The analytic calculations,[12,15] which are equivalent to the calculation presented here, include dephasing uniformly throughout the system as in present experiments.

With the multiterminal Landauer formula in a four-terminal geometry, there are four electron reservoirs which are characterized by chemical potentials and taken to be perfect conductors. The current flowing into reservoir i, denoted I_i, is related to the voltage drops between the reservoirs via the transmission probabilities, T_{ij},

$$I_i = 2\frac{e^2}{h}\sum_{j\,(\neq i)} T_{ij}(V_j - V_i)\ . \tag{1}$$

(The 2 here comes from the spin degeneracy— we do not include the complication of spin-flip scattering.) A real experiment, of course, does not have perfect conductors, so we view the perfect conductors as mathematical constructs which simplify the problem by limiting the scattering to a finite region. As such, the result for the resistance should not depend on the length of the leads. Our results are, for the first time, independent of where the reservoirs are placed for long leads.

The average or classical transmission probabilities may be calculated from Eq. (1) using the rules for adding classical resistors. For example, in the geometry shown in Fig. 1 there are four probes of lengths $L_1, \ldots, L_4$ and a sample of length L_s. All of the wires have the same cross-sectional area, A. For any given set of currents coming out of the wires, the knowledge of the classical voltage drops allows one to solve for the transmission probabilities. One can also solve for the transmission probabilities using the Fisher and Lee[17] relation between the transmission probability and the nonlocal conductivity,

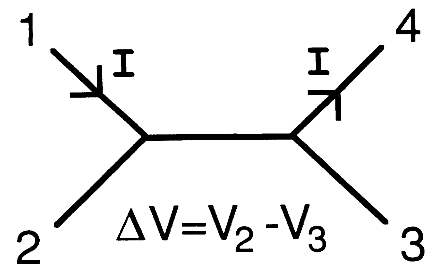

FIG. 1. Schematic of the four-terminal geometry used here. The reservoirs are labeled by 1, . . . , 4.

$$T_{ij}=(2e^2/h)^{-1}\int d^2x\, d^2x'\sigma_{zz}(\mathbf{x}\in i, \mathbf{x}'\in j)\ , \qquad (2)$$

and using the lowest-order current-conserving approximation for the average conductivity (the ladder graphs). The transmission probabilities are not simply related to the resistance of the leads. For example, the transmission probability to go from reservoir 1 to reservoir 2 depends on the resistance of all the leads, not just leads 1 and 2. Nonetheless, they combine to give the classical result for the resistance: $(V_2-V_3)/I=(m/ne^2\tau)(L_s/A)$.

For a particular configuration of the disorder, as opposed to the average case described above, each of the transmission probabilities deviates by some small amount, δT_{ij}, from its average value. Using the calculation of the average T_{ij} above and Eq. (1), the change in the resistance computed to linear order in the δT_{ij}'s is

$$\delta R=2\frac{e^2}{h}\left[\frac{m}{ne^2\tau}\frac{L}{A}\right]^2(\delta T_{21}-\delta T_{31}-\delta T_{24}+\delta T_{34})\ . \qquad (3)$$

To make the calculation more tractable, all the leads are taken to have the same length, L, which is much greater than L_s. As with the average resistance, the resistance measured for a particular configuration of the disorder depends in a complex way on the transmission probabilities. Here, the information about fluctuations in the resistance of the leads far away from the sample is canceled by taking this particular combination of the δT_{ij}'s. For example, suppose that one makes lead one slightly longer than the other leads in the case of classical conductors with $L_s=0$. By symmetry, pairs of the transmission probabilities are equal, $\delta T_{21}=\delta T_{31}$ and $\delta T_{24}=\delta T_{34}$, and hence the resistance change, δR, is zero. Had we made the sample longer instead and kept all of the lead lengths the same, then symmetry implies that $\delta T_{21}=\delta T_{34}$ and $\delta T_{31}=\delta T_{24}$. Because one also expects $\delta T_{21}>0$ and $\delta T_{31}<0$ when the length of the sample section is increased, the resistance change in this case is positive.

Although we cannot analytically compute the δT's for a particular configuration of the disorder, we can calculate the mean-square fluctuation about average in a magnetic field using the so-called ergodic hypothesis.[4] This hypothesis is based on the idea that changing the magnetic field by a sufficiently large amount randomizes the phases between the different scattering paths which an electron can undergo. Changing the distribution of impurities has the same effect. The ergodic hypothesis states that changing the magnetic field and changing the distribution of impurities produce fluctuations of the same size. Thus, to obtain the mean-square resistance fluctuations we need the average of δR^2 over all configurations of the disorder.

Our calculation uses the ergodic hypothesis to compute the mean-square fluctuation in the transmission probabilities and hence to compute δR^2. A typical diagram used in computing the averages of products of the δT's is shown in Fig. 2. The dashed and wavy lines denote elastic and dephasing scattering, respectively. There are an infinite number of diagrams like that shown in Fig. 2. The facts that the Fermi wavelength is much smaller than the mean free path, and that the quantum deviations in the transmission probabilities are small, restricts the number of graphs which must be summed. The most restrictive requirement, however, is that the conductivity be divergenceless: $\nabla_\alpha\sigma_{\alpha\beta}=\nabla_\beta\sigma_{\alpha\beta}=0$. This implies that the divergence of the current density is zero. From Eq. (2) we can see that it is crucial that current be conserved in this manner to ensure that the transmission probabilities do not depend on where the point z and z' are placed in the leads. We have produced a set of diagrams which explicitly conserve current: taking the divergence of the contributions of these diagrams produces a set of diagrams which cancel one another and a set of diagrams which can be shown to give zero.[18] The set of current conserving diagrams includes all those that have been previously used plus large classes not considered before. This class of diagrams is generated by inserting current vertices, both with and without vertex corrections, in all possible positions in graphs like that shown in Fig. 2—a result which is formally quite similar to the problem of current conservation in paraconductivity.[19] Besides choosing which diagrams to evaluate, a number of approximations are made to simplify the evaluation of these diagrams: the temperature is taken to be much less than the dephasing rate, the width of the wires is taken to be much less than the dephasing length, and the magnetic field is taken to be large enough to suppress weak locali-

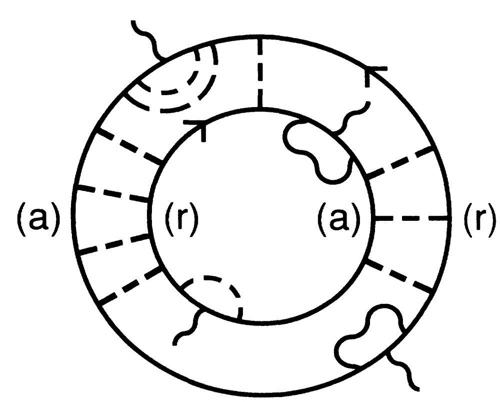

FIG. 2. Typical diagram in calculating the correlations between transmission probabilities. The wavy and dashed lines represent dephasing and elastic scattering, respectively. Retarded and advanced Green's functions are denoted by (r) and (a).

zation. These approximations do not restrict the applicability of these results to the experiments on small metal wires.[7]

The dephasing scattering included here, as in other conductance-fluctuation calculations, introduces a finite lifetime to the quasiparticles. It is not, however, inelastic scattering. The dephasing is produced by averaging over the positions of a small fraction of the elastic scatterers *before* one computes the mean-square deviation from average. In the graph in Fig. 2 this is indicated by the wavy (dephasing) lines only connecting the same loop or conductivity measurement, while the dashed (elastic) lines can connect the two loops. Spatial averaging in this way partially, but not completely, randomizes the phases of the electrons producing dephasing.

The result of evaluating the diagrams is given by

$$\delta R_A^2 = \left[2\frac{e^2}{h}\right]^2 \left[\frac{m}{ne^2\tau}\frac{1}{A}\right]^4 4\left[\int_{z\in 1,\, z'\in 2} dz\, dz'[D\tau P(z,z')]^2 + \int_{z\in 1,\, z'\in 3} dz\, dz'[D\tau P(z,z')]^2 + 2\int_{z\in 1,\, z'\in s} dz\, dz'[D\tau P(z,z')]^2\right], \tag{4a}$$

$$\delta R_S^2 - \delta R_A^2 = \left[2\frac{e^2}{h}\right]^2 \left[\frac{m}{ne^2\tau}\frac{1}{A}\right]^4 6\int_{z\in s,\, z'\in s} dz\, dz'[D\tau P(z,z')]^2 . \tag{4b}$$

The antisymmetric and symmetric parts of the resistance fluctuations in a magnetic field are referred to as δR_A and δR_S, respectively. The square of P is proportional to the mean-square size of the quantum correction to the probability to go from z to z' in the quasi-one-dimensional wires. It satisfies the differential equation,

$$\delta(z-z') = -D\tau\frac{\partial^2}{\partial z^2}P(z,z') + \frac{\tau}{\tau_\phi}P(z,z') , \tag{5}$$

so that it decays over a dephasing length, making $|z-z'| \lesssim L_\phi$ is the above integrals. In Eq. (4) the symmetry between the different leads after impurity averaging has been used to express the integrals more compactly. Solving for P and doing the integrals in Eq. (4) gives the curves shown in Fig. 3. This result is in qualitative agreement with experiment,[1,7,8] since the symmetric part of the voltage fluctuations increases as the square root of the sample length for $L_s > L_\phi$ and remains roughly constant for $L_s < L_\phi$. The antisymmetric portion of the resistance fluctuations is roughly constant for all sample lengths.

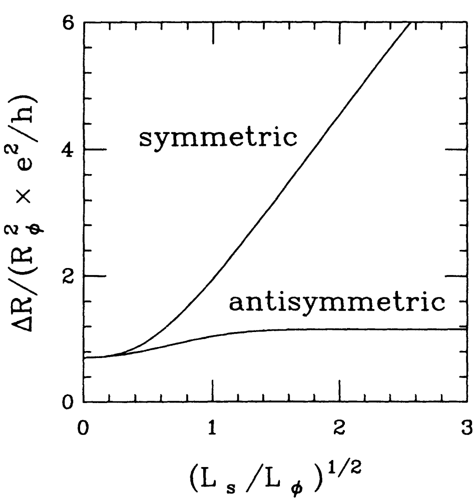

FIG. 3. Calculated result for the symmetric and antisymmetric part of the resistance fluctuations. R_ϕ is the average resistance of a wire one dephasing length long and L_s is the distance between the voltage probes.

From Eq. (4) the resistance fluctuations are due to quantum deviations in the probability to go from a point in a lead to somewhere in the sample, from one lead to another, or between two points in the sample. There is no contribution to the resistance fluctuation from quantum deviations in the probability to go between two points within the same lead. For points far away from the sample compared to L_ϕ, such deviations can be associated with fluctuations in the resistance of the leads, which we do not expect to affect the resistance measured in a four-terminal measurement. For points near the junctions between the leads, these deviations do depend on the disorder in the lithographical sample, but still do not contribute to the resistance measured. Hence, it is still meaningful to associate them with resistance fluctuations in the leads.

Similarly, deviations in the probability to go between two points in the sample can be associated with resistance fluctuations in the lithographical sample and do contribute to the resistance measured. In Eq. (4) such deviations enter only through $\delta R_S^2 - \delta R_A^2$, not δR_A^2. That this is the relevant combination for a resistance fluctuation in the sample can be seen using the Onsager relations introduced in this context by Buttiker.[16] According to these Onsager relations the symmetric part of the resistance fluctuations can also be obtained by averaging the resistance of the geometry in Fig. 1 with the resistance obtained when one interchanges the current and voltage probes. Likewise, the antisymmetric part is equal to half the difference of the resistances for the two configurations of the voltage and current leads. Thus, $\delta R_S^2 - \delta R_A^2$ is just the average of the product of the resistance measured in the two configurations. Classically, a change in the resistance of the sample changes the resistance measured in these two configurations by the same amount. Hence,

$\delta R_S^2 - \delta R_A^2$ will contain such a change, but δR_A^2 will not. From Eq. 4(b) we can make the stronger statement that on the average, there is no correlation between these two measurements due to the quantum deviations in the probability to go between leads or from a lead to the sample.

While the antisymmetric part does not contain information about the fluctuations in the resistance of the lithographical sample, it does contain information on what might be called the contact resistance. These are fluctuations in the probability to go between two points in different wire segments which are within L_ϕ of the junctions. Because the region of interest is restricted to within L_ϕ of the intersections, the size of these fluctuations remains roughly constant as L_s is increased. The mean-square–resistance fluctuations of the lithographical sample, on the other hand, grow linearly with the sample length for samples larger than L_ϕ because the fluctuations separated by more than L_ϕ are uncorrelated.

It is too early to discuss the quantitative agreement between experiment and theory because the dephasing length is not known precisely. The best test of experiment and theory will probably be to compare the values for the dephasing length determined in a number of independent ways from the resistance fluctuations themselves. For example, the magnitude of the fluctuations and the correlations of the fluctuations in a magnetic field both give estimates of the dephasing length. For now, there is clearly qualitative and semiquantitative agreement between theory and experiment without any *ad hoc* assumptions. We have also shown that when dephasing scattering is included in the four-terminal Landauer formula, the placement of the reservoirs drops out of the calculation for long leads.

This work was supported by the Cornell University Materials Science Center through the U. S. National Science Foundation under Grant No. DMR-82-17227.

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