Density matrices for states in the lowest Landau level of a two-dimensional electron gas

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(Received 23 March 1988)

We show that possible forms for the density matrices of states of a two-dimensional electron gas (2D EG) are profoundly limited when the electrons are constrained to lie in the lowest Landau level. In particular for any number of particles the matrices may be expressed in terms of their diagonal elements in a position representation. The simplification which is known to occur for Hartree-Fock calculations in a single Landau level of the 2D EG is a specific consequence of this general property.

Following the discovery of the fractional quantum Hall effect[1] (FQHE) interest in the strong-field two-dimensional-electron-gas system has increased. In the limit where the field is sufficiently strong that only the lowest Landau level is occupied, the electrons in the gas may be thought of as executing cyclotron orbits with radius $l=(\hbar c/eB)^{1/2}$ where B is the magnetic field strength. This restriction to the lowest Landau level greatly constrains the form of the many-electron wave functions in a way which, as discovered by Laughlin,[2,3] leads to a theoretical explanation for the FQHE. In this Brief Report we show that the N-particle density matrices for this system are also constrained by the restriction to the lowest Landau level. In particular, we show that they are uniquely determined by their diagonal elements in a coordinate representation. This fact has been used implicitly by Laughlin[3] in a study of the hierarchy of fractional states. We also rederive the Hartree-Fock equations for this system and demonstrate that it is this property for the one-particle density matrix which leads to the unusual situation that the Fock operator can be expressed explicitly and solely in terms of the electron density. We also comment on the off-diagonal density matrices with $N>1$.

We begin by defining our notation. At zero temperature the N-particle density matrix for a system of M electrons is given by

$$\rho_N(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N;\mathbf{r}'_1,\mathbf{r}'_2,\ldots,\mathbf{r}'_N)=M(M-1)\cdots(M-N+1)\int d^2\mathbf{r}_{N+1}\cdots\int d^2\mathbf{r}_M\psi_0^*(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N,\mathbf{r}_{N+1},\ldots,\mathbf{r}_M)\times\psi_0(\mathbf{r}'_1,\mathbf{r}'_2,\ldots,\mathbf{r}'_N,\mathbf{r}_{N+1},\ldots,\mathbf{r}_M)\,, \quad (1)$$

where $\psi_0[\mathbf{r}]$ is the M-electron ground-state wave function. The diagonal elements of $\rho_N[\mathbf{r};\mathbf{r}']$ give the n-particle distribution function which we denote by

$$n_N[\mathbf{r}]=\rho_N[\mathbf{r};\mathbf{r}]\,. \quad (2)$$

The restriction to the lowest Landau level is most conveniently expressed in the symmetric gauge where it implies that[2,4]

$$\psi_0[\mathbf{r}]=f_0[z]\exp\left[-\sum_k |z_k|^2/4\right], \quad (3)$$

where $z_k=x_k-iy_k$, we are using l as the unit of length and $f[z]$ is analytic in each of the z_k's. Using Eq. (3) in Eq. (1) we have

$$\rho_N[\mathbf{r},\mathbf{r}']=\prod_{k=1}^{N}\exp\left[\frac{-|z_k|^2}{4}\right]\exp\left[\frac{-|z'_k|^2}{4}\right]F_N[z^*,z'] \quad (4a)$$

and

$$n_N[\mathbf{r}]=\rho_N[\mathbf{r},\mathbf{r}]=\sum_{k=1}^{N}\exp\left[\frac{-|z_k|^2}{2}\right]F_N[z^*,z]\,. \quad (4b)$$

In Eqs. (4) $F_N[s,t]$ is a function analytic in each s_k and t_k and is uniquely determined by Eq. (4b) if only the diagonal elements of $\rho_N[\mathbf{r},\mathbf{r}]$ are known. By substituting $(z_k+z_k^*)/2$ for x_k and $(z_k^*-z_k)/2i$ for y_k we can write $n_N[\mathbf{r}]$ in the form

$$n_N[\mathbf{r}]=G_N[z^*,z]\,, \quad (5)$$

where $G_N[z^*,z]$ is also an analytic function of each z_k and z_k^*. It follows that

$$F_N[z^*,z]=\prod_{k=1}^{N}\exp\left[\frac{z_k^*z_k}{2}\right]G_N[z^*,z]\,, \quad (6a)$$

and hence that

$$\rho_N[\mathbf{r},\mathbf{r}']=\prod_{k=1}^{N}\exp\left[\frac{-|z_k|^2}{4}\frac{-|z'_k|^2}{4}+\frac{z_k^*z'_k}{2}\right]\times G_N[z^*,z']\,. \quad (6b)$$

Equations (6) are our basic result. To illustrate their power we consider the case of a state of uniform electron density. For this case, $n_1[\mathbf{r}]=N/A=\nu/2\pi$ where A is the system area and ν is the Landau-level filling factor. It follows from Eq. (6b) that

$$\rho_1(\mathbf{r},\mathbf{r}')=(\nu/2\pi)e^{-|z^2|/4}e^{-|z'|^2/4}e^{z^*z'/2}$$
$$=(\nu/2\pi)e^{-|z-z'|^2/4}e^{(z^*z'-zz'^*)/4} \quad (7)$$

for any such state. We note that the second factor in the last form of Eq. (7), which is a phase factor of unit magnitude, may be removed by modifying the definition of the density matrix to make it gauge invariant. From Eq. (7) we see that no uniform-density state can have off-diagonal long-range order in the one-particle density matrix such as exists in superfluids.

The nonlocal effective potential representing Coulomb interactions in the Hartree-Fock approximation may be expressed as

$$V_{\rm HF}(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r},\mathbf{r}')\int d\mathbf{r}''\frac{e^2}{|\mathbf{r}-\mathbf{r}''|}\rho_1(\mathbf{r}'',\mathbf{r}'')$$
$$-\frac{e^2}{|\mathbf{r}-\mathbf{r}'|}\rho_1(\mathbf{r}',\mathbf{r}) . \quad (8)$$

We can in general write

$$n_1(\mathbf{r})=\frac{1}{A}\sum_{\mathbf{q}}n(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}}=\frac{1}{A}\sum_{\mathbf{q}}n(\mathbf{q})e^{i(q^*z+qz^*)/2} , \quad (9)$$

where A is the system area and $q=q_x-iq_y$ is the wave vector in complex notation. Using Eq. (6) it follows that

$$\rho_1(\mathbf{r}',\mathbf{r})=\frac{e^{-|z|^2/4}e^{-|z'|^2/4}e^{z'^*z/2}}{A}$$
$$\times\sum_{\mathbf{q}}n(\mathbf{q})e^{i(q^*z+qz'^*)/2}$$
$$=\frac{2\pi}{A}\sum_{\mathbf{q}}n(\mathbf{q})\sum_{m',m}\theta^*_{m'}(\mathbf{r}')G_{m',m}(-q)\theta_m(\mathbf{r}), \quad (10)$$

where $\{\theta_m(\mathbf{r})\}$ are the symmetric-gauge angular-momentum eigenstates in the lowest Landau level,

$$\theta_m(\mathbf{r})=\frac{z^m e^{-|z|^2/4}}{(2\pi 2^m m!)^{1/2}} , \quad (11)$$

$$G_{m,m'}(q)=e^{|q|^2/2}\langle\theta_{m'}|e^{-iq\cdot\mathbf{r}}|\theta_m\rangle$$
$$=\left[\frac{m!}{m'!}\right]^{1/2}\left[\frac{-iq}{\sqrt{2}}\right]^{m'-m}L_m^{m'-m}(|q|^2/2) , \quad (12)$$

and $L_m^{\alpha}(x)$ is a generalized Laguerre polynomial. The second form for the right-hand side of Eq. (10) follows from the first by expanding each of the exponentials and comparing with Eqs. (11) and (12).

Using Eq. (10) in Eq. (8) and Fourier expanding $e^2/|\mathbf{r}-\mathbf{r}'|$ gives

$$\langle\theta_{s'}|V_{\rm HF}|\theta_s\rangle=\frac{2\pi}{A^2}\sum_{\mathbf{p}}n(\mathbf{p})\sum_{\mathbf{q}}\frac{2\pi e^2}{|q|}\sum_{m',m}G_{m',m}(-p)(\langle\theta_{s'}|e^{+i\mathbf{q}\cdot\mathbf{r}}|\theta_s\rangle\langle\theta_{m'}|e^{-i\mathbf{q}\cdot\mathbf{r}}|\theta_m\rangle$$
$$-\langle\theta_{s'}|e^{-i\mathbf{q}\cdot\mathbf{r}}|\theta_m\rangle\langle\theta_{m'}|e^{+i\mathbf{q}\cdot\mathbf{r}}|\theta_s\rangle) . \quad (13)$$

This can be simplified using Eq. (12) and the sum rule,

$$\sum_l G_{m',l}(k_1)G_{l,m}(k_2)=e^{-k_1^*k_2/2}G_{m',m}(k_1+k_2) , \quad (14)$$

to obtain

$$\langle\theta_{s'}|V_{\rm HF}|\theta_s\rangle=\int\frac{d\mathbf{q}}{(2\pi)^2}\langle\theta_{s'}|e^{i\mathbf{q}\cdot\mathbf{r}}|\theta_s\rangle n(\mathbf{q})V_{\rm eff}(\mathbf{q}) \quad (15a)$$

with

$$V_{\rm eff}(q)=\frac{2\pi e^2}{|\mathbf{q}|}-2\pi e^2 l(\pi/2)^{1/2}e^{|q|^2l^2/4}I_0(|q|^2l^2/4) . \quad (15b)$$

[Note that the magnetic length has been exhibited explicitly in Eq. (15b).] Thus, because of the restriction to the lowest Landau level which leads to Eq. (1), the Fock term depends only on the electron density and leads to an effective electron-electron interaction which is attractive at short distances (large q). This result [Eqs. (15)] has been obtained previously[5-7] in terms of Landau-gauge eigenstates but the present derivation makes it clear that it is a consequence of the special properties of the one-particle density matrix in the lowest Landau level. Equations (15) gives an explicit and *exact* expression for the exchange potential which may be contrasted with approximate expressions based on the local-density approximation.[8]

Finally we discuss approximations for the N-particle off-diagonal density matrices of uniform electron density states, like the incompressible fluid states[2,9-11] responsible for the fractional quantum Hall effect. For $N=1$ a rigorously exact expression has already been given [Eq. (7)]. For $N=2$ Girvin[12] has shown that the two-particle distribution function can always be expressed in the form

$$n_2(\mathbf{r}_1,\mathbf{r}_2)=\left[\frac{\nu}{2\pi}\right]\left[1-e^{-|\mathbf{r}_1-\mathbf{r}_2|^2/2}\right.$$
$$\left.-2\sum_l{}'\frac{C_l}{4^l l!}|\mathbf{r}_1-\mathbf{r}_2|^{2l}e^{-|\mathbf{r}_1-\mathbf{r}_2|^2/4}\right] , \quad (16)$$

where the sum is over odd values of l only. $\{C_l\}$ values for Laughlin's incompressible fluid states at $\nu=\frac{1}{3}$ and $\frac{1}{5}$ are given in Ref. 13. Using Eqs. (5) and (6b) it follows from Eq. (16) that

$$\rho_2(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_1',\mathbf{r}_2')=\prod_{k=1}^{2} e^{-|z_k|^2/4} e^{-|z_k'|^2/4} e^{z_k^* z_k'/2} \left[\frac{\nu}{2\pi}\right]^2 \times \left[1-e^{-(z_1-z_2)^*(z_1'-z_2')/2}-2\sum_l{}' \frac{C_l}{4^l l!}(z_1-z_2)^*(z_1'-z_2')^l e^{-(z_1-z_2)^*(z_1'-z_2')/4}\right] . \quad (17)$$

For larger N there are circumstances where the superposition approximation may be expected to be reliable. For the N-particle distribution function this implies that

$$n_N[\mathbf{r}]=\left[\frac{\nu}{2\pi}\right]^N \prod_{i<j}^{N} g(|\mathbf{r}_i-\mathbf{r}_j|^2) , \quad (18)$$

where $g(|\mathbf{r}_i-\mathbf{r}_j|^2)=(2\pi/\nu)^2 n_2(\mathbf{r}_i,\mathbf{r}_j)$ is the pair correlation function which depends only on the square of the difference coordinates [see Eq. (16)]. Using Eqs. (5) and (6) again gives as the corresponding approximation for the density matrix

$$\rho_N[\mathbf{r},\mathbf{r}'] \simeq \left[\frac{\nu}{2\pi}\right]^N \prod_{k=1}^{N} e^{-(z_k^* z_k + z_k'^* z_k' + 2z_k^* z_k')/4} \times \prod_{\substack{i=1 \\ i<j}}^{N} g((z_i-z_j)^*(z_i'-z_j')) . \quad (19)$$

In closing we emphasize that the special properties of the one-particle density matrix discussed here are unique to the strong magnetic field limit where Landau-level mixing may be ignored. We find an *exact* expression for the exchange potential in the strong-field limit which is a nonlocal functional of the electron density. Our results will be useful for testing the accuracy of local-density approximations and also for suggesting nonlocal corrections.

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