Comments

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Comment on "Random-field Ising model as a dynamical system"

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It is analytically shown that the gap which produces the fractal structure of the attractor of the dynamical system vanishes linearly if the exchange reaches a critical value in contrast to the $\frac{5}{2}$ power law claimed to be observed numerically by Satija [Phys. Rev. B 35, 6877 (1987)]. Several other statements of this paper are critically discussed.

We show *analytically* that the gap which generates the fractal structure of the attractor vanishes *linearly* if the exchange reaches a critical value. The fixed point of Eq. (3) in Ref. 1, $x^* = h + \frac{1}{2} g(x^*)$, is given by

$$2x^* = h + \operatorname{arcsinh}[e^{2J}\sinh(h)] . \tag{1}$$

The first gap closes, $\Delta = 2(2h - x^*) = 0$, if the exchange is[2,3] $2J_c^0 = \ln[\sinh(3h)/\sinh(h)]$. Near the critical exchange we find

$$x^*(J \to J_c^0) = 2h + (J - J_c^0)\tanh(3h) . \tag{2}$$

Thus the linear dependence of the gap, $\Delta(J \to J_c^0) = -2(J - J_c^0)\tanh(3h)$, is a simple fact resulting from the analytic form of the recursion around the fixed point x^* in contrast to the $\frac{5}{2}$ power law claimed to be observed numerically in Ref. 1.

The frustrated region was defined in Ref. 3 by the condition that the probability density of the local magnetization $m = \langle s_n \rangle$ is nonzero at $m = 0$ and *not* by the condition that the gap closes as stated in Ref. 1. Thus $J_f = \frac{1}{2}\ln[2\cosh(h)] \neq J_c^0$. For $T = 0$ the correct threshold is found: For $h/J \leq h/J_f = 2$ the residual entropy is nonzero.

For nonzero temperature the attractor constitutes a multifractal which is more complex than a simple Cantor set due to the nonlinear character of the map, only for a linear approximation of $g(x)$ it is self-similar at every scale.[4,5] It is possible to encode all the points of this set, and its bands and gaps at the corresponding level of the hierarchy in an unique way by symbolic dynamics.[4–6] In the nth iteration of the Chapman-Kolmogorov equation [Eq. (5) in Ref. 1] the measure consists of 2^n bands labeled by sequences of n signs $\pm$ characterizing its history.[6] For $n \to \infty$ we have a *continuous* number of bands. Therefore, statements like "the number of bands jump by unity" make no sense without a reference to a relevant length scale [e.g., one sees in Fig. 3(a) of Ref. 1 four or eight bands or even more depending on the reference length].

For high temperatures it is justified to replace $g(x)$ by $2x(x^* - h)/x^*$ which gives a self-similar Cantor set, the first gap of which has the exact value.[4] The fractal dimension is $d_f^{\text{Cantor}} = \ln 2/\ln[x^*/(x^* - h)]$ if $\Delta \geq 0$ and one otherwise. Also the Lyapunov exponent can be calculated analytically[6]

$$\delta_{\text{Ly}}^{\text{Cantor}} = \langle \ln[g'(x)/2] \rangle = \ln[(x^* - h)/x^*] . \tag{3}$$

Near the critical exchange we find, inserting Eq. (2),

$$\delta_{\text{Ly}}^{\text{Cantor}}(J \to J_c^0) = -\ln 2 + (J - J_c^0)\tanh(3h)/(2h) . \tag{4}$$

The Lyapunov exponent reaches the value $-\ln 2$ if NN bands overlap in a *linear* way in contrast to Eq. (11) in Ref. 1 (note also the misprint in the definition of δ_{Ly}). For $\Delta \geq 0$ we can write $\delta_{\text{Ly}}^{\text{Cantor}} = -\ln 2/d_f^{\text{Cantor}}$ which elucidates that $-\delta_{\text{Ly}}^{\text{Cantor}}$ increases as the attractor decreases. To characterize fractal properties avoiding this approximation one needs more sophisticated methods.[7]

To discuss zero-temperature properties it is more appropriate to study the map which results dividing Eq. (3) of Ref. 1 by β and renaming $J \to \beta J$, $x \to \beta x$, and $h \to \beta h$.[4] The thus obtained map is for $T = 0$ piecewise linear,

$$x_n = h_n + A(x_{n-1})$$
$$A(x) = \begin{cases} \pm J & \text{if } x \gtrless \pm J, \\ x & \text{if } |x| < J. \end{cases} \tag{5}$$

and generates only a finite number of possible values. The corresponding fractal dimension is zero. The measure consists of weighted δ functions located at these possible values and is *not* smooth. To calculate the Lyapunov exponent we observe that $A'=1$ for $|x|<J$ and $A'=0$ otherwise, and that the measure reaches from $-h-J$ to $h+J$. Thus $\delta_{\rm Ly}(T=0)=-\infty$ and not zero as stated in Ref. 1. For $T=0$ the *finite* number of states jumps at some critical values of the parameters and causes, for a field with nonzero mean, a discontinuous behavior of magnetization and residual entropy.[6] For $T\neq0$ these discontinuities are expected to be smoothed.

In Ref. 1 it was claimed that the NNN bands overlap at $J=J_c^1$ determined by $x^*(J_c^1)=\frac{5}{2}h$. We show that this is only true for $T=0$. The overlap condition for NNN bands leads to $h+\frac{1}{2}g[h+\frac{1}{2}g(-x^*)]=-h+\frac{1}{2}g(x^*)$. Injecting the definition of the fixed point yields $2x^*-6h=g(2h-x^*)$, resulting in

$$2x^*=5h+\operatorname{arcsinh}[e^{-2J}\sinh(h)]\,. \qquad (6)$$

Obviously, $x^*=\frac{5}{2}h$ would mean for $T\neq0$, $J\to\infty$ or $h=0$ in contrast to the observed finite values (cf. Fig. 1 in Ref. 1). For $T\to0$, however, $e^{-2J}\sinh(h)\to0$ if $2J>h$ so that with $x^*(T=0)=h+J$ from (1) we find in this case the correct result $J_c^1=\frac{3}{2}h$.[5]

In a similar way it can be shown that Eq. (8) in Ref. 1 defines *for zero temperature* the critical values $J_c^n=\frac{1}{2}(n+2)h$, $n=0,1,2,\ldots$, for which the number of states jumps by *two*, at $J_f=\frac{1}{2}h$ this number jumps by *four* (cf. Fig. 2 in Ref. 5). For J_f as well as for the J_c^n the residual entropy exhibits spikes.

We agree with Satija that for $d_f=1$ where the measure constitutes a fat fractal, generalized scaling exponents[8,9] are of interest. The explicit representation of the measure in the nth iteration of the Chapman-Kolmogorov equation obtained in Ref. 6 could be useful to characterize the multifractal and to calculate the spectrum of singularities avoiding previous simplifications.[10]

[1] I. I. Satija, Phys. Rev. B **35**, 6877 (1987).
[2] G. Györgyi and P. Rujan, J. Phys. C **17**, 4207 (1984).
[3] R. Bruinsma and G. Aeppli, Phys. Rev. Lett. **50**, 1494 (1983).
[4] U. Behn and V. A. Zagrebnov, J. Stat. Phys. **47**, 939 (1987).
[5] U. Behn and V. A. Zagrebnov, Commun. J. Inst. Nucl. Res. Dubna, E17-87-138 (1987).
[6] U. Behn and V. A. Zagrebnov, J. Phys. A **21**, 2151 (1988).
[7] P. Szepfalusy and U. Behn, Z. Phys. B **65**, 337 (1987).
[8] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).
[9] R. Eykholt and D. K. Umberger, Phys. Rev. Lett. **57**, 2333 (1986).
[10] S. N. Evangelou, J. Phys. C **20**, L511 (1987).