Family of diamond-type hierarchical lattices

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A family of the diamond-type hierarchical lattices as a kind of fractals is proposed, on which the Ising model is exactly solved. The convergent condition of the free energy per bond in the thermodynamic limit is obviously given. We find that the unstable fixed point of the renormalization-group transformation moves toward $K=0$ ($T=\infty$) from $K=\infty$ ($T=0$) as the number of branches P is increased, and when $P=\infty$ (i.e., $d_f=\infty$) the unstable fixed point $K=0$ exactly, in contrast with that of the Bethe lattice. We also calculate the critical exponent of the correlation length; we find that in the $d_f=2$ and 3 cases the exponent is different from that of the regular lattice with $d=2$ and 3, respectively, which seems to imply that more general criteria for the classification of universality should be proposed. We have also discussed the upper critical fractal dimension.

I. INTRODUCTION

Recently hierarchical lattices have attracted much attention in the study of statistical mechanics of phase transitions, because classical spin models, such as the Ising and Potts models, are exactly soluble for these lattices. In the field of critical phenomena, the bridge from microscopic to macroscopic studies can be realized by spin-lattice models through statistical mechanics; therefore, an exactly soluble spin-lattice model becomes very significant. So far only a few exact solutions, such as Onsager's solution[1] of the two-dimensional (2D) Ising model and Baxter's solution[2] of the eight-vertex model, have been obtained. For the spin models associated with hierarchical lattices thus far certain exact results have been produced. Among these, the following are very important: the thermodynamic limit of the zeros of the partition function Z_N on, for instance, the diamond hierarchical lattice forms the Julia set[3] of the renormalization transformation of the Ising model, the free energy per bond has a well-defined thermodynamic limit[4] for a large class of discrete spin models under general conditions, the oscillatory critical amplitude that replaces a fixed constant of the free energy exists, and a finite temperature phase transition occurs.[3-5]

In this paper we will construct a family of the diamond-type hierarchical lattices; examples of these lattices will be found in Sec. II. We emphasize that not only the diamond-type hierarchical lattices but also regular (i.e., translational symmetry) lattices can form in a self-similar manner. Therefore whether the dimensionality takes integer or noninteger values is not of great importance.

The decimation renormalization-group transformation is especially appropriate to apply to the Ising model for fractal structure, and results in an exact partition function and free energy. We find that the partition function, the free energy, and the recursion relation are simply associated with the number of branches (see Sec. III). In particular, we give a general condition for convergence of the free energy in the thermodynamical limit. This condition is much more obvious and concrete than that of Griffiths and Kaufman.

The study of the properties of phase transition and universality is central to the field of critical phenomena. In this aspect for a regular lattice there have been many studies; however, not much work has been done on the fractal lattice.[6-10] The latter seems much more complicated. In the present work, by means of finding out the fixed point (see Sec. IV) and calculating the critical exponent of correlation length (see Sec. V), we find that general improved criteria for classification of universality are necessary when fractal and regular lattices are both considered. We also discuss the upper critical dimension.

II. EXAMPLES OF A FAMILY OF THE DIAMOND-TYPE HIERARCHICAL LATTICES

As we know, the hierarchical lattice is generated in an iterative manner, starting from a two-point lattice joined by a single bond. One then repeatedly uses an operation of replacing a bond by P branches of q bonds. Figure 1 shows a family of the diamond-type hierarchical lattices, in which $q=2$ and $P=1$, 2, 3, and 4.

Griffiths and Kaufman[4] gave two interpretations for the construction of the hierarchical lattices, which are called "aggregation" and "miniaturization." In essence, these two explanations reflect the self-similar property of the structure. It is worth mentioning that an infinite translational symmetry lattice, i.e., regular lattice, can also be constructed by the "aggregation" or "miniaturization" procedure in terms of a primitive unit; a translational symmetry lattice then is also self-similar. Examples of constructing the translational symmetry lattice are shown in Fig. 2.

For self-similar structure a well-defined fractal dimension $d_f=\ln N/\ln b$ is used to describe their geometrical feature; here N denotes the total mass (or the total

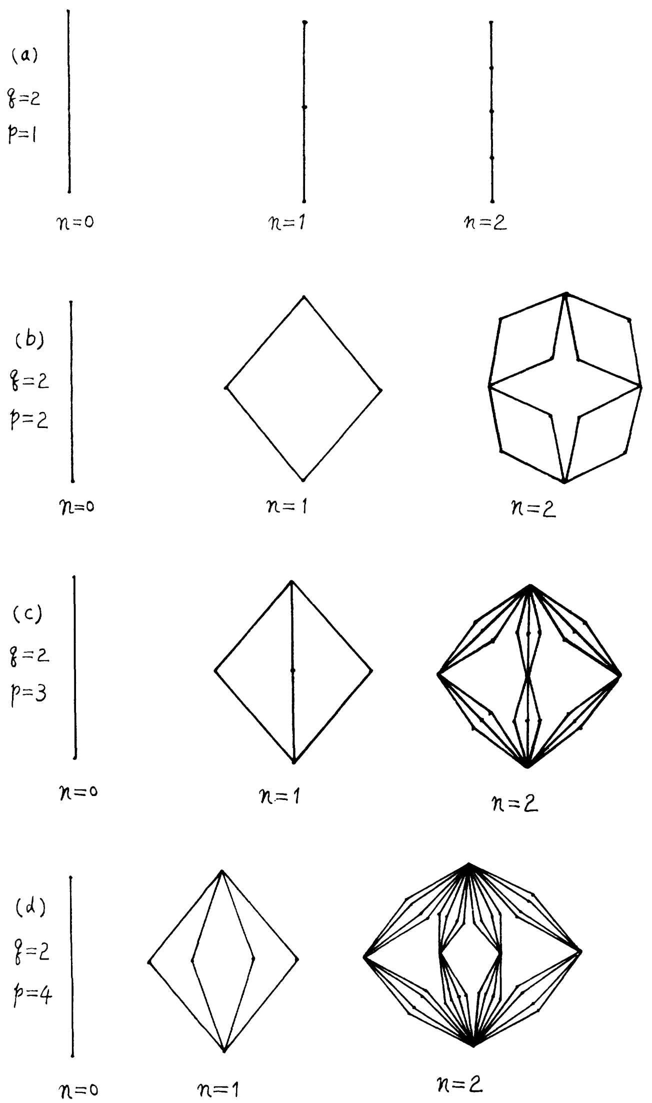

FIG. 1. A family of the diamond-type hierarchical lattices. n, q, and p denote the order of construction, the number of bonds per branch, and the number of branches, respectively.

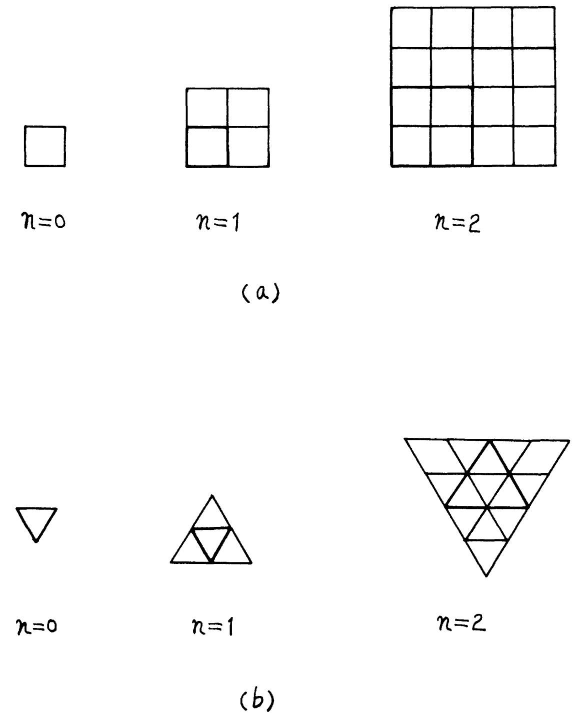

FIG. 2. Examples of constructing square and triangle lattices as a kind of fractal in the "aggregation" manner.

volume) and b the rescaling factor (dilation factor). In fact, this definition can act as the general one, which not only applies to fractal but also to the regular structure. Therefore, if one regards dimensionality as a continuous variation, Euclidean dimension only is a discrete set of integers on the dimensionality axis. In this sense, it is not significant to distinguish fractal dimension and Euclidean dimension. Moreover, Gefen *et al.*[8] and Lin and Yang[9] have already emphasized that the translationally invariant lattice is a particularly self-similar structure with lacunarity $L=0$.

III. ISING PARTITION FUNCTION AND FREE ENERGY

The Ising model is given by the following Hamiltonian:

$$-\beta\mathcal{H}=\sum_{\langle i,j\rangle} K s_i s_j \ , \qquad (1)$$

where s_i is an Ising spin associated with site i; the values take ± 1. We only consider the interaction between nearest-neighbor spins, and zero field is applied. The partition function is written as

$$Z^{(n)}(K_n)=\sum_{\{s\}=\pm 1}\exp(-\beta\mathcal{H}) \ , \qquad (2)$$

where the parameter β is inversely proportional to the temperature and proportional to the exchange integral. $\{s\}$ represents all site spins. For the family of the diamond-type hierarchical lattices, using a series of decimation operations will result in the following expressions:

$$Z^{(n)}(K_n)=A_1(K_0)\prod_{i=1}^{n}[A(K_i)]^{(Pq)^{i-1}} \ , \qquad (3)$$

where $A(K_i)$ is a constant associated with the ith decimation procedure and K_i is the renormalized interaction parameter. $A_1(K_0)$, $A(K_i)$, and K_i are, respectively, given as follows:

$$A_1(K_0)=4\cosh K_0 \ , \qquad (4)$$

$$A(K_i)=[2\sqrt{\cosh(2K_i)}]^P \ , \qquad (5)$$

$$K_{i-1}=P\ln\sqrt{\cosh(2K_i)} \ . \qquad (6)$$

The latter is the recursion relation of renormalization-group (RG) transformation. Bessis *et al.* have proved that the thermodynamic limit of the zeros of $Z^{(n)}(K_n)$ gives the Julia set[3] of transformation R.

The free energy per bond can be expressed as

$$f=\lim_{n\to\infty}\frac{1}{(Pq)^n}\ln Z^{(n)}(K_n)$$
$$=\lim_{n\to\infty}\frac{1}{(Pq)^n}\left[\sum_{i=1}^{n}(Pq)^{i-1}\ln A(K_i)+\ln A_1(K_0)\right]. \quad (7)$$

Griffiths and Kaufman have shown that the free energy has a well-defined thermodynamic limit[4] for a large class of discrete spin models on hierarchical lattices. Here we will find a concrete convergent condition. According to expression (5) and Eq. (6), employing the convergent condition of the series on the right-hand side of (7), we found

$$\ln\{4\cosh[P\ln\cosh(2K)]\}<(Pq)\ln[4\cosh(2K)]\ . \quad (8)$$

This condition is always satisfied in the region $0\le K\le\infty$. A sketch is plotted in Fig. 3. At the fixed point the free energy per bond is exactly given by

$$f=\frac{1}{Pq-1}\ln A(K^*)\ . \quad (9)$$

IV. FIXED POINT ASSOCIATED WITH THE NUMBER OF BRANCHES

We now study how the nontrivial fixed point varies with the number of branches P. From the recursion relation of RG transformation (6), we plot the dependence of the unstable fixed point on the number of branches P in Fig. 4. A novel feature is found: When $P=1$, then $d_f=1$; the unstable fixed point is at $K^*=\infty$ ($T=0$). The unstable fixed point moves toward $K=0$ as P increases. When P is sufficiently large, the fixed point is very close to $K=0$.

It will be interesting to compare the hierarchical lattice with $d_f=\infty$ with the Bethe lattice; the latter is also $d=\infty$. We know that on the Bethe lattice the Ising model exhibits a finite-temperature phase transition and mean-field behavior.[11] However, for the diamond-type hierarchical lattice with $d_f=\infty$ there only occurs a trivial phase transition with $T=\infty$. This distinction reflects the difference between the two constructions. In fact, for the Bethe lattice there is neither translational symmetry nor dilation symmetry. Therefore, the definition $d_f=\ln N/\ln b$ cannot apply to the Bethe lattice. As we know, the dimensionality for the Bethe lattice is given by $d=\lim_{n\to\infty}\ln c_n/\ln n$, where c_n is the number of sites within n steps of a given site.[11]

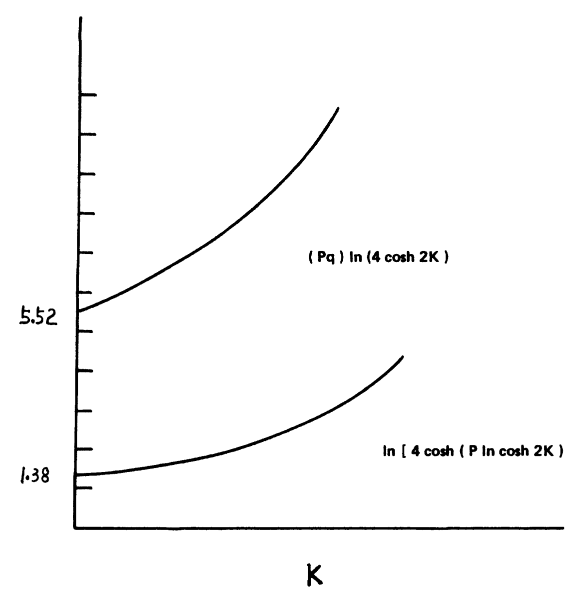

FIG. 3. A sketch of functions $\ln\{4\cosh[P\ln\cosh 2K]\}$ and $(Pq)\ln\{4\cosh 2K\}$ with $P=q=2$ shown in Eq. (8) vs K.

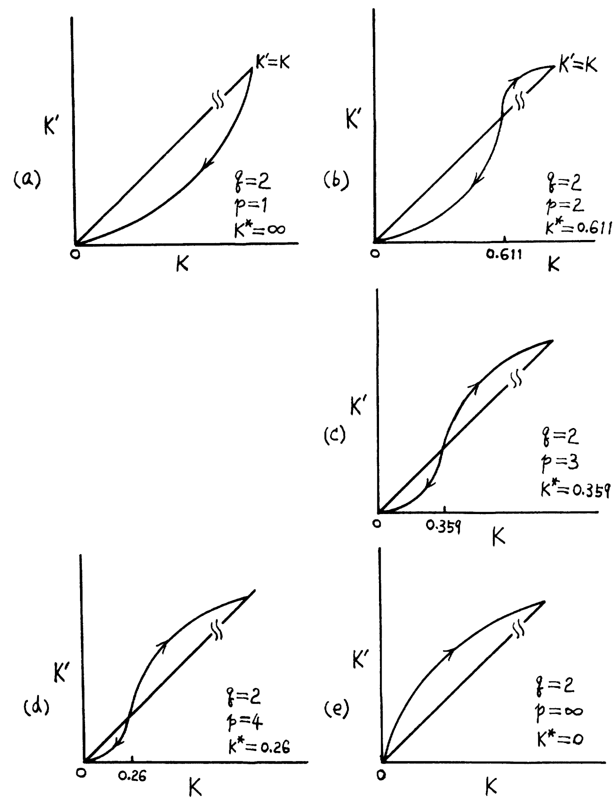

FIG. 4. The dependence of unstable fixed point K^* on the number of branches, P.

V. CRITICAL EXPONENT

We now turn to the critical exponent; it can help us to gain a deep understanding of the universality. Following the renormalization theory, linearize the RG transformation R around the fixed point K in the form

$$R^{(L)}(K)=K_c+\left.\frac{\partial R}{\partial K}\right|_{K_c}(K-K_c)\ . \quad (10)$$

TABLE I. The values of d_f and ν for the family of the diamond-type hierarchical lattices.

d_f:	1	2	2.585	3	3.323	4.323
ν:		1.336	1.126	1.0715	1.045	1.019

The correlation length critical exponent is defined as

$$\nu=\frac{\ln b}{\ln(\partial R/\partial K)_{K_c}}\ , \tag{11}$$

where $(\partial R/\partial K)_{K_c}$ is an eigenvalue of R, and b is a dilation factor. The values for different d_f are summed in Table I.

The following two points are worth noting: First of all, the value of the critical exponent shown in Table I is different from that of all the two- or three-dimensional regular lattices; this seems to imply that a more general universality hypothesis can be stated as follows: All critical problems may be divided into classes according to (a) the dimensionality of the system which can be on regular lattices or fractals; (b) the symmetry group of the order parameter; and (c) other geometric factors associated with the construction feature. Within each class, the critical properties are identical.

A possible example can be mentioned: For Sierpinski carpets an additional geometrical factor mentioned above is lacunarity.[8,9] However, for our hierarchical lattices such elements as lacunarity are to be found. When we are only interested in a regular lattice, criterion (c) will not be considered.

The other notable point is that the Ising critical behavior on the lattices with $d_f=4$ does not exhibit mean-field behavior, in contrast with that of the regular lattices. This immediately gives rise to questions: Does the critical fractal dimension exist? What is the value of the critical dimension if it exists?

In summary, in this paper we propose a family of the diamond-type hierarchical lattices and study the Ising critical behavior on such. We regard the regular lattice as a special fractal lattice. In a unified view considering both regular lattice and fractal lattice, we propose a general criteria of classification of universality, in which certain geometric factors describing the distinction between regular and fractal lattices have to be added. The existence of an upper critical fractal dimension is still an open problem.

ACKNOWLEDGMENT

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