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Absence of the Hopf invariant in the long-wavelength action of two-dimensional quantum antiferromagnets

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Contrary to recent theoretical suggestions, we show that the long-wavelength action of a two-dimensional quantum antiferromagnet on a square lattice can be mapped onto a classical (2+1) nonlinear σ model without any additional topological term.

Interest in two-dimensional quantum antiferromagnets has considerably revived, when it was recognized theoretically[1] and experimentally[2] that they might give a clue to the understanding of the normal phase of undoped lanthanum copper oxides. The long-wavelength action of the d-dimensional Heisenberg antiferromagnet is known[3] to be given in the large S limit by a $(d+1)$-dimensional nonlinear σ model with the following Lagrangian density

$$\mathcal{L}=\tfrac{1}{2}g^{-1}[c^{-1}(\partial_t\mathbf{n})^2-c(\nabla\mathbf{n})^2]\,, \qquad (1)$$

where $\mathbf{n}$ is a three-dimensional unit vector, representing the magnetization on one of the two sublattices for a bipartite lattice. In Eq. (1), $c=2^{(d+1)/2}JSa$ is the spin wave velocity and the coupling constant g is equal to $2^{(d+1)/2}a^{d-1}/S$.

This mapping has been extensively used for spin chains ($d=1$) and has revealed striking differences between the cases of integer or half-integer spin S[3]. This is linked mathematically to the appearance, for the quantum spin problem of a topological term in the action

$$\theta Q=(\theta/4\pi)\int\epsilon^{abc}n_a\partial_x n_b\partial_t n_c\,dx\,dt\,, \qquad (2)$$

with $\theta=2\pi S$. Q is the winding number on the unit sphere of any smooth configuration $\mathbf{n}(x,t)$, tending to a constant at infinity, and may take only integer values. The existence of topologically nontrivial configurations is implied by the homotopy group $\pi_2(S^2)=Z$. As a consequence of this θ term, the ground-state properties of the quantum spin chain are believed to depend in a crucial way on the value of the angle,[3] $\theta=0$ or $\theta=\pi$.

The extension of this mapping to two space dimensions seems a straightforward task but it was pointed out recently[4] that one may expect in this case the occurrence of a new topological term

$$\theta H=(-\theta/2\pi)\int A_\mu J^\mu d^2x\,dt\,. \qquad (3)$$

In this formula H is the so-called Hopf invariant, defined in terms of the topological current $J^\mu=(1/8\pi)\epsilon^{\mu\nu\lambda}\times\epsilon^{abc}n_a\partial_\nu n_b\partial_\lambda n_c$ [J^0 is associated to the conserved charge Q of a two-dimensional slice at fixed t analogous to Eq. (2)] and its vector potential A_μ ($J^\mu=\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$). At this stage θ is a phenomenological quantity which should be extracted from the underlying microscopic theory. H is the unique topological invariant in three dimensions, assuming integer values on smooth mappings $S^3\to S^2$ or Hopf textures. It was first introduced by Wilczek and Zee,[5] who discussed its physical meaning in a very illuminating way: using the linking-number interpretation of the Hopf invariant they showed that the extra phase in the action $i\theta H$ confers to skyrmions (i.e., configurations possessing at a given time a nonzero topological charge Q) a spin $\theta/2\pi$ as well as corresponding exotic statistics. In particular, for $\theta=\pi$, skyrmions become fermions and, according to scenarios proposed by Wiegmann and coworkers,[4] could be identified, after condensation, with the "spinons" of the resonating-valence-band (RVB) theory.[6]

Therefore, it seems important to see whether the action of a two-dimensional quantum antiferromagnet on a lattice contains in the continuum-limit an extra term as given in Eq. (3) and, if yes, to calculate the phenomenological angle θ. In this note, we investigate the question for the square lattice and find $\theta=0$. This means that topological excitations around the Néel classical ground state should be considered as bosons, irrespective of the value of S.

In order to get our result, we adopt a coherent state path-integral formalism for interacting quantum spins[7] and follow a derivation given by Haldane[8] for the $d=1$ chain. For a single spin parametrized by a unit vector $\hat{\Omega}$ ($\mathbf{S}=S\hat{\Omega}$), in a potential $V(\hat{\Omega})$, we write the action in such a path integral as

$$\int dt[S\mathbf{A}(\hat{\Omega})\cdot\partial_t\hat{\Omega}-V(\hat{\Omega})]\,, \qquad (4)$$

where $\mathbf{A}(\mathbf{r})$ is the singular vector potential of a magnetic monopole of flux 4π at the origin ($\nabla\times\mathbf{A}=\mathbf{r}/r^3$ or, on the unit sphere, $\epsilon^{abc}\partial A_b/\partial\Omega_c=\Omega_a$). An analytic expression of $\mathbf{A}$ in terms of spherical variables is given for instance by $\mathbf{A}(\mathbf{r})=(\cos\theta+1)/r\sin\theta\hat{\boldsymbol{\phi}}$. We note that the quantity $\int\mathbf{A}(\hat{\Omega})\cdot d\hat{\Omega}$ measures the area covered by the spin on the unit sphere in function of time. It is discontinuous, changing by 4π every time the spin crosses the singularity of $\mathbf{A}$ (within our particular choice of gauge, this happens for $\hat{\Omega}=\hat{\mathbf{z}}$). It can easily be checked that the action given in Eq. (4) leads to the right classical equations of motion for a single spin.

Taking up now the case of an antiferromagnet on a two-dimensional lattice with interaction

$$h=+J\sum_{\langle i,j\rangle}\mathbf{S}_i\cdot\mathbf{S}_j \qquad (5)$$

(where the summation is on pairs of nearest neighbors), we shall restrict our attention, in the same spirit as for the spin chain, to the low-energy modes around the classical Néel ground state. On a square lattice of spacing a, those are spin waves of wave vector $\mathbf{q}=(0,0)$ or $\mathbf{q}=(\pi/a,\pi/a)$. Therefore, we shall write the spin at site (p,q) as

$$\mathbf{S}_{pq}=(-1)^{p+q}S(1-a^2L^2/S^2)^{1/2}\mathbf{n}+a\mathbf{L}\,, \tag{6}$$

where $\mathbf{n}$ and $\mathbf{L}$ are two fields defined at each site and slowly varying with p and q. $\mathbf{n}$ is a unit vector, $\mathbf{L}$ a small staggered fluctuation component ($a|L|\ll S$) which we have assumed to be perpendicular to $\mathbf{n}$ as it will turn out to be true at the order considered in this paper. It will become obvious in the following that $\mathbf{L}$ is of the same order as a first-order derivative of $\mathbf{n}$: this has to be kept in mind in the forthcoming expansion of the action.

A brief inspection of the Hopf invariant as given in Eq. (3) shows that it is third order in derivative of $\mathbf{n}$ [this is perhaps more clearly seen in the CP^1 formulation of the O(3) nonlinear σ model, see Ref. 9]. Thus, in order to seek for its presence in the action of the two-dimensional antiferromagnet, we have to expand the Lagrangian at least to third order in powers of space-time derivatives or $\mathbf{L}$. We first consider the kinetic term. By exploiting the freedom that is left to us to choose either $\mathbf{A}(\hat{\Omega})$ or $\mathbf{A}(-\hat{\Omega})$ in Eq. (4), we get on each site

$$\mathbf{A}[(-1)^{p+q}\mathbf{S}_{pq}/S]\cdot\partial_t\mathbf{S}_{pq}=(-1)^{p+q}S\mathbf{A}(\mathbf{n})\cdot\partial_t\mathbf{n}+a\mathbf{L}\cdot(\mathbf{n}\times\partial_t\mathbf{n})+2a\partial_t[\mathbf{A}(\mathbf{n})\cdot\mathbf{L}]\,. \tag{7}$$

This result is obtained by injecting into the left-hand side of Eq. (7) the definition (6) and expanding around $\mathbf{n}$. Note that third-order corrections appear with the alternating factor $(-1)^{p+q}$ and give rise in the continuum limit to vanishing surface terms

$$\left[\sum_{pq}(-1)^{p+q}\rightarrow\tfrac{1}{4}\int dx\,dy\,\partial_{xy}^2\right].$$

The next corrections are fourth order and have not been displayed here for the sake of conciseness. In the following, we shall also discard the last term of Eq. (7) as a total time derivative.

Special care has to be exercised about the first term in Eq. (7). Remember that the quantity $\int dt\mathbf{A}(\mathbf{n})\cdot\partial_t\mathbf{n}$ measures the total area on the unit sphere swept by the spin at site (p,q). Consider now a given row of spins along the x direction (for a fixed value of q) and assume that a soliton is present in the corresponding (x,t) plane. This may be described for instance (in the continuum limit) by

$$\mathbf{n}(x,t)=(\hat{\rho}\sin f,\cos f)\,, \tag{8}$$

where $\hat{\rho}$ is the two-dimensional unit radial vector in the (x,t) plane and $f[\rho=(x^2+t^2)^{1/2}]$ is a smooth function varying monotonically from $f(0)=0$ to $f(\infty)=\pi$, so that we recover at infinity the ground state $\mathbf{n}=(0,0,-1)$. We want to calculate the alternating sum $\sum_p(-1)^p\times\int dt\mathbf{A}(\mathbf{n})\cdot\partial_t\mathbf{n}=l_q$. Starting from $x=-\infty$ where spins sweep a very small area around $\mathbf{n}=(0,0,-1)$, and moving along the x direction, we see that this area undergoes a sudden jump of 4π at $x=0$ [where $\mathbf{n}$ crosses the singular value (0,0,1)], before shrinking again to zero at $x=\infty$. Due to this discontinuity, we get for l_q a nonzero result in the continuum limit

$$l_q=\cong\tfrac{1}{2}\int dx\,\partial_x\left[\int dt\mathbf{A}(\mathbf{n})\cdot\partial_t\mathbf{n}\right]=2\pi\,. \tag{9}$$

Since the argument may be obviously extended to more general configurations, it identifies l_q with $2\pi Q(y)$, where $Q(y)$ is the winding number of the $\mathbf{n}$ configuration in the (x,t) plane for a given value of q or y.

What we have found is nothing but the topological term for a single spin chain. But in the two-dimensional case, we have neighboring chains with alternating signs and deduce in the continuum-limit

$$S\sum_{pq}(-1)^{p+q}\int dt\mathbf{A}(n)\cdot\partial_t\mathbf{n}=\pi S\int dy\partial_yQ(y)$$
$$=\pi S\int dx\partial_xQ(x)\,. \tag{10}$$

In the absence of any pointlike singularity (or hedgehog) violating the assumption of smoothness of the field $\mathbf{n}$, $Q(y)$ or $Q(x)$ are conserved charges and the above integral does not contribute to the action.[10]

Tuning now to the interaction term, we attribute to each site the quantity

$$\tfrac{1}{2}J\sum_{\delta=\pm1,\delta'=\pm1}\mathbf{S}_{pq}\cdot\mathbf{S}_{p+\delta,q+\delta'}\,,$$

use the definition (6), and perform a gradient expansion of the resulting expression. As before, third-order corrections are seen to present the alternating factor $(-1)^{p+q}$ and disappear, once the continuum-limit is taken. Subtracting the result from Eq. (7), we obtain the following expression of the Lagrangian density, up to fourth-order corrections

$$\mathcal{L}=a^{-1}\mathbf{L}\cdot(\mathbf{n}\times\partial_t\mathbf{n})-4JL^2-JS^2/2(\nabla\mathbf{n})^2\,. \tag{11}$$

Extremizing the action with respect to $\mathbf{L}$ yields $\mathbf{L}=(\frac{1}{8}Ja)(\mathbf{n}\times\partial_t\mathbf{n})$, which ensues the final form of $\mathcal{L}$

$$\mathcal{L}=(\tfrac{1}{16}Ja^2)(\partial_t\mathbf{n})^2-J\,S^2/2(\nabla\mathbf{n})^2\,, \tag{12}$$

which is the Lagrangian of the standard nonlinear σ model, given in (1).

In conclusion, we see that the differences existing in one dimension between half-integer and integer spins do not appear to survive in two dimension in the ordered phase. In particular, we have not found any Hopf invariant. It is worth noting that we have not considered in this paper the possibility of hedgehogs or antihedgehogs[11] which should play a role in (2+1) dimensions, as endpoints of skyrmions. It seems unlikely that taking them into account would invalidate the conclusions of this work. The question of whether differences between systems of different spin occur in a phase disordered by quantum fluctuations[1] remains open, however.

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[10]Actually, this last result is only true for an even number of rows, which is the physically correct situation to consider. For an odd number of rows and a nontrivial configuration of charge Q, extending throughout the y axis, we would get from the last row the contribution $i2\pi SQ$ in the action. We also note that an equally acceptable formula for the topological term could be $(2\pi S/a)\int dyQ(y)$, rather than the one given in the text [Eq. (10)]. This is because, at the discrete level, $2\pi S\sum_q(-1)^q Q(q) = 2\pi S\sum_q Q(q)$ (mod2π), for any integer value of Q and any half-integer or integer value of S.

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