Inhomogeneity expansion for the incommensurate charge-density-wave system

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We propose an effective-Lagrange description together with a practical calculational method, the inhomogeneity expansion, for the quasi-one-dimensional incommensurate charge-density-wave system. Corrections next to the leading order for the observables are calculated and estimated. We show that in the lowest-order approximation with the pinning potential being neglected the resultant expressions for those physical quantities of interest are consistent with known results.

I. INTRODUCTION

Since the discovery of the synthesis of $NbSe_3$ in 1975, extensive studies for the quasi-one-dimensional incommensurate charge-density-wave (ICDW) system are reported in the literature. A rich variety of very interesting experimental phenomena have been discovered. Meanwhile they were more or less successfully interpreted by various kinds of phenomenological or semiphenomenological theories.[1] But it seems to us that a unified and completely microscopic quantum description for the ICDW is still left open.

Recently, it was reported[2,3] that from a field-theoretical point of view, the 1 + 1 abelian chiral anomaly does make sense for the quasi-one-dimensional ICDW system, and chiral symmetry may play an important role in the microscopic description for the ICDW.[4] Pursuing along this line, in this paper, we would like to propose a systematic approximation scheme, the inhomogeneity expansion, as a concrete calculational scheme for observables in the ICDW system. Corrections next to the leading order for the observables are calculated and estimated. We show further that if we neglect the pinning potential for simplicity, in the lowest-order approximation we obtain the expressions for the observables which are consistent with the known and generally accepted results.[5]

In order to make this paper self-contained, we first summarize some of the discussions in Ref. 2, and then develop a simple but rather elaborate field-theoretical technique for the ICDW.

II. CHIRAL SYMMETRY AND CHIRAL ANOMALY IN ICDW

Following the main lines of the Lee-Rice-Anderson[5] approach we can derive the following Lagrange density for the quasi-one-dimensional ICDW system from the Fröhlich Lagrangian of interacting electron-phonon system.[6] Here we neglect the external random potential for simplicity, and we have

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\mathrm{ph}} + \mathcal{L}_{\mathrm{el}} , \\
\mathcal{L}_{\mathrm{ph}} &= \rho_0 \left[\left| \frac{\partial \phi}{\partial t} \right|^2 - v_Q^2 \left| \frac{\partial \phi}{\partial x} \right|^2 - \omega_0^2 |\phi|^2 \right] , \\
\mathcal{L}_{\mathrm{el}} &= \psi^\dagger \left[i\hbar \frac{\partial}{\partial \hbar} - e\Phi - \tau_3 v_F \left[\frac{\hbar}{i}\frac{\partial}{\partial x} - \frac{e}{c} A \right] \right. \\
&\quad \left. + \frac{G}{\sqrt{2}} (\tau_1 \phi_1 + \tau_2 \phi_2) \right] \psi ,
\end{aligned} \tag{2.1}$$

where ρ_0 is the linear density of the ion masses, v_F is the Fermi velocity, ω_0 is the optical phonon frequency and v_Q^2 is defined by $\partial^2\omega^2(Q)/2\partial Q^2$, ψ is the two component spinor

$$\psi(x) = \begin{bmatrix} \psi_R(x) \\ \psi_L(x) \end{bmatrix} , \tag{2.2}$$

τ_i's are Pauli matrices, and ϕ_1 and ϕ_2 are the real and imaginary parts of

$$\phi(x) = \phi_1(x) + i\phi_2(x) = \eta(x) e^{i\chi(x)} \tag{2.3}$$

with $\eta(x)$ and $\chi(x)$ being the corresponding modulus and phase of the $\phi(x)$. These fields are related respectively to the nonrelativistic electron Schrödinger wave field $\Psi(x)$ (the spin degree of freedom is kept implicitly) and the phonon wave field $u(x)$ (i.e., the one-dimensional ion displacement) by

$$\begin{aligned}
\Psi(x) &= \psi_R(x) e^{iQx/2} + \psi_L(x) e^{-iQx/2} , \\
u(x) &= \phi(x) e^{iQx} + \phi^*(x) e^{-iQx} ,
\end{aligned} \tag{2.4}$$

where $Q/2 = p_F$ is the Fermi momentum which is incommensurate with the lattice spacing and the acoustic part of the phonon field has been neglected.

In this derivation we keep only the electron modes which are near the Fermi surface as well as the phonon modes of momenta near $\pm Q$. So we assume $\psi_R(x)$, $\psi_L(x)$, and $\phi(x)$ are all slowly varying functions, namely,

$$|\partial_t \phi| \ll \omega_0 |\phi| , \quad |\partial_x \phi| \ll Q\, |\phi| , \text{ etc .} \tag{2.5}$$

In order to relate Lagrangian Eq. (2.1) to the standard (1 + 1)-dimensional relativistic field theory expression, we introduce

$$\gamma^0=\tau_1,\ \ \gamma^1=-i\tau_2,\ \ \gamma_5=\gamma^0\gamma^1=\tau_3\ ,$$
$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu},\ \ \gamma_5\gamma^\mu=\epsilon^{\mu\nu}\gamma_\nu\ ,$$

with

$$g^{\mu\nu}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\ \ \epsilon^{\mu\nu}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

and furthermore, $\bar{\psi}=\psi^\dagger\gamma^0$,

$$\partial_\mu=\left[\frac{1}{v_F}\frac{\partial}{\partial t},\ \frac{\partial}{\partial x}\right],$$

$$A^\mu(x)=\left[\frac{c}{v_F}\Phi(x),\ A(x)\right],\ \ E(x)=\frac{c}{v_F}\epsilon^{\mu\nu}\partial_\mu A_\nu(x)\ ,$$

then the expression of the Lagrangian $\mathcal{L}$, Eq. (2.1), becomes

$$\begin{aligned}\mathcal{L}_{\rm ph}&=\tfrac{1}{2}\rho_0\left[\left[\frac{\partial\eta}{\partial t}\right]^2-v_Q^2\left[\frac{\partial\eta}{\partial x}\right]^2-\omega_0^2\eta^2\right]\\&\quad+\tfrac{1}{2}\rho_0\eta^2\left[\left[\frac{\partial\chi}{\partial t}\right]^2-v_Q^2\left[\frac{\partial\chi}{\partial x}\right]^2\right],\\ \mathcal{L}_{\rm el}&=\bar{\psi}\left[i\hbar v_F\gamma^\mu\left[\partial_\mu+\frac{ie}{\hbar c}A_\mu\right]+\frac{G}{\sqrt{2}}\eta e^{i\gamma_5\chi}\right]\psi\ ,\end{aligned}\tag{2.6}$$

where $A^\mu(x)$ is the vector potential and $E(x)$ the electric field.

The Lagrangian (2.6) is invariant under the following global gauge and chiral transformations:

$$\begin{aligned}&\psi(x)\to e^{i\alpha}\psi(x),\ \ \bar{\psi}(x)\to\bar{\psi}(x)e^{-i\alpha}\ ,\\&\psi(x)\to e^{i\gamma_5\beta}\psi(x),\ \ \bar{\psi}(x)\to\bar{\psi}(x)e^{i\gamma_5\beta},\ \ \chi\to\chi-2\beta\ .\end{aligned}\tag{2.7}$$

The global gauge invariance merely reflects the charge conservation. But the unexpected chiral invariance is a dynamical one and it is based among others on the incommensurability of the system.

The Noether current, induced by the Gauge transformation, is the usual conserved electric current and has its "relativistic" expression as

$$j^\mu(x)\equiv(v_F\rho(x),j(x))=-\frac{c}{e}\frac{\partial\mathcal{L}}{\partial A_\mu(x)}=v_F\bar{\psi}(x)\gamma^\mu\psi(x)\ ,\tag{2.8}$$

with charge density $\rho(x)=\psi^\dagger(x)\psi(x)$ and $j(x)=v_F\psi^\dagger(x)\tau_3\psi(x)$. The Noether current induced by the chiral transformation

$$j_5^\mu(x)=j_{5,\rm el}^\mu(x)+j_{5,\rm ph}^\mu(x)\tag{2.9}$$

with

$$j_{5,\rm el}^\mu(x)=-\epsilon^{\mu\nu}j_\nu(x)=v_F\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)\ ,\tag{2.10}$$

$$j_{5,\rm ph}^\mu(x)=\frac{2}{\hbar}\frac{\partial\mathcal{L}_{\rm ph}}{\partial[\partial_\mu\chi(x)]}\ ,\tag{2.11}$$

are conserved only in the classical case and it was shown in Ref. 2 that

$$\partial_\mu\langle j_5^\mu(x)\rangle=\frac{2e}{\pi\hbar}E(x)\tag{2.12}$$

where $\langle\cdots\rangle$ means the quantum average over the ground state and $E(x)$ is the electric field (in this note, we discuss only the zero temperature case but the generalization to the finite temperature is straightforward). The right-hand side of Eq. (2.12) has the correct chiral anomaly. It is due to the quantum fluctuations of the ground state, i.e., the Fermi sea, and breaks the chiral symmetry. Expressing Eq. (2.12) in terms of the observable quantities $\rho(x)$ and $j(x)$, we have equivalently

$$\begin{aligned}&\frac{1}{v_F}\frac{\partial\langle j(x)\rangle}{\partial t}-v_F\frac{\partial\langle\rho(x)\rangle}{\partial x}\\&=\frac{2e}{\pi\hbar}E(x)-\frac{2}{\hbar}\rho_0\left[\frac{\partial}{\partial t}\left\langle\eta^2(x)\frac{\partial\chi(x)}{\partial t}\right\rangle\right.\\&\qquad\left.-v_Q^2\frac{\partial}{\partial x}\left\langle\eta^2(x)\frac{\partial\chi(x)}{\partial\chi}\right\rangle\right]\ .\end{aligned}\tag{2.13}$$

III. EFFECTIVE LAGRANGIAN

In order to develop a systematic approximation scheme, we first derive an effective Lagrangian which involves only the phonon variables by carrying out the functional integral for the electron variables. Here we follow the chiral transformation method discussed by Bardeen *et al.*[7] and applied to the ICDW system by Krive and Rozhavsky.[3]

We introduce the path integral expression for the generating functional of the system

$$Z[A]=\int\cdots\int[d\psi][d\bar{\psi}][d\chi][d\eta]\exp\left[\frac{i}{\hbar}\int d^2x\,\mathcal{L}\right]\tag{3.1}$$

where $\mathcal{L}$ is given by Eq. (2.6). And then, we change the variables into the chiral transformed field $\psi',\bar{\psi}'$ as

$$\psi'=e^{i\gamma_5\chi/2}\psi,\ \ \bar{\psi}'=\bar{\psi}e^{i\gamma_5\chi/2}\ ,$$

so the electron Lagrangian $\mathcal{L}_{\rm el}$, Eq. (2.6), can be now written as

$$\mathcal{L}_{\rm el}=\bar{\psi}'\left[i\hbar v_F\gamma^\mu\left[\partial_\mu+\frac{ie}{\hbar c}B_\mu\right]+\frac{G}{\sqrt{2}}\eta\right]\psi'\tag{3.2}$$

where

$$B^\mu(x)=A^\mu(x)-\frac{\hbar c}{2e}\epsilon^{\mu\nu}\partial_\nu\chi(x)\ .\tag{3.3}$$

The generating functional Eq. (3.1) then becomes

$$Z[A]=\int\cdots\int[d\psi'[d\bar{\psi}'][d\eta][d\chi]\exp\left[\frac{i}{\hbar}\int d^2x(\mathcal{L}+\mathcal{L}')\right],\tag{3.4}$$

where $\mathcal{L}'$ is the contribution from the Jacobian of the transformation

$$[d\psi][d\bar{\psi}]=[d\psi'][d\bar{\psi}']\exp\left[\frac{i}{\hbar}\int d^2x\,\mathcal{L}'\right] \tag{3.5}$$

and it can be proved as

$$\begin{aligned}\mathcal{L}'&=\frac{\hbar v_F}{\pi}\left[\frac{e}{\hbar c}\right]^2(A^\mu A_\mu-B^\mu B_\mu)\\&=\frac{\hbar v_F}{\pi}\left[\frac{1}{4}(\partial^\mu\chi)(\partial_\mu\chi)+\frac{e}{\hbar c}\chi\epsilon^{\mu\nu}\partial_\mu A_\nu\right]\\&=\frac{\hbar v_F}{4\pi}\left[\frac{1}{v_F^2}\left[\frac{\partial\chi}{\partial t}\right]^2-\left[\frac{\partial\chi}{\partial x}\right]^2\right]+\frac{e}{\pi}\chi E\ .\end{aligned} \tag{3.6}$$

This is a well known result in field theory.[8] But since all the procedures which we are going through are rather involved, we prefer to prove it by utilizing the following known formula appeared in the Schwinger model[9]

$$\begin{aligned}\int[d\psi][d\bar{\psi}]\exp\left[i\int d^2x\,\bar{\psi}i\gamma^\mu(\partial_\mu+iA_\mu)\psi\right]&=\exp\left[i\int d^2x\frac{1}{2\pi}A^\mu A_\mu\right]\\&\quad\times\int[d\psi][d\bar{\psi}]\exp\left[i\int d^2x\,\bar{\psi}i\gamma^\mu\partial_\mu\psi\right]\end{aligned} \tag{3.7}$$

where we fixed the gauge as the Lorentz gauge, i.e., $\partial_\mu A^\mu=0$. In Eq. (3.7) we have also taken account of both the pseudospin and the spin degrees of freedom. Replacing A_μ by B_μ and $\psi,\bar{\psi}$ by $\psi',\bar{\psi}'$ just to change the notation, we may also have

$$\begin{aligned}\int[d\psi'][d\bar{\psi}']\exp\left[i\int d^2x\,\bar{\psi}'i\gamma^\mu(\partial_\mu+iB_\mu)\psi'\right]&=\exp\left[i\int d^2x\frac{1}{2\pi}B^\mu B_\mu\right]\\&\quad\times\int[d\psi'][d\bar{\psi}']\exp\left[i\int d^2x\,\bar{\psi}'i\gamma^\mu\partial_\mu\psi'\right]\ .\end{aligned} \tag{3.8}$$

Since the functional integral on the right-hand side of Eqs. (3.7) and (3.8) are exactly the same, by comparing these two equations and rescaling them with appropriate constants, we obtain Eq. (3.5) with $\mathcal{L}'$ in the form of Eq. (3.6). We note that Eq. (3.6) is gauge invariant although we had used the Lorentz gauge for the derivation.

The integration over ψ' and $\bar{\psi}'$ is more involved. However, formally one can write

$$Z[A]=\int\cdots\int[d\eta][d\chi]\exp\left[\frac{i}{\hbar}\int d^2x\tilde{\mathcal{L}}_{\rm eff}\right] \tag{3.9}$$

with

$$\tilde{\mathcal{L}}_{\rm eff}=\tilde{\mathcal{L}}_\Delta+\tilde{\mathcal{L}}_\chi+\tilde{\mathcal{L}}_{\rm el}+\tilde{\mathcal{L}}_{\rm an}\ , \tag{3.10}$$

$$\tilde{\mathcal{L}}_\Delta=\frac{\rho_0}{G^2}\left[\left[\frac{\partial\Delta}{\partial t}\right]^2-v_Q^2\left[\frac{\partial\Delta}{\partial x}\right]^2-\omega_0^2\Delta^2\right]\ , \tag{3.11}$$

$$\tilde{\mathcal{L}}_\chi=\frac{\rho_0}{G^2}\Delta^2\left[\left[1+\frac{\lambda\hbar^2\omega_0^2}{4\Delta^2}\right]\left[\frac{\partial\chi}{\partial t}\right]^2-\left[v_Q^2+v_F^2\frac{\lambda\hbar^2\omega_0^2}{4\Delta^2}\right]\left[\frac{\partial\chi}{\partial x}\right]^2\right]\ , \tag{3.12}$$

$$\tilde{\mathcal{L}}_{\rm an}=\frac{e}{\pi}\chi E\ , \tag{3.13}$$

and $\tilde{\mathcal{L}}_{\rm el}$ is formally defined as

$$\int d^2x\,\tilde{\mathcal{L}}_{\rm el}=-i\hbar{\rm Tr}\ln\hat{D}^{-1}\ , \tag{3.14}$$

where

$$\hat{D}^{-1}=i\hbar v_F\gamma^\mu\left[\partial_\mu+\frac{ie}{\hbar c}B_\mu\right]+\Delta\ . \tag{3.15}$$

In the above equations we have introduced the gap parameter Δ to replace the modulus of the phonon variables η and the dimensionless coupling constant λ as

$$\Delta=\frac{G}{\sqrt{2}}\eta,\quad\lambda=\frac{G^2}{2}N(\varepsilon_F)\frac{4}{\rho_0\omega_0^2}\ ,$$

where $N(\varepsilon_F)$ is the density of states at the Fermi surface. $N(\varepsilon_F)=1/2\pi\hbar v_F$.

After this procedure, the original electron Lagrangian $\mathcal{L}_{\rm el}$, Eq. (2.6), is split effectively into three parts.

(1) $\tilde{\mathcal{L}}_{\rm an}$, which represents a direct phonon-phase–electric-field interaction with an universal coefficient. It provides the anomaly physics.

(2) An effective free Lagrangian of phonon-phase variable χ, $\tilde{\mathcal{L}}_\chi$, in which the corresponding original free Lagrangian has been combined together. This Lagrangian describes exactly the main physics of the phase mode propagation which were worked out by Lee, Rice, and Anderson.[5]

(3) $\tilde{\mathcal{L}}_{\rm el}$, for which we shall develop a systematic inhomogeneity expansion in the following section. From the definition of $\tilde{\mathcal{L}}_{\rm el}$, Eqs. (3.14) and (3.15), it is clear that the A_μ and χ dependence of $\tilde{\mathcal{L}}_{\rm el}$ is only through B_μ, which is defined in Eq. (3.3).

Next we examine whether we can derive the anomalous Ward-Takahashi identity, Eq. (2.13), from the effective Lagrangian. Taking the functional derivatives of $\int d^2x\,\tilde{\mathcal{L}}_{\rm el}$ with respect to $B_\mu(x)$ and $\Delta(x)$, respectively, we have

$$\frac{\delta}{\delta B_\mu(x)}\int d^2x\,\tilde{\mathcal{L}}_{\rm el}\equiv\frac{\partial\tilde{\mathcal{L}}_{\rm el}}{\partial B_\mu(x)}=i\hbar v_F\frac{e}{c}{\rm Sp}[\gamma^\mu\hat{D}(x,x)]\ , \tag{3.16}$$

$$\frac{\delta}{\delta\Delta(x)}\int d^2x\,\tilde{\mathcal{L}}_{\rm el}\equiv\frac{\partial\tilde{\mathcal{L}}_{\rm el}}{\partial\Delta(x)}=-i\hbar{\rm Sp}[\hat{\mathcal{D}}(x,x)]\ ,$$

where Sp means tracing both the spin and pseudospin indices, and

$$\hat{D}(x,x)=\langle x\,|\left[v_F\gamma^\mu\left[\hat{p}_\mu-\frac{e}{c}B_\mu(\hat{x})\right]+\Delta(\hat{x})\right]^{-1}|\,x\rangle\ , \tag{3.17}$$

which is in accordance with Eq. (3.15). In Eq. (3.17), $\hat{x}$ and $\hat{p}$ are operators, and

$$[\hat{p}_\mu, \hat{x}^\nu] = -i\hbar\delta_\mu{}^\nu, \quad \hat{x}^\mu | x \rangle = x^\mu | x \rangle . \tag{3.18}$$

Moreover, Eq. (3.17) and all the relevant previous expressions are understood as being defined in the Euclidean space, although we write those expressions formally in terms of the Minkowsky notation. It is important to note that Eqs. (3.16)–(3.18) now fix the precise meaning of the $\mathcal{L}_{\rm el}$ which is originally introduced formally by a formal functional integration over the ψ' and $\bar{\psi}'$ variables.

Taking derivatives of $\mathcal{L}_{\rm eff}$ with respect to $A_\mu(x)$, $\Delta(x)$, and $\chi(x)$, we obtain the following generalized Ward-Takahashi relations:

$$\langle \rho(x) \rangle = -\frac{1}{\pi}\frac{\partial\langle \chi(x)\rangle}{\partial x} - v\hbar\langle {\rm Sp}[\gamma^0\hat{D}(x,x)]\rangle$$

$$\langle j(x) \rangle = \frac{1}{\pi}\frac{\partial\langle \chi(x)\rangle}{\partial t} - v\hbar v_F\langle {\rm Sp}[\gamma'\hat{D}(x,x)]\rangle \tag{3.19}$$

and

$$\left[\frac{\partial^2}{\partial t^2} - v_Q^2\frac{\partial^2}{\partial x^2} + \omega_0^2\right]\langle \Delta(x)\rangle - \left\langle \Delta(x)\left[\frac{\partial\chi(x)}{\partial t}\right]^2\right\rangle + v_Q^2\left\langle \Delta(x)\left[\frac{\partial\chi(x)}{\partial x}\right]^2\right\rangle + i\hbar\frac{G^2}{2\rho_0}\langle {\rm Sp}[\hat{D}(x,x)]\rangle = 0 . \tag{3.20}$$

$$\frac{2\rho_0}{G^2}\left[\frac{\partial}{\partial t}\left\langle \Delta^2\left[1+\frac{\lambda\hbar^2\omega_0^2}{4\Delta^2}\right]\frac{\partial\chi}{\partial t}\right\rangle - v_F^2\frac{\partial}{\partial x}\left\langle \Delta^2\left[\frac{v_Q^2}{v_F^2}+\frac{\lambda\hbar^2\omega_0^2}{4\Delta^2}\right]\frac{\partial\chi}{\partial x}\right\rangle\right] = \frac{e}{\pi}E(x) - i\hbar^2 v_F\epsilon^{\mu\nu}\partial_\mu\langle {\rm Sp}[\gamma_\nu\hat{D}(x,x)]\rangle . \tag{3.21}$$

Using Eq. (3.19) we can easily verify that the equation of motion (3.21) for the phase variable $\chi(x)$ is precisely the anomalous Ward-Takahashi identity, Eq. (2.13), which is independent of the details of $\hat{D}(x,x)$.

IV. INHOMOGENEITY EXPANSION

In this section we discuss the calculation of $\hat{D}(x,x)$, which appears in the equations of motion. Since it is derived from the gauge invariant Eq. (3.14) by a functional derivative, therefore $\hat{D}(x,x)$ should also be gauge invariant.

We use the following identity:

$$\begin{aligned}\hat{D}(x,x) &= \langle x \,|\, [v_F\gamma^\mu\hat{p}_\mu + M(\hat{x})]^{-1} \,|\, x\rangle \\ &= \int\frac{d^2p}{(2\pi\hbar)^2}\sum_{n=0}^{\infty}\frac{(-)^n}{v_F\gamma^\mu p_\mu + M(x)} \times\left[\Delta M\left[\frac{\hbar}{i}\frac{\partial}{\partial p};x\right]\times\frac{1}{v_F\gamma^\nu p_\nu + M(x)}\right]^n\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}M(\hat{x}) &\equiv -v_F\frac{e}{c}\gamma^\mu B_\mu(\hat{x}) + \Delta(\hat{x}) \\ &= M(x) + (\hat{x}-x)^\mu\partial_\mu M(x) \\ &\quad + \tfrac{1}{2}(\hat{x}-x)^\mu(\hat{x}-x)^\nu\partial_\mu\partial_\nu M(x) + \cdots \\ &\equiv M(x) + \Delta M[\hat{x}-x;x]\end{aligned} \tag{4.2}$$

and $d^2p = v_F\,dp^0\,dp^1$.

Because our whole discussion is under the condition that only the electrons near the Fermi surface are taken account of, the spatial and temporal variation of the physical quantities are accordingly smooth compared to the inverse of the Fermi wave vector or that of the optical phonon frequency. Although $\hat{D}(x,x)$ is used inside the expectation bracket, i.e., the functional integration over Δ and χ, since the frequency and the wave number of Δ and χ are restricted to satisfy Eq. (2.5), the order of the spatial and temporal derivatives for the physical quantity could be taken as the order of a convergent expansion. Then Eqs. (4.1) and (4.2) provide a systematic expansion scheme: inhomogeneity expansion for the practical calculation. Moreover, examining the relevant expressions in Eqs. (2.13), (3.3), and (3.21), it is not difficult to verify that such an inhomogeneity expansion would be consistent with the following estimation: The derivative of the vector potential should be one order smaller than the derivative of the phase order parameter. It is also obvious that the derivative of the amplitude of the order parameter should be the smallest one.

We regularize the integration in Eq. (4.1) to keep the gauge invariance. Then the integration variable p_μ can be shifted to a finite amount. Especially if we shift p_μ by $v_F eB_\mu(x)/c$, we can eliminate $B_\mu(x)$ in the denominator. We then obtain

$$\hat{D}(x,x) = \int\frac{d^2p}{(2\pi\hbar)^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{v_F\gamma^\mu p_\mu + \Delta(x)}\times\left[\Delta M\left[\frac{\hbar}{i}\frac{\partial}{\partial p};x\right]\times\frac{1}{v_F\gamma^\mu p_\mu + \Delta(x)}\right]^n , \tag{4.3}$$

where

$$\begin{aligned}\Delta M\left[\frac{\hbar}{i}\frac{\partial}{\partial p};x\right] &= \frac{\partial}{\partial x^\mu}\left[-v_F\frac{e}{c}\gamma^\rho B_\rho(x) + \Delta(x)\right]\frac{\hbar}{i}\frac{\partial}{\partial p_\mu} \\ &\quad + \frac{1}{2}\frac{\partial^2}{\partial x^\mu\partial x^\nu}\left[-v_F\frac{e}{c}\gamma^\rho B_\rho(x) + \Delta(x)\right]\times\left[\frac{\hbar}{i}\right]^2\frac{\partial^2}{\partial p_\mu\partial p_\nu} + \cdots .\end{aligned} \tag{4.4}$$

We consider the $\mathrm{Sp}[\gamma^\mu \hat{D}(x,x)]$ first. The $n=0$ term of Eq. (4.3) will not contribute since

$$\hat{D}(x,x)\Big|_{n=0} = \int \frac{d^2p}{(2\pi\hbar)^2}\frac{1}{v_F\gamma^\mu p_\mu + \Delta(x)} ,$$

which is independent of B_μ, and

$$\mathrm{Sp}[\gamma^\mu \hat{D}(x,x)]_{n=0} = 0 . \tag{4.5}$$

Moreover, it can be easily seen that for all the terms with even number n we have

$$\mathrm{Sp}[\gamma^\mu \hat{D}(x,x)]_{n=\mathrm{even}} = 0 ,$$

which is a part of the Furry's theorem. Therefore, a straightforward calculation for the contribution of the $n=1$ term gives

$$\begin{aligned}\mathrm{Sp}[\gamma^\mu \hat{D}(x,x)]_{n=1} &= \frac{iev_F^2}{3\pi c\Delta^2}(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)B_\nu(x)\\ &= \frac{iev_F}{3\pi\Delta^2}\epsilon^{\mu\nu}\partial_\nu\\ &\quad\times\left[E(x) - \frac{\hbar v_F}{2e}\Box\chi(x)\right] ,\end{aligned} \tag{4.6}$$

where the d'Alembertian $\Box$ means $\partial_\mu\partial^\mu$. Eq. (4.6) is really a gauge invariant form.

Substituting Eq. (4.6) into Eqs. (3.19) and (3.21), the charge-current expression has the form

$$\begin{aligned}\langle\rho(x)\rangle &= -\frac{1}{\pi}\frac{\partial}{\partial\chi}\left[\left\langle\chi(x) + \frac{\hbar^2 v_F^2}{6\Delta^2}\Box\chi(x)\right\rangle - \left\langle\frac{e\hbar v_F}{3\Delta^2}E(x)\right\rangle\right] ,\\ \langle j(x)\rangle &= \frac{1}{\pi}\frac{\partial}{\partial t}\left[\left\langle\chi(x) + \frac{\hbar^2 v_F^2}{6\Delta^2}\Box\chi(x)\right\rangle - \left\langle\frac{e\hbar v_F}{3\Delta^2}E(x)\right\rangle\right] ,\end{aligned} \tag{4.7}$$

and the equation of motion for the phase variable

$$\frac{2\rho_0}{G^2}\left[\frac{\partial}{\partial t}\left\langle\Delta^2\left[1 + \frac{\lambda\hbar^2\omega_0^2}{4\Delta^2}\right]\frac{\partial\chi}{\partial t}\right\rangle - v_F^2\frac{\partial}{\partial x}\left\langle\Delta^2\left[\frac{v_Q^2}{v_F^2} + \frac{\lambda\hbar^2\omega_0^2}{4\Delta^2}\right]\frac{\partial\chi}{\partial x}\right\rangle\right] = \frac{e}{\pi}\left\langle\left[1 + \frac{\hbar^2 v_F^2}{3\Delta^2}\Box\right]E(x)\right\rangle - \frac{\hbar v_F}{2\pi}\left\langle\frac{\hbar^2 v_F^2}{3\Delta^2}\Box\Box\chi(x)\right\rangle . \tag{4.8}$$

In fact, all the contribution of Eq. (4.6) can be described by an effective Lagrangian as

$$\begin{aligned}&-\frac{e^2\hbar v_F^3}{6\pi c^2\Delta^2}B_\mu(x)(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)B_\nu(x)\\ &\qquad = -\frac{e^2\hbar v_F}{6\pi\Delta^2}\left[E(x) - \frac{\hbar v_F}{2e}\Box\chi(x)\right]^2\end{aligned} \tag{4.9}$$

which is the integration of Eq. (3.16).

The contribution of $\mathrm{Sp}[\gamma^\mu \hat{D}(x,x)]$ to the equation of motion for the phase order parameter has roughly the order of $(\hbar\omega/\Delta)^2$ where ω is the characteristic frequency for the temporal variation of the system. In our discussion, ω should be kept much smaller than the optical phonon frequency ω_0, therefore, so far $\hbar\omega_0$ is not larger than Δ, such corrections are not important. For the charge-current expressions Eq. (4.7), we should further estimate the electric-field-dependent term compared to the diamagnetic current $eA(x)\psi^\dagger(x)\psi(x)/mc$, since the latter has been neglected at the very beginning of our approach. The ratio of the two is of the order of

$$\frac{mv_F}{\hbar N}\left[\frac{\hbar\omega}{\Delta}\right]^2 , \tag{4.10}$$

where m is the electron mass and N the linear density of the conduction electrons. It is again proportional to $(\hbar\omega/\Delta)^2$.

Utilizing Eq. (4.3) we can also calculate $\mathrm{Sp}[\hat{D}(x,x)]$ order by order in a similar way. The contribution of the $n=0$ term will no longer vanish and gives the well known leading order contribution to the gap equation. The next-to-leading order of $\mathrm{Sp}[\hat{D}(x,x)]$ can also be integrated according to Eq. (3.16), and therefore can be described by an effective Lagrangian which has the following form

$$\sim\frac{\hbar v_F}{\pi\Delta^2}\Delta(x)\Box\Delta(x) . \tag{4.11}$$

The ratio of Eq. (4.11) to the amplitude part of the free-phonon Lagrangian, $\tilde{\mathcal{L}}_\Delta$, Eq. (3.11), is of the order of

$$\frac{\lambda\hbar^2\omega_0^2}{4\Delta^2} .$$

As discussed above, the derivative of the amplitude of the order parameter is the smallest among the others, the effective Lagrangian, Eq. (4.11), is also unimportant.

Since all the corrections are inversely proportional to Δ^2, it might be interesting to see the effect of the temperature-dependent cases where the Δ will decrease as the temperature increases.

According to the above discussion, the leading-order contribution of $\tilde{\mathcal{L}}_{\mathrm{el}}$, Eq. (3.14), incorporated with the other parts of the Lagrangian as shown in Eqs. (3.10)–(3.13), gives a fairly good description for the zero-temperature ICDW system with pinning potential being neglected. Then Eqs. (4.7), (3.20), and (4.8) become

$$\rho(x)=-\frac{1}{\pi}\frac{\partial\chi(x)}{\partial x}\ , \tag{4.12}$$

$$j(x)=\frac{1}{\pi}\frac{\partial\chi(x)}{\partial t}\ , \tag{4.13}$$

$$\rho_0\omega_0^2\Delta(x)+\frac{i\hbar G^2}{2}\int\frac{d^2p}{(2\pi\hbar)^2}\mathrm{Sp}\left[\frac{1}{v_F\gamma^\mu p_\mu+\Delta(x)}\right]=0\ , \tag{4.14}$$

$$\frac{2\rho_0}{G^2}\Delta^2\left[\left[1+\frac{\lambda\hbar^2\omega_0^2}{4\Delta^2}\right]\frac{\partial^2}{\partial t^2}-v_F^2\left[\frac{v_Q^2}{v_F^2}+\frac{\lambda\hbar^2\omega_0^2}{4\Delta^2}\right]\frac{\partial^2}{\partial x^2}\right]\chi(x)=\frac{e}{\pi}E(x)\ . \tag{4.15}$$

Notice in Eqs. (4.12)–(4.15) we have also taken the mean field approximation. Equation (2.13) as well as Eq. (4.15) [in connection with Eqs. (4.12) and (4.13)] reveal the interesting intuition of the classical approach, e.g., Ref. 1, that of a sliding charge density wave being accelerated by an applied electric field. The ICDW is in fact a collective degree of freedom of the Fermi sea below the Peierls gap. It is a Goldstone mode intimately connected to the incommensurately symmetry broken Peierls' ground state. The 1 + 1 abelian chiral anomaly describes the acceleration mechanism of the Goldstone mode in an applied electric field.

The solution for Δ from Eq. (4.14) for the static, homogeneous case is straightforward, we can easily have that

$$\Delta=2\varepsilon_F e^{-1/\lambda}\ .$$

If we solve the equation for the phase variable, Eq. (4.15) formally for the spatial homogeneous case, and substitute it into the current expression, Eq. (4.13), we can immediately derive

$$ej(\omega)=\sigma(\omega)E(\omega) \tag{4.16}$$

with

$$\sigma(\omega)=\frac{ie^2n}{m^*\omega}\ , \tag{4.17}$$

where

$$m^*=\left[1+\frac{4\Delta^2}{\lambda\hbar\omega_0^2}\right]m\ ,$$

$$n=2\int_{-p_F}^{p_F}\frac{dp}{2\pi\hbar}=\frac{2mv_F}{\pi\hbar}\ . \tag{4.18}$$

Equation (4.17) is just the low-frequency limit of the conductivity derived by Lee, Rice, and Anderson,[5] where m^* is the effective mass first introduced by Frohlich.[6] Moreover, with the formal solution of Eq. (4.15) for the phase order parameter, the second term on the right-hand side of Eq. (2.13) will have the following form:

$$\frac{2}{\hbar}\rho_0\eta^2\left[\frac{\partial^2}{\partial t^2}-v_Q^2\frac{\partial^2}{\partial x^2}\right]\chi(x)$$
$$=\left[\frac{\partial^2}{\partial t^2}-v_Q^2\frac{\partial^2}{\partial x^2}\right]\times\left[\left[1+\frac{\lambda\hbar^2\omega_0^2}{4\Delta^2}\right]\left[\frac{\partial^2}{\partial t^2}-\frac{m}{m^*}v_F^2\frac{\partial^2}{\partial x^2}\right]-v_Q^2\frac{\partial^2}{\partial x^2}\right]^{-1}\frac{2e}{\pi\hbar}E(x)$$

where we have utilized the expression for the m^*, Eq. (4.18). Usually it is reasonable to have

$$v_Q^2 \ll \frac{m}{m^*}v_F^2 \ll v_F^2\ .$$

Therefore, so far we have our system that

$$\frac{\partial^2}{\partial t^2}\gtrsim v_F^2\frac{\partial^2}{\partial x^2}\ ;$$

we could neglect $v_Q^2\partial^2/\partial\chi^2$ and $(m/m^*)v_F^2\partial^2/\partial\chi^2$ compared to $v_F^2\partial^2/\partial\chi^2$, and the above equation simplifies to

$$\frac{2}{\hbar}\rho_0\eta^2\left[\frac{\partial^2}{\partial t^2}-v_Q^2\frac{\partial^2}{\partial x^2}\right]\chi(x)\cong\left[1+\frac{\lambda\hbar^2\omega_0^2}{4\Delta^2}\right]^{-1}E(x)\ . \tag{4.19}$$

Substituting Eq. (4.19) into the anomalous Ward identity Eq. (2.13) with the same approximation, it becomes

$$\frac{1}{v_F}\frac{\partial\langle j(x)\rangle}{\partial t}-v_F\frac{\partial\langle\rho(x)\rangle}{\partial x}\cong\frac{m}{m^*}\left[\frac{2e}{\pi\hbar}\right]E(x)\ . \tag{4.20}$$

The above discussion indicates the interesting physics of the phase variable which propagates as a Goldstone mode with the phason propagator as

$$\left\{\frac{2\rho_0}{G^2}\Delta^2\left[\left[1+\frac{\lambda\hbar^2\omega_0^2}{4\Delta^2}\right]\frac{\partial^2}{\partial t^2}-v_F^2\left[\frac{v_Q^2}{v_F^2}+\frac{\lambda\hbar^2\omega_0^2}{4\Delta^2}\right]\frac{\partial^2}{\partial x^2}\right]\right\}^{-1}\ .$$

The term of Eq. (4.19) thus contributes an interesting "renormalization" effect, m/m^*, to the anomaly in accord with the Goldstone mode mechanism. This is in fact a phonon drag effect to the sliding charge density waves. Equation (4.20) is an interesting equation which connects mainly the observable quantities so far the random pinning potential can be neglected.

In summary, in this note, we suggest a well defined *ab initio* effective Lagrangian incorporated with a practically applicable calculation method. It provides an interesting way for the further microscopic studies, especially for the future applications of the field theoretic technique to the ICDW system with random pinning potentials.

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