Field theory for growth kinetics

Gene F. Mazenko
Department of Physics and James Franck Institute, University of Chicago, Chicago, Illinois 60637

Oriol T. Valls
School of Physics, University of Minnesota, Minneapolis, Minnesota 55455

Marco Zannetti*
James Franck Institute, University of Chicago, Chicago, Illinois 60637
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We present a new field-theory method for growth-kinetics problems which describes the entire time evolution of the system from the early stage after the quench until final equilibrium is attained. The method is developed for a scalar order parameter (conserved or nonconserved) with dynamics of the Langevin type and a systematic low-temperature perturbation scheme is constructed. The main results obtained in lowest order are as follows: (i) a reduced singlet probability distribution which evolves from a Gaussian at early times to a bimodal distribution at late times; (ii) the dynamical separation of two characteristic lengths $L(t)$ and $\xi(t)$ associated, respectively, with the domain size and with the correlation length of fluctuations within a domain; (iii) scaling behavior for the structure factor at long times and a growth law $L(t)\sim t^n$ with $n=\frac{1}{4}$ for conserved order parameter and $n=\frac{1}{2}$ for nonconserved order parameter; and (iv) the realization of the exact equilibrium state, free of spurious Nambu-Goldstone modes, as $t\to\infty$. First-order corrections to the structure factor are computed and it is found that they lead to no change in the growth law and to the appropriate first-order temperature corrections in the final equilibrium quantities. Finally the implications of these results for future work are briefly discussed.

I. INTRODUCTION

Substantial progress has been made on growth-kinetics problems[1] over the past ten years. Most of this progress has been driven by the results of Monte Carlo simulations[1,2] of kinetic Ising models. This development was a reversal of the earlier evolution of this field which was dominated by studies[3] of field theoretical models of the Langevin type. In this paper we pick up and extend this earlier development with a low-temperature theoretical study of the growth of order in the time-dependent Ginzburg-Landau (TDGL) model for a scalar field. In particular we consider the TDGL model with Gaussian noise in the case where the system is quenched from an initial symmetric disordered equilibrium state into a nonequilibrium state driven by parameters corresponding to some new final ordered equilibrium state with broken symmetry. We are interested here in treating the time evolution which characterizes the ordering of the system—the growth of the domains of the new ordered phases.

We have gained a significant understanding of the phenomenology of growth kinetics from simulations of both kinetic Ising models[1] and Langevin equations.[4] The main measure of the domain growth is the order parameter structure factor $C(\mathbf{q},t)$ where $\mathbf{q}$ is a wave vector and t is the time after the quench. Ordering is reflected in the growth in time of a Bragg peak in the structure factor. One of the major discoveries[2,5] in this field was that the peak contributions to the structure factor scale with a single characteristic length $L(t)$ which is a measure of a typical domain size in the system. In particular, for $\mathbf{q}$ near the ordering wave vector,

$$C(\mathbf{q},t)=L(t)^d F(qL(t)) , \tag{1.1}$$

where d is the spatial dimensionality of the system. Considerable effort has been spent trying to determine the growth law, how L depends on t for various systems and the form of the scaling function F in (1.1). This work has been based on either direct numerical simulation,[6] renormalization-group calculations,[7–9] or on simple model calculations[10] with an assumed morphological structure for the evolving domains at late times. For the case of kinetic Ising models, evidence is mounting[9] that there are several types of growth-kinetics classes depending on the defect structure of the evolving domains and whether there are quenched impurities or not. The associated growth laws are either power laws or logarithmic in time (depending again on whether there are quenched impurities or not). Much less work has been carried out for the Langevin models, primarily because they are more costly to simulate.

The earliest theoretical work on these growth-kinetics problems was that of Cahn and coworkers[11] and Lifshitz and co-workers.[12] The problem of spinodal decomposition was essentially defined in the the early work of Cahn and Hilliard[11] in terms of a linearized treatment of the TDGL model for a conserved order parameter without

thermal noise. This work led to the prediction of exponential growth of a particular wave-number component of the structure factor. While this theory reflected the underlying unstable nature of the system, it did not include any appropriate stabilizing mechanism for later times and could only be viewed as a very early time approximation. Cook[13] later included the effects of thermal noise in the problem, but this did not change the basic structure of the approximation. A further substantial advance in the theory was due to Langer[14] and coworkers who took into account nonlinear feedback terms which had the effect of stabilizing the later stage growth. The resulting theory was a considerable improvement on that of Cahn-Hilliard and Cook[11,13] since it led to a peaked form for the structure factor which moved to lower values of the wave number as time evolved and the peak grew with a rate much slower than exponential. Both effects were in qualitative agreement with experimental observation[15] of spinodally decomposing fluid systems using light-scattering techniques.

While the work of Langer, Bar-on, and Miller[14] (LBM) represented a significant advance for the field, there were some important drawbacks to their theory. The approximation developed was somewhat *ad hoc* and difficult to treat systematically. More specifically, Binder and coworkers[3,16] pointed out conceptually important problems associated with the long-time behavior of the theory. We restate these problems here in a somewhat different language. In the original high-temperature equilibrium state there is a single correlation length ξ_I which characterizes the structure factor of the system. As the system evolves toward its final broken symmetry state there are two independent lengths which characterize the system: the typical domain size $L(t)$ and the final equilibrium correlation length ξ. The LBM theory, as described below in Sec. II, can accomodate only one length at any given time, and therefore can not describe both $L(t)$ and ξ. The result is that the theory does not lead to the appropriate final equilibrium state. While this type of theory has been extended to more complicated physical systems,[17] they all share this same defect. This problem of developing a theory capable of treating two characteristic lengths (or masses in field theoretic language) is the fundamental unsolved problem that we address in this paper.

Another line of theoretical development has been to avoid the problem of early time evolution and jump to the later stages where one imposes a certain morphological structure on the system. The work of Cahn and Allen[11] and Lifshitz and Slyozov[12] was along these lines and led to very useful predictions for the growth laws of a variety of systems. These theories focused on the evolution of a single droplet or domain and did not worry about the distribution of these domains (Lifshitz and Slyozov, for example, restricted their analysis to the case of a very dilute solution of droplets). Kawasaki and coworkers[10] have extended these techniques to the case of the late-stage evolution of interacting defects. It is not clear in this method how one takes into account temperature effects, equilibration, and the evolution toward this late stage. In particular it is difficult to determine self-consistently the distribution of defects. In the work of Ohta, Jasnow, and Kawasaki,[10] for example, it was necessary to postulate that the distribution of interfaces satisfied a Gaussian distribution.

In recent years there has been rather little effort to construct theories which follow the evolution of systems from early through late times. One of the reasons is that much of the focus has been on kinetic Ising models and it is quite difficult to develop analytical methods for treating such systems. Glauber's exact solution[18] for the one-dimensional kinetic Ising model was used[19] to analyze spinodal decomposition, but no exact results exist for kinetic Ising systems which show true ordering. Approximate renormalization-group treatments[20] for such systems were developed, but these analytical treatments required knowledge of the time rescaling parameter which had to be determined using auxiliary methods (typically Monte Carlo results).

During the time when attention has been primarily focused on the *universal* properties associated with the late-stage growth of ordering, it has seemed less pressing to develop a first-principles theory governing the entire evolution of a system. However, with our increasing understanding of the pinning role of defects and their dependence on local mechanisms and the parallel development of interest in glassy behavior, the importance of developing a more general theory has become clearer. We choose in this paper to study Langevin equations rather than kinetic Ising models for several reasons. The first is that it seems more feasible to develop an analytic treatment for these models since they have an extra parameter, the quartic coupling, which may be used to advantage. Second, these models appear to be more generally applicable. They can, for example, be generalized to the treatment of fluids[17,21] and they also include Ising variables as a special limit. Third, a continuous order parameter can be treated using field-theory techniques. Finally, we were encouraged by the work[22,23] treating the N-vector model generalization of the model studied here. In Ref. 22 this model was solved in the large-N limit and the time evolution of the system was analyzed in detail over the entire time regime from the time of quench to the late-time scaling regime. This system is qualitatively different from the scalar order parameter case we study here since there is a broken continuous symmetry in the final state of the N-vector model for $N > 1$. Thus one generates Nambu-Goldstone (NG) modes[24] in the final state as the system evolves. A relevant observation made in Ref. 22 is that the solution looks very similar to that found by LBM. This comes about in the large-N limit because the transverse modes dominate over the single longitudinal mode and the only length in the problem during the later stages of growth is the characteristic domain size $L(t)$. The problem for N finite, and in particular for $N = 1$, is more complicated since any simple theory of the LBM type will incorrectly yield a solution for the equilibrated part of the structure factor with a massless (q^{-2}) NG mode.

The method presented in this paper is designed to deal with the problem of separating the two length scales $L(t)$ and ξ as the system evolves. In the very long time limit

the method gives the exact equilibrium result without spurious NG modes. As described in Sec. II, the method makes systematic use of field-theory techniques for classical fields as well as new techniques required for treating the ordering component of the order parameter field. The main physical motivation behind the formalism is the recognition that the order parameter field $\psi(\mathbf{R},t)$ can be decomposed into the sum of two fields. Early in the evolution of the system these fields are strongly coupled. However, as time passes, they assume separate roles and, for sufficiently long times, they become essentially independent. One of the two fields is then associated with the domain growth and the peak in the structure factor and the other is associated with the fluctuations of the order parameter within an ordered domain. The characteristic size associated with the "peak" field is $L(t)$, while that for the "fluctuation" field is ξ. At long times $L(t) \gg \xi$ and the two fields decouple. It seems reasonable that the fluctuating field can be treated at low temperatures as a field governed by Gaussian statistics. The peak field can be visualized at long times in coordinate space as being uniform almost everywhere except near domain boundaries where the system rapidly changes its orientation from one type of domain to another. It is therefore appealing to think of these variables as being somewhat Ising-like in character.

It should be emphasized that this separation of variables governing growth and fluctuations is more generally useful than for the particular application discussed here. The peak variable is the vehicle which passes information from the microscopic to the macroscopic scale. Therefore it should be a useful device for studying a variety of growth problems.

The main difficulty in treating this problem is that it is not at all obvious how to systematically implement these physical ideas. Specifically, the construction of an Ising constraint for the peak component in the framework of a field theory is a very complex problem. In this work we develop a solution in the context of a classical field theory which uses the methodology originally introduced by Martin, Siggia, and Rose (MSR).[25] We set up in Secs. II and III a systematic perturbation theory scheme valid for quenches to low temperatures. Detailed results for the lowest-order approximation are given in Sec. IV. Among these are (i) a reduced singlet probability distribution for the field $\psi(\mathbf{R},t)$ which naturally evolves (see Fig. 1) from a Gaussian at early times to the expected bimodal distribution at late times; (ii) an accurate treatment of the very early time behavior; (iii) a structure factor which exhibits scaling behavior of the form (1.1) at long times and with a growth law given by a power law, $L(t) \sim t^n$, where $n = \frac{1}{2}$ when the order parameter is not conserved, in agreement with the well-known Lifshitz-Cahn-Allen[11,12] law, and $n = \frac{1}{4}$ when the order parameter is conserved; (iv) we obtain the exact lowest-order equilibrium state, free of spurious NG modes, for $t \to \infty$.

The result $n = \frac{1}{4}$ for a conserved order parameter differs from the Lifshitz-Slyozov result ($n = \frac{1}{3}$). There are conflicting reports in the literature[26] as to whether simulation results are in better agreement with $\frac{1}{3}$ or with $\frac{1}{4}$. This is a very difficult question to resolve from purely numerical methods, since the small difference between 0.25 and 0.33 is easily buried in numerical uncertainty and the high cost of the numerical calculations makes it prohibitively expensive to reach the indisputably asymptotic time region. The method presented here, being purely theoretical, is obviously quite free of these limitations. As we shall see below, the perturbation theory is systematic and well defined, and we can show formally that higher-order terms in perturbation theory decay to zero at long times. Consequently we expect that our low-order results should not be affected by higher-order terms for low enough temperatures and long enough times. It is nevertheless impossible to conclusively rule out *a priori* the possibility that this is one of the problems for which perturbation theory fails and the growth law could then differ from the $n = \frac{1}{4}$ found here.

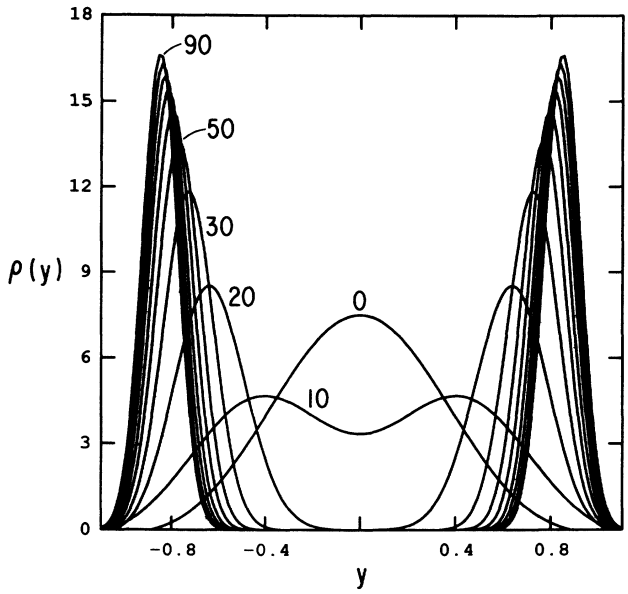


FIG. 1. Reduced singlet distribution function with COP in $d=2$ for a quench from $\epsilon_I = 0$ to $\epsilon = 0.1$. Time steps of 10 units. The quantity plotted is the right-hand side of (3.38) times 2π.

Summarizing, we present here for the first time a global solution of a nonequilibrium problem with two competing length scales which reproduces all the essential features of the physical process. Prominent among these is the dynamical separation of the two characteristic lengths and the subsequent generation of the correct final equilibrium state. In other words, we have constructed the framework upon which a systematic perturbation theory is based. The method presented here is therefore expected to prove useful in different areas of physics where one is confronted with simultaneous phenomena taking place over different length scales.

II. MODEL AND DEVELOPMENT OF PERTURBATION THEORY

A. Definition of model

The model we study is a standard TDGL model for scalar field. The dynamics are driven by the Langevin equation

$$\frac{\partial\psi(\mathbf{R},t)}{\partial t}=-\Gamma(\mathbf{R})\frac{\delta F}{\delta\psi(\mathbf{R},t)}+\eta(\mathbf{R},t)\ , \tag{2.1}$$

where the field $\psi(\mathbf{R},t)$ is defined either on the continuum or on a lattice[27] characterized by a set of lattice vectors $\mathbf{R}$. In (2.1), $\Gamma(\mathbf{R})$ is a constant kinetic coefficient, Γ, in the case of a nonconserved order parameter (NCOP), while for a conserved order parameter (COP) we have

$$\Gamma(\mathbf{R})=-D\nabla_R^2\ , \tag{2.2}$$

where D is a transport coefficient and ∇_R^2 is the Laplacian or its lattice version. The noise $\eta(\mathbf{R},t)$ appearing in (2.1) is Gaussian and white and satisfies

$$\langle\eta(\mathbf{R},t)\eta(\mathbf{R}',t')\rangle=2T\Gamma(\mathbf{R})\delta(\mathbf{R}-\mathbf{R}')\delta(t-t')\ , \tag{2.3}$$

where T is the temperature[28] of the thermal bath in contact with the system. We assume that the effective free energy is of the Ginzburg-Landau-Wilson form

$$F=\tfrac{1}{2}\int d^dR\left[c[\nabla\psi(\mathbf{R})]^2+r\psi^2(\mathbf{R})+\frac{u}{2}\psi^4(\mathbf{R})\right]\ . \tag{2.4}$$

B. Static equilibrium behavior

The static equilibrium properties of this model are governed by the Boltzmann probability distribution proportional to $e^{-F[\psi]/T}$. The two independent variable parameters in the equilibrium theory are chosen to be the coupling r and the temperature T. The coefficient c and the nonlinear coupling u are kept fixed and positive. By varying r from positive to negative values the local potential $V(\psi)=(r/2)\psi^2+(u/4)\psi4$ changes from a single- to a double-well form. For $r>0$ one has only a single disordered phase for all temperatures greater than zero. However, for $r<0$, there will be some transition temperature $T_c=T_c(r,u,c)$, below which ($T<T_c$) the system develops a spontaneous magnetization $\langle\psi\rangle=M(T,r,u,c)\neq0$. In our work here we will be concerned with quenches to final states where $T\ll T_c$ and a low-temperature theory can be developed since one expects fluctuations about the ordered state to be small.

The basic nature of the low-temperature theory can be seen by shifting and rescaling the field appearing in the free energy.

$$\psi(\mathbf{R})=M+\sqrt{T}\,\phi(\mathbf{R})\ , \tag{2.5}$$

where $\langle\psi(\mathbf{R})\rangle=M$ and expanding in powers of $\sqrt{T}$. Since this expansion is well known,[29] we give here only the results. For the structure factor $C(\mathbf{q})$, the Fourier transform of $\langle\psi(\mathbf{R})\psi(\mathbf{R}')\rangle$, one finds

$$C(\mathbf{q})=M^2(2\pi)^d\delta(\mathbf{q})+C_E(\mathbf{q})\ , \tag{2.6}$$

where the spontaneous magnetization is given by

$$M=M_0\left[1-\frac{3}{2M_0^2}T\langle\phi^2(\mathbf{R})\rangle+O(T^2)\right] \tag{2.7}$$

with

$$M_0^2=\frac{|r|}{u}\ , \tag{2.8}$$

$C_E(\mathbf{q})$ is the fluctuating part of the correlation function given by

$$C_E(\mathbf{q})=\frac{T}{cQ^2(\mathbf{q})+r-\Sigma(\mathbf{q})} \tag{2.9}$$

where $Q^2(\mathbf{q})$ (q^2 in the continuum) is the Fourier transform of the Laplacinan and, to $O(T)$,

$$\Sigma(\mathbf{q})=-3uM^2-3uT\langle\phi^2\rangle+\frac{18u^2M^2}{T}\Pi(\mathbf{q}) \tag{2.10}$$

with

$$\Pi(\mathbf{q})=\int\frac{d^dk}{(2\pi)^d}C_E(\mathbf{q}-\mathbf{k})C_E(\mathbf{k})\ . \tag{2.11}$$

C. Functional integral formulation for quench problems

We are interested in studying the nonequilibrium properties of our dynamical model. Specifically, we want to consider the case where the system is initially in equilibrium at some high temperature T_I and at time $t=t_0$ is quenched to some low temperature T. The parameters r and u may also suddenly change at $t=t_0$ from initial values R_I and u_I to final values r,u. It is now well understood from studies of dynamic correlations in equilibrium, that it is advantageous in developing a systematic perturbation theory, to recast the problem in functional integral form.[30] As first pointed out in Ref. 25, this is most conveniently carried out by introducing a response field $\hat{\psi}(\mathbf{R},t)$ conjugate to $\psi(\mathbf{R},t)$. The transformation from averages over the noise to averages over ψ and $\hat{\psi}$ is well described in Refs. 30 and 31. We follow here the conventions developed in Ref. 32. The additional ingredient in our development here is the quench at $t=t_0$ and the imposition of an initial condition at that time. This new feature is taken into account[33] by rewriting the Langevin equation in the form

$$\frac{\partial\psi(\mathbf{R},t)}{\partial t}=-\Gamma(\mathbf{R})\frac{\delta F}{\delta\phi(\mathbf{R},t)}+\eta(\mathbf{R},t)+\delta(t-t_0)\psi_0(\mathbf{R})\ , \tag{2.12}$$

where $\psi(\mathbf{R},t)$ and $\eta(\mathbf{R},t)$ are zero for $t<t_0$ and the initial value $\psi(\mathbf{R},t)$ is imposed as a constraint. In carrying out averages $\psi_0(\mathbf{R})$ is treated as an independent field with its own probability distribution $\sim e^{-F_I\psi_0]/T_I}$. In this paper we take F_I to be a symmetric quadratic form:

$$F_I[\psi_0]T_I=\tfrac{1}{2}\int d^dR_1d^dR_2\psi_0(\mathbf{R}_1)g_0^{-1}\times(\mathbf{R}_1-R_2)\psi_0(\mathbf{R}_2)\ . \tag{2.13}$$

The generating functional of correlation functions is given by

$$Z[U]=\int D[\psi_0]D[\hat{\psi}]D[\psi]\times\exp\left[-A[\hat{\psi},\psi_0,\psi]+\int d1\,U(1)\psi(1)\right]\ , \tag{2.14}$$

where the action A is of the form

$$A[\psi,\hat{\psi},\psi_0]=\frac{1}{T_I}F_I[\psi_0]+\int d1\,d2[\hat{\psi}(1)\pi(12)\hat{\psi}(2)+\hat{\psi}(1)G_0^{-1}(12)\psi(2)$$
$$+iu\,\hat{\psi}(1)\Gamma(12)\psi^3(2)+\hat{\psi}(1)I(12)\psi_0(2)]+A_J[\psi] \quad (2.15)$$

with the following specifications: The indices 1 and 2 stand, respectively, for $(\mathbf{R}_1,t_1)$ and $(\mathbf{R}_2,t_2)$, the integral

$$\int d1=\int d^d\mathbf{R}_1\int_{-\infty}^{+\infty}dt_1\ ,$$

and the quantities under the integral are given by

$$\Gamma(12)=\Gamma(\mathbf{R}_1)\delta(12)\ , \quad (2.16)$$

$$\delta(12)=\delta(\mathbf{R}_1-\mathbf{R}_2)\delta(t_1-t_2)\ , \quad (2.17)$$

$$\pi(12)=\theta(t_1-t_0)T\Gamma(12), \quad (2.18)$$

$$I(12)=-i\delta(12)\delta(t_2-t_0)\ , \quad (2.19)$$

$$G_0^{-1}(12)=\left[i\frac{\partial}{\partial t_1}+i\Gamma(\mathbf{R}_1)(-c\nabla_{R_1}^2+r)\right]\delta(12)\ . \quad (2.20)$$

In (2.15) A_J is the Jacobian associated with the transformation from a functional integral over the noise to one over the field ψ and is given[30] in the case of (2.12) by

$$A_J[\psi]=-\frac{3u}{2}\int d1\Gamma(11)\psi^2(1)\ . \quad (2.21)$$

This term ensures[31] the normalization $Z(0)=1$, or equivalently, we can take

$$P_\psi[\hat{\psi},\psi_0,\psi]=e^{-A[\hat{\psi},\psi_0,\psi]} \quad (2.22)$$

as the functional probability measure associated with the stochastic equation (2.12).

The advantage of this formalism in the case of fluctuations in equilibrium is that one has a formulation of the standard field-theoretical type and one can conveniently develop perturbation theory.

D. Naive perturbation theory

Starting with the action given by (2.15), it is straightforward to develop perturbation theory directly in terms of the coupling u. We skip the detailed derivation here and simply write down the associated equation of motion satisfied by the Fourier transform of the equal time correlation function

$$C(\mathbf{R}-\mathbf{R}',t)=\langle\psi(\mathbf{R},t)\psi(\mathbf{R}',t)\rangle\ , \quad (2.23)$$

given by

$$\left[\frac{\partial}{\partial t}+2\Gamma(\mathbf{q})[cQ^2(\mathbf{q})+r+3uS(t)]\right]C(\mathbf{q},t)=2T\Gamma(\mathbf{q})\ , \quad (2.24)$$

where

$$S(t)=\int\frac{d^dq}{(2\pi)^d}C(\mathbf{q},t) \quad (2.25)$$

which is identical in structure to Eq. (2.31) found in Ref. 24 in the large-N limit except that the 3 multiplying $uS(t)$ in (2.24) is replaced by 1. These equations are very similar in structure to those found by LBM.[34] The important aspect of the solution of these equations for our purposes here is that

$$\lim_{t\to\infty}r+3uS(t)=0 \quad (2.26)$$

and in the long-time limit

$$C(\mathbf{q})=M^2(2\pi)^d\delta(\mathbf{q})+\frac{T}{cQ^2(\mathbf{q})}\ . \quad (2.27)$$

Inserting (2.27) into (2.25) and using (2.26), one obtains

$$M^2=-\frac{r}{3u}-\int\frac{d^dq}{(2\pi)^d}\frac{T}{cQ^2(\mathbf{q})}\ . \quad (2.28)$$

Comparing with (2.9), we see that (2.27) gives a very poor approximation for the final state. It gives an unphysical NG mode ($\sim T/q^2$) in the fluctuation spectrum. It also gives an incorrect value for the zero-temperature magnetization [compare (2.28) with the correct result (2.7)] and an incorrect first-order temperature correction.

As mentioned in the Introduction, the problem with theories of this type is that they allow for only a single length or mass. In (2.24) we can identify a single mass term W and a length L via

$$W(t)/c=r+3uS(t)=-L^{-2}(t) \quad (2.29)$$

and the dynamics drives $W\to 0$ and $L\to\infty$ as $t\to\infty$. To overcome these fundamental defects we must construct a theory which naturally allows for two masses and the two corresponding lengths.

E. Decomposition of the field

In order to make progress we now implement the idea brought forward in the Introduction of decomposing $\psi(\mathbf{R},t)$ as the sum of two fields, one of which becomes associated, as times evolves, with the growth of domains, and the other with fluctuations within a domain. In other words, we must introduce the appropriate dynamical generalization of the shift (2.5) used in the analysis of the equilibrium state, with the requirement of keeping the symmetry unbroken at all finite times.

The first step is to enlarge the function space by introducing, in addition to the order parameter $\psi(\mathbf{R},t)$, a new independent stochastic field $\sigma(\mathbf{R},t)$ with its own normalized distribution $P_\sigma[\sigma]$, which, for the moment, is not specified, except for the condition $\langle\sigma(\mathbf{R},t)\rangle=0$. The joint distribution for the pair (ψ,σ) then takes the product form

$$P[\hat{\psi},\psi_0,\psi,\sigma]=P_\psi[\hat{\psi},\psi_0,\psi]P_\sigma[\sigma] \quad (2.30)$$

and it is clear that at this stage the field σ does not enter into the order parameter correlation functions.

Next, we introduce the field $\phi(\mathbf{R},t)$ through the translation

$$\psi(\mathbf{R},t)=\phi(\mathbf{R},t)+\sigma(\mathbf{R},t) \tag{2.31}$$

and, switching to the pair (ϕ,σ) we obtain the joint distribution

$$P[\hat{\psi},\psi_0,\phi,\sigma]=P_\psi[\hat{\psi},\psi_0,\phi+\sigma]P_\sigma[\sigma]\ . \tag{2.32}$$

The scheme becomes nontrivial when we use $\sigma(\mathbf{R},t)$, whose dynamics is governed by $P_\sigma[\sigma]$, to model the growth of order. The field $\phi(\mathbf{R},t)$ is left to describe fluctuations about order, which become less and less important as time proceeds. Eventually, in the asymptotic regime, the field $\phi(\mathbf{R},t)$ is expected to become Gaussian and to decouple from $\sigma(\mathbf{R},t)$. It is clear that the careful construction of the appropriate form for $P_\sigma[\sigma]$ is crucial to the structure of our theory.

According to the physical picture described above, $\sigma(\mathbf{R},t)$ should be a two-valued variable, which keeps the same magnitude inside a given domain and changes its sign across an interface. We formalize this by taking

$$\sigma(\mathbf{R},t)=\sqrt{S}\,(t)\mu(\mathbf{R},t)\ , \tag{2.33}$$

where $S(t)$ is some time dependent quantity to be discussed below, and $\mu(\mathbf{R},t)=\pm 1$. We can then rewrite the action in terms of ϕ and σ as

$$\begin{aligned}A[\hat{\psi},\psi_0,\phi,\sigma]=\frac{1}{T_I}F_I[\psi_0]+\int d1\,d2(\hat{\psi}(1)\pi(12)\hat{\psi}(2)\\ +\hat{\psi}(1)I(12)\psi_0(2)+\hat{\psi}(1)\{G_f^{-1}(12)\phi(2)+G_p^{-1}(12)\sigma(2)\\ +iu\Gamma(12)[3\sigma(2)\phi^2(2)+\phi^3(2)]\})+A_J(\sigma+\phi)\ ,\end{aligned} \tag{2.34}$$

where the new propagators, after a spatial Fourier transform, are given by

$$G_p^{-1}(\mathbf{q},t_1,t_2)=i\left[\frac{\partial}{\partial t_1}+\Gamma(\mathbf{q})[cQ^2(\mathbf{q})+W_p(t)]\right]\delta(t_1-t_2)\ , \tag{2.35}$$

$$G_f^{-1}(\mathbf{q},t_1,t_2)=i\left[\frac{\partial}{\partial t_1}+\Gamma(\mathbf{q})[cQ^2(\mathbf{q})+W_f(t)]\right]\delta(t_1-t_2)\ , \tag{2.36}$$

and the mass terms are now different and given by

$$W_p(t)=r+uS(t)\ , \tag{2.37}$$

$$W_f(t)=r+3uS(t)\ , \tag{2.38}$$

where the subscripts p and f will be used to denote *peak* and *fluctuating* contributions.

Before going further with the specification of the $\sigma(\mathbf{R},t)$ variable and its governing probability distribution $P_\sigma[\sigma]$, we need some results for Gaussian functional integrals which we will refer to repeatedly as we proceed and will help us in motivating the next step in our development of perturbation theory.

F. Gaussian functional integrals

Let us consider the structure of response and correlation functions within a functional integral approach with a quadratic action of the general form:

$$\begin{aligned}A_q[\hat{\Psi},\Psi,\Psi_0]=\tfrac{1}{2}\int d^dR_1 d^dR_2\Psi_0(\mathbf{R}_1)g^{-1}(\mathbf{R}_1-\mathbf{R}_2)\Psi_0(\mathbf{R}_2)\\ +\int d1\,d2\,\hat{\Psi}(1)\pi(12)\hat{\Psi}(2)+\int d1\,d2\,\hat{\Psi}(1)[G^{-1}(12)\Psi(2)+I(12)\Psi_0(2)]\ ,\end{aligned} \tag{2.39}$$

where g^{-1}, π, and G^{-1} are arbitrary functions, translationally invariant in space, and $I(12)$ is given by (2.19). It is not hard to show [essentially by recalling the equivalence of (2.12) and (2.15)] that (2.39) leads to a dynamics of the field Ψ equivalent to that generated by the Langevin equation

$$G^{-1}(1\bar{1})\Psi(\bar{1})=-I(1\bar{1})\Psi_0(\bar{1})+i\eta(1)\ , \tag{2.40}$$

where, here and below, repeated barred indices indicate integration over space and time and the Gaussian noise satisfies $\langle\eta(1)\eta(2)\rangle=2\pi(12)$. Thus, one expects that π is proportional to the temperature and drives the system to-

TABLE I. Correlation matrix for the quadratic action (2.39). $C(12)$ is defined by (2.42).

	$\hat{\psi}(1)$	$\psi(1)$	$\psi_0(1)$
$\hat{\psi}(2)$	0	$G(12)$	0
$\psi(2)$	$G(21)$	$C(12)$	$i\int d^dR_3\, g(\mathbf{R}_1-\mathbf{R}_3)G(2,\mathbf{R}_3t_0)$
$\psi_0(2)$	0	$i\int d^dR_3\, G(1,\mathbf{R}_3t_0)g(\mathbf{R}_3-\mathbf{R}_2)$	$g(\mathbf{R}_1-\mathbf{R}_2)$

ward equilibrium.

It is not difficult to derive the correlation matrix given in Table I by computing the averages over the fields Ψ, Ψ_0, and $\hat{\Psi}$ through the identity

$$\int d[\hat{\Psi}]d[\Psi]d[\Psi_0]\frac{\delta}{\delta\Psi(1)}[e^{-A_q}f(\hat{\Psi},\Psi,\Psi_0)]=0 \quad (2.41)$$

and similar ones obtained by differentiation with respect to $\hat{\Psi}$ and Ψ_0. In Table I,

$$C(12)=-\int d\bar{1}\,d\bar{2}\,G(1\bar{1})G(2\bar{2})2\tilde{\pi}(\bar{1}\bar{2}) \quad (2.42)$$

and

$$2\tilde{\pi}(12)=2\pi(12)\delta(t_1-t_0)\delta(t_2-t_0)g(\mathbf{R}_1-\mathbf{R}_2)\ . \quad (2.43)$$

These results can also be obtained directly from the Langevin equation. Restricting the analysis to inverse propagators G^{-1} of the general form (2.35)–(2.36) with a mass term $W(t)$, it is easy to show that

$$G(\mathbf{q},t_1,t_2)=-i\theta(t_1-t_2)\times\exp-\int_{t_2}^{t_1}d\bar{t}\,\Gamma(\mathbf{q})[cQ^2(\mathbf{q})+W(\bar{t})]\ , \quad (2.44)$$

which is identified as a retarded propagator with the equal-time value

$$G(\mathbf{q},t,t)=-i/2\ . \quad (2.45)$$

In the important special case where $W(t)\to W_E>0$ as $t\to\infty$ and $\pi(\mathbf{q},t_1t_2)=T\Gamma(\mathbf{q})\delta(t_1-t_2)$, then, for large $t_1=t_2=t$, the Fourier transform of the correlation function (2.42) reduces to its equilibrium value

$$\lim_{t\to\infty}C(\mathbf{q},t,t)=\frac{T}{cQ^2(\mathbf{q})+W_E}\ . \quad (2.46)$$

G. Separation of variables

Armed with these results we return to the specification of $P_\sigma[\sigma]$. We must, of course, be guided by the physics described by the original Langevin equation (2.12) and our intuition that ϕ becomes in some sense "small" at long times. Let us assume for the moment that we can ignore the field ϕ in (2.12) and replace $\psi\to\sigma$. If we multiply the resulting equation by i and recognize that the noise θ associated with σ should differ from η, we obtain immediately the equation of motion

$$G_p^{-1}(1\bar{1})\sigma(\bar{1})=-I(1\bar{1})+i\theta(1)\ , \quad (2.47)$$

where the noise θ is Gaussian and satisfies

$$\langle\theta(1)\theta(2)=2\Pi(12)\ , \quad (2.48)$$

where Π will be discussed below and differs from $\pi(12)$. Suffice it to say here that Π and θ can be set to zero as $T\to 0$. Comparing (2.47) and (2.40), we are led to the identification of G_p as the propagator associated with the peak contribution.

There is another way of viewing this discussion. Let us consider $P_\psi[\hat{\psi},\psi_0,\phi+\sigma]$ as the probability distribution for $\phi(\mathbf{R},t)$, parametrized by the stochastic process $\sigma(\mathbf{R},t)$. If the dynamics of $\sigma(\mathbf{R},t)$ governed by $P_\sigma[\sigma]$, are not properly chosen, then, in the long-time limit, ϕ will not be appropriately enslaved by $\sigma(\mathbf{R},t)$ and will not represent small fluctuations on an ordered background.

If this picture is to be valid, then it is crucial that the coefficient of the term in (2.34) linear in σ and proportional to G_p^{-1}, vanishes in the long time and distance limit. It is, of course, this vanishing of G_p^{-1} and W_p for small wave numbers and long times which contributes to the building of a Bragg peak in the solution of (2.47) for the structure factor.

The choice of G_p^{-1} as the propagator associated with the σ dynamics leads not only to the separation of the variables σ and ϕ at long times, but also to the correct separation of length scales. The characteristic length $L(t)=[-W_p(t)/c]^{-1/2}$ can then be associated with the domain size, and $\xi(t)=[W_f(t)/c]^{-1/2}$ associated with the correlation length of fluctuations. As order develops in the peak, we expect $L(t)\to\infty$ ($W_p=r+uS\to 0$) and $W_f(t)\to 2\,|\,r\,|>0$. In this case, with $W_f(\infty)=2\,|\,r\,|$, we see, using (2.36) and (2.46), that the equal time correlation function associated with the ϕ variable will reach, as $t\to\infty$, its correct equilibrium value (2.9):

$$\lim_{t\to\infty}\langle\,|\,\phi(\mathbf{q},t)\,|^2\rangle=\frac{T}{cQ^2(\mathbf{q})+2\,|\,r\,|}\ . \quad (2.49)$$

This indicates that the variable ϕ can be taken as $O(\sqrt{T})$ for sufficiently long times.

The above schematic analysis, although very appealing on physical grounds, is mathematically flawed, since (2.47) cannot hold for a discrete variable. However, one can require that it be satisfied on average. If we formally solve (2.47) for σ and compute the average two-point correlation function, we must average over ψ_0 and θ. Using (2.48) and (2.13), so that

$$\langle\psi_0(\mathbf{R}_1)\psi_0(\mathbf{R}_2)\rangle=g_0(\mathbf{R}_1-\mathbf{R}_2)\ , \quad (2.50)$$

we immediately obtain

$$\langle\sigma(1)\sigma(2)\rangle=-G_p(1\bar{1})G_p(2\bar{2})2\tilde{\Pi}(\bar{1}\bar{2}) \quad (2.51)$$

where

$$2\tilde{\Pi}(12)=2\Pi(12)+\delta(t_1-t_0)\delta(t_2-t_0)g_0(\mathbf{R}_1-\mathbf{R}_2)\ . \tag{2.52}$$

For consistency, we must remember that (2.51) must be supplemented by the crucial constraint

$$S(t)=\langle\sigma^2(1)\rangle \tag{2.53}$$

which follows from (2.33).

H. Construction of $P[\sigma]$

Our analysis in Sec. II G has led us to a set of conclusions which are difficult to reconcile.

(i) The fundamental field ψ should contain an Ising-like component σ whose identification allows us to separate the two mass scales W_p and W_f and suggests that we identify the propagators G_p and G_f with the variables σ and ϕ.

(ii) This Ising-field σ however, cannot directly satisfy (2.47) derived by setting $\psi=\sigma$ in the original Langevin equation *because* of its Ising nature. The best we can do is to require that $\langle\sigma(1)\sigma(2)\rangle$ satisfies (2.51) and (2.53).

These apparently conflicting conclusions can be reconciled through the proper construction of the distribution $P_\sigma[\sigma]$. As a first step in constructing $P_\sigma[\sigma]$, we introduce[35,36]

$$Q[\sigma\mid m]=\prod_l \tfrac{1}{2}[1+m(l)\sigma(l)/S(t_l)]\ , \tag{2.54}$$

where $m(\mathbf{R},t)$ is a continuous field, and Q, which satisfies,

$$\sum_{\{\sigma\}} Q[\sigma\mid m]=1\ , \tag{2.55}$$

$$\sum_{\{\sigma\}} \sigma(1)Q[\sigma\mid m]=m(1)\ , \tag{2.56}$$

$$\sum_{\{\sigma\}} \sigma(1)\sigma(2)Q[\sigma\mid m]=m(1)m(2)+\delta(12)[S(t_1)-m^2(1)]\ , \tag{2.57}$$

maps σ onto m except for space-time points which coincide, where, for example, $\sigma^2(i)=S(t_i)$.

Next we endow the field $m(\mathbf{R},t)$ with dynamical behavior through a probability distribution of the form

$$P_m[\hat{m},m_0,m]=e^{-A_m[\hat{m},m_0,m]}\ , \tag{2.58}$$

where $A_m[\hat{m},m_0,m]$ is an action of the MSR type and we define[37]

$$P_\sigma[\sigma]=\int D[\hat{m}]D[m_0]D[m]Q[\sigma\mid m]P_m[\hat{m},m_0,m]\ . \tag{2.59}$$

If we reorganize the product in Q in terms of a sum ordered by the number of σ variables we obtain

$$P_\sigma[\sigma]=\frac{1}{2^N}\left[1+\sum_l \frac{\langle m(l)\rangle}{S(t_l)}\sigma(l)+\tfrac{1}{2}\sum_{\substack{l_1,l_2\\ l_1\neq l_2}} \frac{\langle m(l_1)m(l_2)\rangle}{S(t_{l_1})S(t_{l_2})}\sigma(l_1)\sigma(l_2)+\frac{1}{3!}\sum_{\substack{l_1,l_2,l_3\\ l_1\neq l_2\neq l_3}} \frac{\langle m(l_1)m(l_2)m(l_3)\rangle}{S(t_{l_1})S(t_{l_2})S(t_{l_3})}\sigma(l_1)\sigma(l_2)\sigma(l_3)+\cdots\right], \tag{2.60}$$

where the averages in the coefficients are taken with respect to P_m. We note, however, that a normalized distribution for an Ising-like field is of the general form

$$P_\sigma[\sigma]=\frac{1}{2^N}\left[1+\sum_l \frac{\langle\sigma(l)\rangle}{S(t_l)}\sigma(l)+\tfrac{1}{2}\sum_{\substack{l_1,l_2\\ l_1\neq l_2}} \frac{\langle\sigma(l_1)\sigma(l_2)\rangle}{S(t_{l_1})S(t_{l_2})}\sigma(l_1)\sigma(l_2)+\frac{1}{3!}\sum_{l_1\neq l_2\neq l_3} \frac{\langle\sigma(l_1)\sigma(l_2)\sigma(l_3)\rangle}{S(t_{l_1})S(t_{l_2})S(t_{l_3})}\sigma(l_1)\sigma(l_2)\sigma(l_3)+\cdots\right]. \tag{2.61}$$

Comparing (2.70) and (2.61), we see that we are in the position of modeling the dynamics of the Ising-like field $\sigma(\mathbf{R},t)$ by means of the dynamics of the continuous field $m(\mathbf{R},t)$. In fact, one can see by inspection from (2.60) and (2.61) that the correlation functions of $\sigma(\mathbf{R},t)$ do coincide at distinct points with the correlation functions of $m(\mathbf{R},t)$, and the latter are specified by giving the action $A_m[\hat{m},m_0,m]$. Due to the intrinsic difference between Gaussian and Ising statistics, deviations occur when two or more fields are taken at the same point. Consider, for example, the two-point correlation function $\langle\sigma(1)\sigma(2)\rangle$. If we multiply $\sigma(1)\sigma(2)$ by $Q[\sigma\mid m]P_m[\hat{m},m_0,m]P_\psi[\hat{\psi},\psi_0,\psi]$ and sum over all of the variables, using (2.57), we obtain the general result

$$\langle\sigma(1)\sigma(2)\rangle=\langle m(1)m(2)\rangle+\delta(12)[S(t_1)-\langle m^2(1)\rangle] \tag{2.62}$$

and the two correlation functions will be identical everywhere,

$$C_m(12)=\sigma(1)\sigma(2)\rangle=\langle m(1)m(1)\rangle \ , \tag{2.63}$$

if we enforce the self-consistent constraint (2.53),

$$S(t_1)=\langle m^2(1)\rangle \ . \tag{2.64}$$

One can show in a similar manner, using (2.56), that

$$\langle\sigma(1)m(2)\rangle=C_m(12) \ . \tag{2.65}$$

Higher-order correlation functions cannot be constructed to coincide everywhere.

We can now develop a nontrivial, non-Gaussian dynamics for $\sigma(\mathbf{R},t)$ through a Gaussian dynamics for $m(\mathbf{R},t)$. We simply stipulate that $m(\mathbf{R},t)$ obeys the equation of motion (2.47),

$$G_p^{-1}(1\bar{1})m(\bar{1})=-I(1\bar{1})m_0(\bar{1})+i\theta(1) \ , \tag{2.66}$$

where m_0 is governed by the initial free energy F_I given by (2.13) with ψ_0 replaced by m_0. Clearly, from the discussion of Sec. II F, the quadratic action associated with the field m is given by

$$A_m=\frac{1}{T_I}F_I[m_0]+\int d1\,d2\{\hat{m}(1)\Pi(12)\hat{m}(2)+\hat{m}(1)[G_p^{-1}(12)m(2)+I(12)m_0(2)]\} \ . \tag{2.67}$$

From (2.66) we can identify G_p^{-1} as the propagator for the m field, and, to the extent that the m and σ fields coincide at distinct points, we can also identify G_p^{-1} as the propagator for the σ variable and reconcile the two points listed at the beginning of this section. In particular we have constructed $P_\sigma[\sigma]$ such that (2.51) and (2.53) hold and

$$C_m(12)=-G_p(1\bar{1})G_p(2\bar{2})2\tilde{\Pi}(\bar{1}\bar{2}) \ . \tag{2.68}$$

I. Determination of $\Pi(12)$

The last point to be specified in the determination of the σ and m variables is the noise term $\Pi(12)$. The construction of an acceptable form for Π is based on the observation that unless Π vanishes at long times one will generate the spurious NG modes discussed earlier. In addition, one expects that at short times the peak growth will be influenced by thermal noise and $\Pi\sim T$, while at long times thermal noise should be unimportant in determining the peak contribution.

We construct Π satisfying these requirements by first considering the equation satisfied by C_m obtained by multiplying (2.51) on the left by G_p^{-1}:

$$G_p^{-1}(1\bar{1})C_m(\bar{1}2)=-2\tilde{\Pi}(1\bar{2})G_p(2\bar{2}) \ , \tag{2.69}$$

where Π and $\tilde{\Pi}$ are related by (2.52). Let us now consider the auxiliary correlation function $C_{\rm aux}(12)$ obtained by replacing $\tilde{\Pi}$ by $\bar{\pi}$ in (2.69):

$$G_p^{-1}(1\bar{1})C_{\rm aux}(\bar{1}2)=-2\bar{\pi}(1\bar{2})G_p(2\bar{2}) \ , \tag{2.70}$$

where

$$2\bar{\pi}(12)=2\pi(12)+\delta(t_1-t_0)\delta(t_2-t_0)g_0(\mathbf{R}_1-\mathbf{R}_2) \tag{2.71}$$

and $\pi(12)$ is the usual noise auto correlation given by (2.18). This equation was studied in Ref. 22 and, while leading to an ordering peak, it also generates a NG mode in the final equilibrium state. In the case we study here this mode is spurious and Π must be constructed such that no such mode appears in C_m. The NG mode contained in $C_{\rm aux}$ is associated with the massless limit of G_p^{-1} and consequently we define[38] a new quantity C_N:

$$G_\infty^{-1}(1\bar{1})C_N(\bar{1}2)=-2\pi(1\bar{2})G_\infty(2\bar{2}) \ , \tag{2.72}$$

where G_∞^{-1} is given by (2.32) with $W_p=0$. Since the difference between $C_{\rm aux}(12)$ and $C_N(12)$ will not contain the NG mode, we can set

$$C_m(12)=C_{\rm aux}(12)-C_N(12) \ . \tag{2.73}$$

Subtracting (2.72) from (2.70), using (2.73) and comparing with (2.69) allows us to identify

$$2\tilde{\Pi}=2\bar{\pi}(12)+G_p^{-1}(1\bar{1})G_p^{-1}(2\bar{2})C_N(\bar{1}\bar{2}) \ . \tag{2.74}$$

By making use of (2.71) and the definition (2.35) in (2.74) we obtain, after a spatial Fourier transform,

$$\begin{aligned}2\Pi(\mathbf{q},t_1,t_2)=\Gamma^2(\mathbf{q})[&-2iW_p(t_2)G_\infty(q,t_2,t_1)T\\&-2iW_p(t_1)G_\infty(q,t_1,t_2)T\\&-W_p(t_1)W_p(t_2)C_N(\mathbf{q},t_1,t_2)] \ .\end{aligned} \tag{2.75}$$

The key point for our purposes here is that $\Pi(\mathbf{q},t_1,t_2)$ vanishes as t_1 or $t_2\to\infty$ since $W_p(t)$ vanishes as $t\to\infty$. The resulting peak correlation function $C_m(12)$ will, consequently, not contain any spurious NG modes.

J. Recapitulation

At the beginning of this section the field ψ was split into the two fields ϕ and σ in order to introduce two different "masses" W_p and W_f in the theory. This required that σ be an Ising-like variable, whose dynamics is described by a probability distribution $P_\sigma[\sigma]$ designed to

govern the growth of the peak contribution in the structure factor. The proper mathematical construction of $P_\sigma[\sigma]$ requires the introduction of an additional (continuous) field $m(\mathbf{R},t)$ which, in some sense, is conjugate to $\sigma(\mathbf{R},t)$. In the end, the distribution governing the theory in the expanded function space is given by

$$P_T[\hat{\psi},\psi_0,\phi,\sigma,\hat{m},m_0,m] = e^{-A_T[\hat{\psi},\psi_0,\phi,\sigma,\hat{m},m_0,m]} Q[\sigma \mid m] , \quad (2.76)$$

where the action A_T is given by

$$A_T[\hat{\psi},\psi_0,\phi,\sigma,\hat{m},m,m_0] = A_m[\hat{m},m,m_0] + A[\hat{\psi},\psi_0,\phi,\sigma] \quad (2.77)$$

with A_m given by (2.67) and A by (2.34).

III. PERTURBATION THEORY

A. General considerations

The distribution P_T given by (2.76) is an exact formal rearrangement of the original field theory. We can now proceed to solve for the basic properties of this model in perturbation theory. To do this we start by choosing an appropriate quadratic approximation to A_T. Since A_m is already quadratic in the fields $m,\hat{m},m_0$, we include it completely in our zeroth-order action. There are three sources of nonquadratic terms in A which should be included in the interaction. The obvious terms are those proportional to $3\sigma(2)\phi^2(2)+\phi^3(2)$ and the Jacobian term, which can actually be combined to give one source of nonlinearity. Less obvious is the term[39]

$$\int d1\,d2\,\hat{\psi}(1)G_p^{-1}(12)\sigma(2) . \quad (3.1)$$

This term is quadratic in the fields, but, because of the Ising nature of σ, it does not lead to a Gaussian contribution in the associated probability distribution. Instead we write

$$\int d1\,d2\,\hat{\psi}(1)G_p^{-1}(12)\sigma(2) = \int d1\,d2\,\hat{\psi}(1)G_p^{-1}(12)m(2)+V_\sigma \quad (3.2)$$

and we treat

$$V_\sigma = \int d1\,d2\,\hat{\psi}(1)G_p^{-1}(12)[\sigma(2)-m(2)] \quad (3.3)$$

as part of the perturbation. We expect V_σ to be a small perturbation because $\sigma(1)$ is, on average, very nearly equal to $m(1)$. At long times the two fields coincide except near interfaces. Furthermore, $G_p^{-1}(12)$ goes to zero at long times and distances because $W_p \to 0$. We will later see that this expectation is explicitly verified at first order where the contribution arising from V_σ vanishes. Should one include (3.1) directly in the zeroth-order probability distribution, terms would be generated at first order which are of the same magnitude as the zeroth-order terms.

We can then write down the total action

$$A_T = A_0 + V , \quad (3.4)$$

where

$$A_0 = A_m + \frac{1}{T_I}F_I[\psi_0] + \int d1\,d2\{\hat{\psi}(1)\pi(12)\hat{\psi}(2) + \hat{\psi}(1)[I(12)\psi_0(2)+G_f^{-1}(12)\phi(2) + G_p^{-1}(12)m(2)]\} \quad (3.5)$$

is the zeroth-order contribution, and

$$V = \int d1\,d2\{\hat{\psi}(1)iu\Gamma(12)[3\sigma(2)\phi^2(2)+\phi^3(2)]\} + A_J[\sigma+\phi]+V_\sigma \quad (3.6)$$

is the interaction. We develop perturbation theory by writing the total distribution

$$P_T = P_0 e^{-V} , \quad (3.7)$$

where

$$P_0 = e^{-A_0} Q[\sigma \mid m] \quad (3.8)$$

and expanding in powers of V. A few general comments are in order.

(i) Even though the portion of P_0 proportional to e^{-A_0} is Gaussian, P_0 is certainly not Gaussian with respect to the basic field ψ due to the factor of $Q[\sigma \mid m]$. This will be demonstrated explicitly below when we calculate the singlet distribution function ρ.

(ii) The assumption that V can be treated as small for small T is supported by the expectation that at long times $\hat{\psi} \sim 1/\sqrt{T}$, $\phi \sim \sqrt{T}$, and $m \sim \sigma \sim 1/\sqrt{T}$. Thus the term in V proportional to $\sigma\phi^2$ is of $O(\sqrt{T})$ and the ϕ^3 term of $O(T)$. V_σ has been discussed above.

(iii) Looking at the zeroth-order action A_0, we see that there is an indirect coupling between the fields m and ϕ through the term $\hat{\psi}G_p^{-1}m$. We can achieve a further separation of the peak variable and the equilibrating *phonon* contribution if we make one further shift and define the field $\zeta(1)$ via

$$\phi(1) = \zeta(1) - a(1\bar{1})m(\bar{1}) , \quad (3.9)$$

where

$$a(12) = G_f(1\bar{1})G_p^{-1}(\bar{1}2) . \quad (3.10)$$

The zeroth-order action then decouples into a sum of two parts:

$$A_0 = A_m + A_\zeta , \quad (3.11)$$

where A_m is given by (2.67), and A_ζ has the same general form as A_m:

$$A_\zeta = \tfrac{1}{2}\int d^dR_1\,d^dR_2\,\psi_0(\mathbf{R}_1)g_0^{-1}(\mathbf{R}_1-\mathbf{R}_2)\psi_0(\mathbf{R}_2) + \int d1\,d2\{\hat{\psi}(1)\pi(12)\hat{\psi}(2)+\hat{\psi}(1)[I(12)\psi_0(2)+G_f^{-1}(12)\zeta(2)]\} . \quad (3.12)$$

In terms of these new variables the original field is given by

$$\psi(1)=\sigma(1)-a(1\bar{1})m(\bar{1})+\zeta(1) . \tag{3.13}$$

B. Zero-order theory

In this section we work out the zero-order theory in detail. The results of this computation show that the probability distribution P_0 given by (3.8) contains all the essential features of the physical process that we wish to describe.

Let us then consider the structure factor

$$C(12)=\langle \psi(1)\psi(2)\rangle \tag{3.14}$$

After replacing ψ using (3.13), we evaluate the zeroth-order average with respect to P_0, which we denote by $\langle\ \rangle_0$. Since the variables m and σ are decoupled from ζ at zeroth order, all such cross correlations are zero. Using (2.63) and (2.65), it is easy to show that at zeroth order

$$C^0(12)=C^0_\zeta(12)+C_m(12)-C_a(12)-C_a(21)+C_{aa}(12) , \tag{3.15}$$

where we have introduced the notation

$$C^0_\zeta(12)=\langle \zeta(1)\zeta(2)\rangle_0 , \tag{3.16}$$

$$C_m(12)=\langle m(1)m(2)\rangle_0 , \tag{3.17}$$

$$C_a(12)=a(1\bar{1})C_m(\bar{1}2) , \tag{3.18}$$

$$C_{aa}(12)=a(1\bar{1})a(2\bar{2})C_m(\bar{1}\bar{2}) . \tag{3.19}$$

Let us focus here on the determination of the equal-time correlation functions. Let us start with C^0_ζ. Comparing (3.12) and (2.39) we see that we can evaluate C^0_ζ immediately in the general form given by (2.42) with $G\to G_f$ and $g\to g_0$. Multiplying the result from the left by G_f^{-1}, we obtain

$$G_f^{-1}(1\bar{1})C^0_\zeta(\bar{1}2)=-G_f(2\bar{2})2\tilde{\pi}(1\bar{2}) , \tag{3.20}$$

where $\tilde{\pi}$ is given by (2.71). Rewriting (3.20) with the indices 1 and 2 interchanged, adding the result to (3.20), setting $t_1=t_2=t$, Fourier transforming over space and using (2.45), we obtain for the correlation function $C^0_\zeta(q,t)$

$$\left[\frac{\partial}{\partial t}+2\Gamma(\mathbf{q})[cQ^2(\mathbf{q})+W_f(t)]\right]C^0_\zeta(\mathbf{q},t) =2T\Gamma(\mathbf{q})+\delta(t-t_0)g_0(\mathbf{q}) . \tag{3.21}$$

In complete analogy with the derivation of (3.20) one obtains:

$$G_p^{-1}(1\bar{1})C_m(\bar{1}2)=-G_p(2\bar{2})2\tilde{\Pi}(1\bar{2}) .$$

Using the expression for $\tilde{\Pi}$ given in (2.75) and performing steps very similar to those leading from (3.20) to (3.21), we find

$$\left[\frac{\partial}{\partial t}+2\Gamma(\mathbf{q})[cQ^2(\mathbf{q})+W_p(t)]\right]C_m(\mathbf{q},t) =-2\Gamma(\mathbf{q})W_p(t)C_N(\mathbf{q},t)+\delta(t-t_0)g_0(\mathbf{q}) \tag{3.22}$$

which must be supplemented by the consistency constraint (2.64) given by (2.25) with C replaced by C_m. Similarly, from (2.72) one can derive the equation obeyed by the equal-time quantity $C_N(\mathbf{q},t)$

$$\left[\frac{\partial}{\partial t}+2\Gamma(\mathbf{q})cQ^2(\mathbf{q})\right]C_N(\mathbf{q},t)=-2T\Gamma(\mathbf{q})\theta(t-t_0) , \tag{3.23}$$

which can be integrated explicitly to obtain:

$$C_N(\mathbf{q},t)=-\frac{T\theta(t-t_0)}{cQ^2(\mathbf{q})}[e^{-2\Gamma(\mathbf{q})cQ^2(\mathbf{q})(t-t_0)}-1] . \tag{3.24}$$

Next we consider the contribution to the structure factor given by $C_a(12)$. Inserting (2.51) in (3.18) we obtain

$$C_a(12)=-G_f(1\bar{1})2\tilde{\Pi}(\bar{1}\bar{2})G_p(2\bar{2}) . \tag{3.25}$$

The derivation of the differential equation obeyed by the equal-time quantity $C_a(\mathbf{q},t)$ is considerably more complicated than those discussed above and we illustrate only the key steps. The main new complication is due to the nonlocal structure of $\Pi(12)$ as given by (2.75). If we multiply (3.25) from the left by G_f^{-1} and from the right by G_p^{-1}, add the two resulting equations, set $t_1=t_2$ and Fourier transform over space we obtain

$$\left[\frac{\partial}{\partial t}+2\Gamma(\mathbf{q})[cQ^2(\mathbf{q})+W_a(t)]\right]C_a(\mathbf{q},t) =-\Gamma(\mathbf{q})W_p(t)[H_f(\mathbf{q},t)+H_p(\mathbf{q},t)] +\delta(t-t_0)g_0(\mathbf{q}) , \tag{3.26}$$

where

$$W_a(t)=r+2uS(t) \tag{3.27}$$

and where the quantities H_f and H_p are themselves correlation functions which require a similar analysis to determine the differential equations governing their equal-time behavior:

$$\left[\frac{\partial}{\partial t}+2\Gamma(\mathbf{q})[cQ^2(\mathbf{q})+\tfrac{1}{2}W_f(t)]\right]H_f(\mathbf{q},t) =\Gamma(\mathbf{q})[2T\theta(t-t_0)+C_N(\mathbf{q},t)W_p(t)] , \tag{3.28}$$

$$\left[\frac{\partial}{\partial t}+2\Gamma(\mathbf{q})[cQ^2(\mathbf{q})+\tfrac{1}{2}W_p(t)]\right]H_p(\mathbf{q},t) =\Gamma(\mathbf{q})[2T\theta(t-t_0)+C_N(\mathbf{q},t)W_p(t)] . \tag{3.29}$$

A similar procedure applied to $C_{aa}(\mathbf{q},t)$ yields the determining equation:

$$\left[\frac{\partial}{\partial t}+2\Gamma(\mathbf{q})[cQ^2(\mathbf{q})+W_f(t)]\right]C_{aa}(\mathbf{q},t)$$

$$=-2W_p(t)\Gamma(\mathbf{q})H_f(\mathbf{q},t)+\delta(t-t_0)g_0(\mathbf{q})\ . \quad (3.30)$$

In summary, the set of equations (3.21), (3.22), (3.24), (3.26), and (3.28)–(3.30) give the behavior of the structure factor in the zero-order approximation. It is also possible to derive, without much more effort, the equations for the time-displaced structure factor[40] $C(\mathbf{q},t,t+\tau)$, which, however, will not be studied in this paper.

The solution of the differential equations derived above will be considered in detail in Sec. IV. Here we add a few qualitative comments which elucidate the physical content of the zeroth-order approximation. For simplicity consider a quench to zero temperature ($T=0$) where

$$C_N(\mathbf{q},t)=H_f(\mathbf{q},t)=H_p(\mathbf{q},t)=0\ .$$

The governing differential equations can then be rewritten in the compact form:

$$D_p(\mathbf{q},t)C_m(\mathbf{q},t)=0\ , \quad (3.31a)$$

$$D_f(\mathbf{q},t)C^0_\zeta(\mathbf{q},t)=0\ , \quad (3.31b)$$

$$D_a(\mathbf{q},t)C_a(\mathbf{q},t)=0\ , \quad (3.31c)$$

$$D_f(\mathbf{q},t)C_{aa}(\mathbf{q},t)=0\ , \quad (3.31d)$$

with

$$D_x(\mathbf{q},t)=\frac{\partial}{\partial t}+2\Gamma(\mathbf{q})[cQ^2(\mathbf{q})+W_x(t)] \quad (3.32)$$

and all correlation functions are initially[41] equal to $g_0(\mathbf{q})$. For the particular case of $T=0$, we see that $C^0_\zeta(\mathbf{q},t)=C_{aa}(\mathbf{q},t)$. We show explicitly in Sec. IV that the peak growth in the structure factor is driven by (3.31a), and that it indeed involves only $C_m(\mathbf{q},t)$ and its moment $S(t)$. Starting from the initial instability due to $W_p(t_0)<0$, the system equilibrates by growing a peak until $W_p(t)\to 0$. Hence, (3.31a) has a long-time solution:

$$\lim_{t\to\infty} C_m(\mathbf{q},t)=M_0^2\delta(\mathbf{q}) \quad (3.33)$$

with

$$M_0^2=-r/u\ . \quad (3.34)$$

Furthermore, since $W_f(t)$ and $W_a(t)$ are both positive for sufficiently long times, the other contributions to the structure factor eventually decay exponentially to zero and after the passing of these transients, the structure factor is given by the peak contribution only. This demonstrates that the set (3.31) produces the expected ordering and the correct equilibrium state for $T=0$. The structure of the distribution P_0 underlying the zero-order theory is most effectively illustrated by studying the reduced singlet probability distribution $\rho(y,t)$, defined by

$$\rho(y,t)=\langle\delta(y-\psi(\mathbf{R},t))\rangle_0\ , \quad (3.35)$$

which gives the probability that the field $\psi(\mathbf{R},t)$ at the time t has the value y. Using the integral representation of the δ function and the decomposition (3.13), we have

$$\rho(y,t)=\left\langle\int_{-\infty}^{+\infty}\frac{dx}{2\pi}e^{ix[y-\sigma(1)+a(1\bar{1})m(\bar{1})-\zeta(1)]}\right\rangle_0\ . \quad (3.36)$$

After tracing over σ, which can easily be performed, one is left with Gaussian averages over ζ and m and the integration over x. The Gaussian averages can be carried out using the basic result

$$\langle e^{iB(1\bar{1})\zeta(\bar{1})}\rangle_0=\exp[-(1/2)B(1\bar{1})B(1\bar{2})\langle\zeta(\bar{1})\zeta(\bar{2})\rangle_0]. \quad (3.37)$$

The final result is given by

$$\rho(y,t)=(8\pi b)^{-1/2}\left[\left[1-\frac{S_a(t)}{b\sqrt{S}(t)}[y+\sqrt{S}(t)]\right]e^{-[y+\sqrt{S}(t)]^2/2b}+\left[1+\frac{S_a(t)}{b\sqrt{S}(t)}[y-\sqrt{S}(t)]\right]e^{-(y-\sqrt{S}(t))^2/2b}\right]\ , \quad (3.38)$$

where

$$b=S_\zeta(t)+S_{aa}(t) \quad (3.39)$$

and $S_\zeta(t)$, $S_a(t)$, $S_{aa}(t)$ are moments of $C^0_\zeta(\mathbf{q},t)$, $C_a(\mathbf{q},t)$, $C_{aa}(\mathbf{q},t)$ analogous to $S(t)$ as given by (2.64). The expression (3.38), after computation of these moments, leads to the results shown in Fig. 1, which displays the evolution from the initial Gaussian, associated to the disordered state, to the final bimodal distribution characteristic of the ordered equilibrium state.

C. General structure of perturbation theory

In the fully interacting theory the average giving the structure factor is taken is taken with respect to the total probability distribution P_T, given by (2.76). We now investigate what modifications the interaction produces on the behavior of the structure factor. The first step is to derive a general expression for the structure factor which involves only the variables ζ and m entering the zero-order theory.

As we have shown in Sec. II, the fields $\sigma(\mathbf{R},t)$ and $m(\mathbf{R},t)$ share the same correlation function (2.51) and this is a basic requirement, independent of perturbation theory, which follows from the structure of $P_\sigma[\sigma]$. This is not the case for the mixed term $\langle\sigma(1)\phi(2)\rangle$, since the joint probability of the fields σ and ϕ clearly depends on the interaction. However, a term of this type can be rewritten as an expectation in terms of m and ϕ. From the definition (2.54) it is easy to derive

$$\sigma(1)Q[\sigma\mid m]=\left[m(1)-[S(t_1)-m^2(1)]\frac{\partial}{\partial m(1)}\right]\times Q[\sigma\mid m]\ . \tag{3.40}$$

Inserting this result into $\langle\sigma(1)\phi(2)\rangle$ and integrating by parts with respect to m, we obtain

$$\langle\sigma(1)\phi(2)\rangle=\langle\{3m(1)+[S(1)-m^2(1)]G_p^{-1}(\bar{1}1)\hat{m}(\bar{1})\}\phi(2)\rangle\ . \tag{3.41}$$

Introducing the field

$$N(1)=3m(1)+[S(1)-m^2(1)]G_p^{-1}(\bar{1}1)\hat{m}(\bar{1}) \tag{3.42}$$

and using (3.9), we can write the structure factor in the form

$$\begin{aligned}C(12)=&[\delta(11)-a(1\bar{1})][\delta(2\bar{2})-a(2\bar{2})]C_m(\bar{1}\bar{2})\\&+\langle[N(1)-a(1\bar{1})m(\bar{1})]\zeta(2)\rangle\\&+\langle[N(2)-a(2\bar{2})m(\bar{2})]\zeta(1)\rangle+C_\zeta(12)\ .\end{aligned} \tag{3.43}$$

At zero order the cross terms $\langle N(1)\zeta(2)\rangle$ and $\langle m(1)\zeta(2)\rangle$ vanish and the form (3.15) of the structure factor is recovered.

In order to discuss the perturbation expansion for the structure factor, we start by defining the matrix correlation function

$$G_{\alpha\beta}(12)=\langle\Psi_\alpha(1)\Psi_\beta(2)\rangle\ , \tag{3.44}$$

where the indices α and β run over m, $\hat{m}$, ζ, $\hat{\psi}$ [e.g., $\Psi_{\hat{m}}(1)=\hat{m}(1)$]. The matrix inverse G^{-1} satisfies

$$\sum_\gamma G^{-1}_{\alpha\gamma}(1\bar{1})G_{\gamma\beta}(\bar{1}2)=\delta_{\alpha\beta}\delta(12) \tag{3.45}$$

and can be written as

$$G^{-1}_{\alpha\beta}(12)=G^{0\,-1}_{\alpha\beta}(12)-\Sigma_{\alpha\beta}(12)\ , \tag{3.46}$$

where $\Sigma_{\alpha\beta}(12)$ is the self-energy part which vanishes at zeroth order, and the zeroth-order part of the inverse, $G^{0\,-1}$ is obtained from the zeroth-order action (3.11):

$$G^{0\,-1}_{\hat{m}\hat{\psi}}=G^{0\,-1}_{m\zeta}=G^{0\,-1}_{\hat{\psi}\hat{m}}=G^{0\,-1}_{\zeta m}=0\ , \tag{3.47}$$

$$G^{0\,-1}_{\hat{m}m}(12)=G^{0\,-1}_{m\hat{m}}(12)=G_p^{-1}(12)\ , \tag{3.48}$$

$$G^{0\,-1}_{\hat{\psi}\zeta}(12)=G^{0\,-1}_{\zeta\hat{\psi}}(12)=G_f^{-1}(12)\ , \tag{3.49}$$

$$G^{0\,-1}_{\hat{\psi}\hat{\psi}}(12)=2\tilde{\pi}(12)\ , \tag{3.50}$$

$$G^{0\,-1}_{\hat{m}\hat{m}}(12)=2\tilde{\Pi}(12)\ . \tag{3.51}$$

The inversion of (3.45) for $G_{\alpha\beta}(12)$ is made easier by taking into account (i) from the general structure of the total action (3.4) it follows[25] that $G^{-1}_{\alpha\beta}=0$ when both α and β are unhatted variables; (ii) from the structure of the vertices it follows that $\Sigma_{\alpha\beta}=0$ when either α or β is equal to $\hat{m}$; and (iii) $G^{-1}_{\hat{m}\psi}=G^{-1}_{\hat{m}\zeta}=0$ since the fields $\hat{\psi}$ and $\hat{m}$ are uncorrelated. We then obtain from (3.45):

$$\begin{aligned}G_{\zeta\beta}(12)=&G_R(12)\delta_{\hat{\psi}\beta}+C_\zeta(12)\delta_{\beta\zeta}\\&+G_R(1\bar{1})\Sigma_{\hat{\psi}m}(\bar{1}\bar{2})G_p(\bar{2}2)\delta_{\beta\hat{m}}\\&+G_R(1\bar{1})\Sigma_{\hat{\psi}m}(\bar{1}\bar{2})C_m(\bar{2}2)\delta_{\beta m}\\&+G_R(1\bar{1})\Sigma_{\hat{\psi}m}(\bar{1}\bar{2})C_m(\bar{2}\bar{4})\Sigma_{m\hat{\psi}}(\bar{4}\bar{3})G_R(\bar{3}2)\delta_{\zeta\beta}\end{aligned} \tag{3.52}$$

where

$$G_R^{-1}(12)=G_f^{-1}(12)-\Sigma_{\hat{\psi}\zeta}(12) \tag{3.53}$$

and the interacting C_ζ is given by

$$C_\zeta(12)=-G_R(1\bar{1})[2\tilde{\pi}(\bar{1}\bar{2})-\Sigma_{\hat{\psi}\hat{\psi}}(\bar{1}\bar{2})]G_R(2\bar{2})\ . \tag{3.54}$$

Exploiting the above results, (3.43) can be rewritten in the form

$$\begin{aligned}C(12)=&B(1\bar{1})B(2\bar{2})C_m(\bar{1}\bar{2})+C_\zeta(12)\\&+\langle[N(1)-m(1)]\zeta(2)\rangle\\&+\langle\zeta(1)[N(2)-m(2)]\rangle\ ,\end{aligned} \tag{3.55}$$

where

$$B(12)=\delta(12)-a(12)+G_R(1\bar{3})\Sigma_{\hat{\psi}m}(\bar{3}2)\ . \tag{3.56}$$

This form is useful for a perturbative analysis.

D. First-order correction

The first-order correction to the structure factor (3.55), except for the last two terms which will be discussed separately, requires the evaluation of the self energies. The first step is the construction of the effective interaction, which is obtained by taking the logarithm of the non-Gaussian part of the total distribution (3.7). To first order in the coupling constant we find

$$\begin{aligned}A_I^{\rm eff}&=-\ln{\rm Tr}_\sigma Q[\sigma\mid m](1-V)\\&=A_J(m,\phi)+\tilde{A}_I(m)=A_I^{(1)}\end{aligned} \tag{3.57}$$

where

$$A_J(m,\phi)=-\frac{3u}{2}\int d1\,\Gamma(\mathbf{R}_1,\mathbf{R}_1)[2m(1)\phi(1)+\phi^2(1)]\ , \tag{3.58}$$

$$\tilde{A}_I(m)=\int d1\,d2\,i\hat{\psi}(1)u\,\Gamma(12)[2m(2)\phi^2(2)+\phi^3(2)] \tag{3.59}$$

and ϕ is to be evaluated in terms of ζ and m using (3.9).

The first-order self-energies are then computed using

$$-\Sigma_{\alpha\beta}(12)=\left\langle\frac{\delta^2}{\delta\Psi_\alpha(1)\delta\Psi_\beta(2)}A_I^{(1)}\right\rangle_0\ , \tag{3.60}$$

where Ψ_α is a vector whose components are the independent fields ζ, m, $\hat{\psi}$, and $\hat{m}$. From (3.60), after some straightforward algebra, we find that the only nonvanishing first-order self-energies are

$$\Sigma_{\zeta\hat{\psi}}(12)=-3iu\Gamma(12)[S_\zeta(1)+S_{aa}(1)-2S_a(1)] \tag{3.61}$$

and

$$\Sigma_{m\hat{\psi}}(12)=-3iu\Gamma(12)[S_\zeta(1)+S_{aa}(1)] +3iu\Gamma(2\bar{2})a(\bar{2}1)[S_\zeta(\bar{2})+S_{aa}(\bar{2})-2S_a(\bar{2})] , \tag{3.62}$$

where the moments S_ζ, S_a, S_{aa} have been introduced in Sec. III B.

The correction to the structure factor, which cannot be extracted from a self-energy, is the quantity

$$V(12)=\langle[N(1)-m(1)]\zeta(2)\rangle . \tag{3.63}$$

This vanishes at zeroth order and at first order, before doing the Ising sum, is given by

$$V(12)=-\langle[N(1)-m(1)]\zeta(2)[A_J+\tilde{A}_I+V_\sigma]\rangle_0 . \tag{3.64}$$

The calculation of this contribution is rather lengthy but it leads to the simple final result

$$V(12)=6iu\Gamma(\bar{1}1)G_f(2\bar{1})S_a^2(1) . \tag{3.65}$$

Using the results (3.61), (3.62), (3.65) and that all other self-energies vanish to this order, one can evaluate the structure factor (3.55) to first order in V. We shall settle here for investigating the long-time limit for the Bragg peak. We show in Sec. IV that $S_a(t)$ and $S_{aa}(t)$ vanish as $t\to\infty$, it is also easy to show that $a(\mathbf{q}=0,tt)$, which enters the Fourier transform of $\Sigma_{m\hat{\psi}}$, also vanishes as $t\to\infty$. Clearly at zeroth order in the long-time limit

$$C_\zeta^0(\mathbf{q})=\frac{T}{cQ^2(\mathbf{q})+2|r|} \tag{3.66}$$

and

$$S_\zeta(1)=\langle\zeta^2(1)\rangle\int\frac{d^dq}{(2\pi)^d}C_\zeta^0(\mathbf{q}) . \tag{3.67}$$

We can then evaluate for large t_1 and t_2:

$$\Sigma_{\zeta\hat{\psi}}(\mathbf{q},t_1t_2)=-3iu\Gamma(\mathbf{q})\delta(t_1-t_2)\langle\zeta^2(t_1)\rangle =\Sigma_{m\hat{\psi}}(\mathbf{q},t_1t_2) . \tag{3.68}$$

We then obtain the result for the structure factor

$$C(\mathbf{q})=C_\zeta(\mathbf{q})+M^2\delta(\mathbf{q}) , \tag{3.69}$$

where $C_\zeta(\mathbf{q})$ is given by (3.53), (3.54), and (3.68) and

$$M=M_0\left[1-\frac{3\langle\zeta^2(1)\rangle}{2M_0^2}\right] . \tag{3.70}$$

Comparing with (2.7) and (2.9), we find the same results to first order in the temperature obtained from the equilibrium theory. Therefore, the features of the dynamical solution which we have emphasized in the zero-order theory—growth of a peak and equilibration to the correct final state—do persist when first-order corrections are taken into account.

IV. RESULTS

We now present some of the results that can be obtained from the lowest-order theory. The basic equations for the equal-time correlation functions were derived in Sec. III are given by (3.21), (3.22), (3.24), (3.26), (3.28)–(3.30), and (3.15), which relates the structure factor to its various pieces. We will study both the case where the order parameter is conserved and where it is nonconserved and results in both two and three dimensions will be given. We choose to carry out our calculations in an isotropic continuum, introducing a cutoff Λ_0 in $\mathbf{q}$ space to avoid the ultraviolet divergence in the computation of $S(t)$. Alternatively, we could perform the calculation on a lattice.

Defining the dimensionless variables

$$\mathbf{k}=\left[\frac{c}{|r|}\right]^{1/2}\mathbf{q} \tag{4.1}$$

and

$$\tau=\frac{2\Gamma_a|r|}{c}t , \tag{4.2}$$

with $\Gamma_a=\Gamma c$ for NCOP and $\Gamma_a=D|r|$ for COP, all the differential equations satisfied by the various equal-time correlation functions are of the general form

$$\left[\frac{\partial}{\partial\tau}+\gamma(\mathbf{k},\tau)\right]f(\mathbf{k},\tau)=h(\mathbf{k},\tau) . \tag{4.3}$$

For example, the equation for the peak contribution (3.22) can be rewritten in the dimensionless form

$$\left[\frac{\partial}{\partial\tau}+k^2[k^2-1+\tilde{S}(\tau)]\right]\tilde{C}_m(\mathbf{k},\tau) =\epsilon[1-\tilde{S}(\tau)](1-e^{-k^4\tau})+\tilde{g}(\mathbf{k})\delta(\tau-\tau_0) , \tag{4.4}$$

where

$$\tilde{C}_m(\mathbf{k},\tau)=\frac{u}{|r|}\left[\frac{|r|}{c}\right]^{d/2}C_m(\mathbf{q},t) \tag{4.5}$$

is the dimensionless structure factor, and

$$\tilde{S}(\tau)=\frac{u}{|r|}S(t) \tag{4.6}$$

is the corresponding moment, with a dimensionless cutoff Λ, related to Λ_0 as in (4.1). We have chosen $\Lambda=1$ in all of the calculations reported here. The quantity

$$\epsilon=\frac{uT}{r^2}\left[\frac{|r|}{c}\right]^{d/2} \tag{4.7}$$

is the dimensionless coupling characterizing the final equilibrium state. The shared initial condition for the dimensionless correlation functions $\tilde{C}_m$, $\tilde{C}_\zeta^0$, $\tilde{C}_a$, and $\tilde{C}_{aa}$ is given by

$$\tilde{g}(\mathbf{k})=\frac{u}{|r|}\left[\frac{|r|}{c}\right]^{d/2}g_0(\mathbf{q}) . \tag{4.8}$$

The solution of (4.3), with an initial condition $f(\mathbf{k},\tau_0)$, is given by

$$f(\mathbf{k},\tau)=\exp\left[-\int_{\tau_0}^{\tau}d\bar{\tau}\,\gamma(\mathbf{k},\bar{\tau})\right]f(\mathbf{k},\tau_0)+\int_{\tau_0}^{\tau}d\tau'\,h(\mathbf{k},\tau')\exp\left[-\int_{\tau'}^{\tau}d\bar{\tau}\,\gamma(\mathbf{k},\bar{\tau})\right]. \quad (4.9)$$

Rewriting (4.9) with the time argument $\tau+\Delta$, where Δ is a small time step, it is straightforward to derive

$$f(\mathbf{k},\tau+\Delta)=e^{-\gamma(\mathbf{k},\tau)}f(\mathbf{k},\tau)+\frac{h(\mathbf{k},\tau)}{\gamma(\mathbf{k},\tau)}(1-e^{-\gamma(\mathbf{k},\tau)\Delta}) \quad (4.10)$$

which is correct to $O(\Delta^2)$. Hence the time evolution of $f(\mathbf{k},\tau)$ can be generated by iteration of (4.10), keeping in mind that at each time step the computation of $\gamma(\mathbf{k},t)$ involves an integration over $\mathbf{k}$ to obtain the moment $\tilde{S}(\tau)$. This is the only step which is performed numerically in the following analysis.

The driving terms appearing on the right-hand side of (4.3) are of two types: One is noise driven, proportional to the final quenching temperature (ϵ). The other is proportional to the initial value of the correlations. If both are zero, $\tilde{C}(\mathbf{k},\tau)$ remains zero at all times. If either is nonzero, there is growth.

It is generally believed that the long-time properties of these systems are the same for either type of driving terms: Quenches to finite temperatures with zero initial conditions are expected to give the same long-time behav-

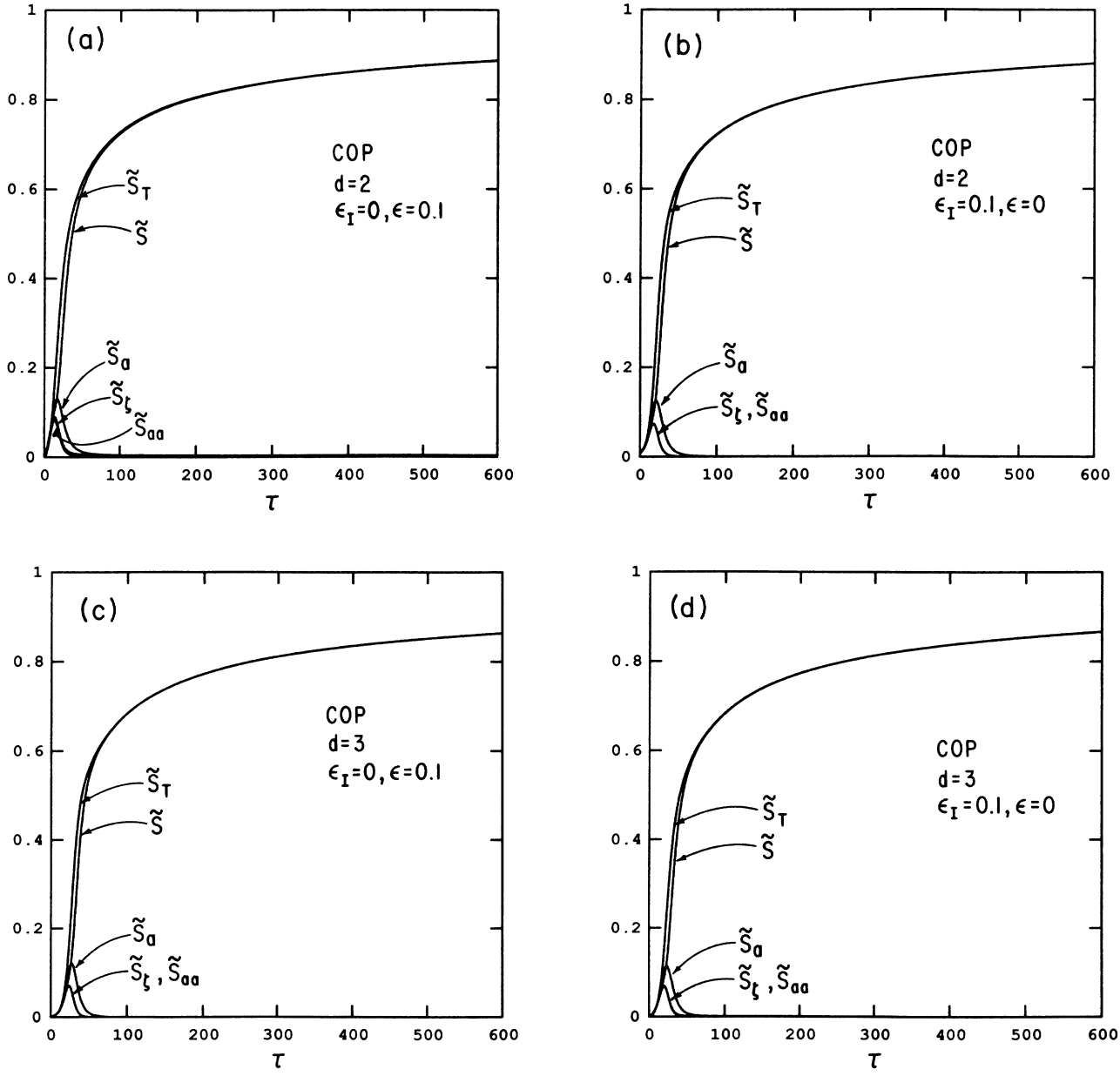


FIG. 2. Moments for COP: $\tilde{S}_{aa}$, $\tilde{S}_\zeta$, $\tilde{S}_a$, $\tilde{S}$, $\tilde{S}_T$. $\tilde{S}_{aa}$ and $\tilde{S}_\zeta$ are distinguishable only in panel (a).

ior as quenches to zero temperature averaged over many randomly chosen initial conditions. We can easily study this question here. As we shall see, our results confirm this widespread belief.

Given the noise level (ϵ), the initial condition $\tilde{g}(\mathbf{k})$ and the cutoff Λ, we can proceed to determine the various correlation functions entering (3.15), which we rewrite in scaled form

$$\tilde{C}(\mathbf{k},\tau)=\tilde{C}_m(\mathbf{k},\tau)+\tilde{C}_{\zeta}^{0}(\mathbf{k},\tau)+\tilde{C}_{aa}(\mathbf{k},\tau)-2\tilde{C}_a(\mathbf{k},\tau) \tag{4.11}$$

and the corresponding dimensionless moments

$$\tilde{S}_T(\tau)=\tilde{S}(\tau)+\tilde{S}_{\zeta}(\tau)+\tilde{S}_{aa}(\tau)-2\tilde{S}_a(\tau)\ . \tag{4.12}$$

In Fig. 2 we examine the moments appearing in (4.12) for a COP over a time range ($\tau \lesssim 600$) which, although short compared with the times we consider later on, is already comparable to the longest times ever considered in computer simulations.[42] The panels (a) and (b) correspond to $d=2$, while (c) and (d) correspond to $d=3$. The panels (a) and (c) correspond to the initial condition $\tilde{g}(\mathbf{k})=0$ and $\epsilon=0.1$. while (b) and (d) examine the zero-temperature case $\epsilon=0$ with initial conditions give by:

$$\tilde{g}(\mathbf{k})=\epsilon_I\ . \tag{4.13}$$

These uniform initial conditions correspond to an equilibrium state with $u_I=0$ and $r_I \gg cQ^2(\mathbf{k})$ for all $|k|<\Lambda$. From (4.7), (4.8), and (2.9), it follows that

$$\epsilon_I=\frac{u}{|r|}\left[\frac{|r|}{c}\right]^{d/2}\frac{T_I}{r_I} \tag{4.14}$$

and to the usual assumption of a spatially disordered initial field:

$$\langle \psi_0(\mathbf{R})\psi_0(\mathbf{R}')\rangle=\frac{T_I}{r_I}\delta(\mathbf{R}-\mathbf{R}')\ . \tag{4.15}$$

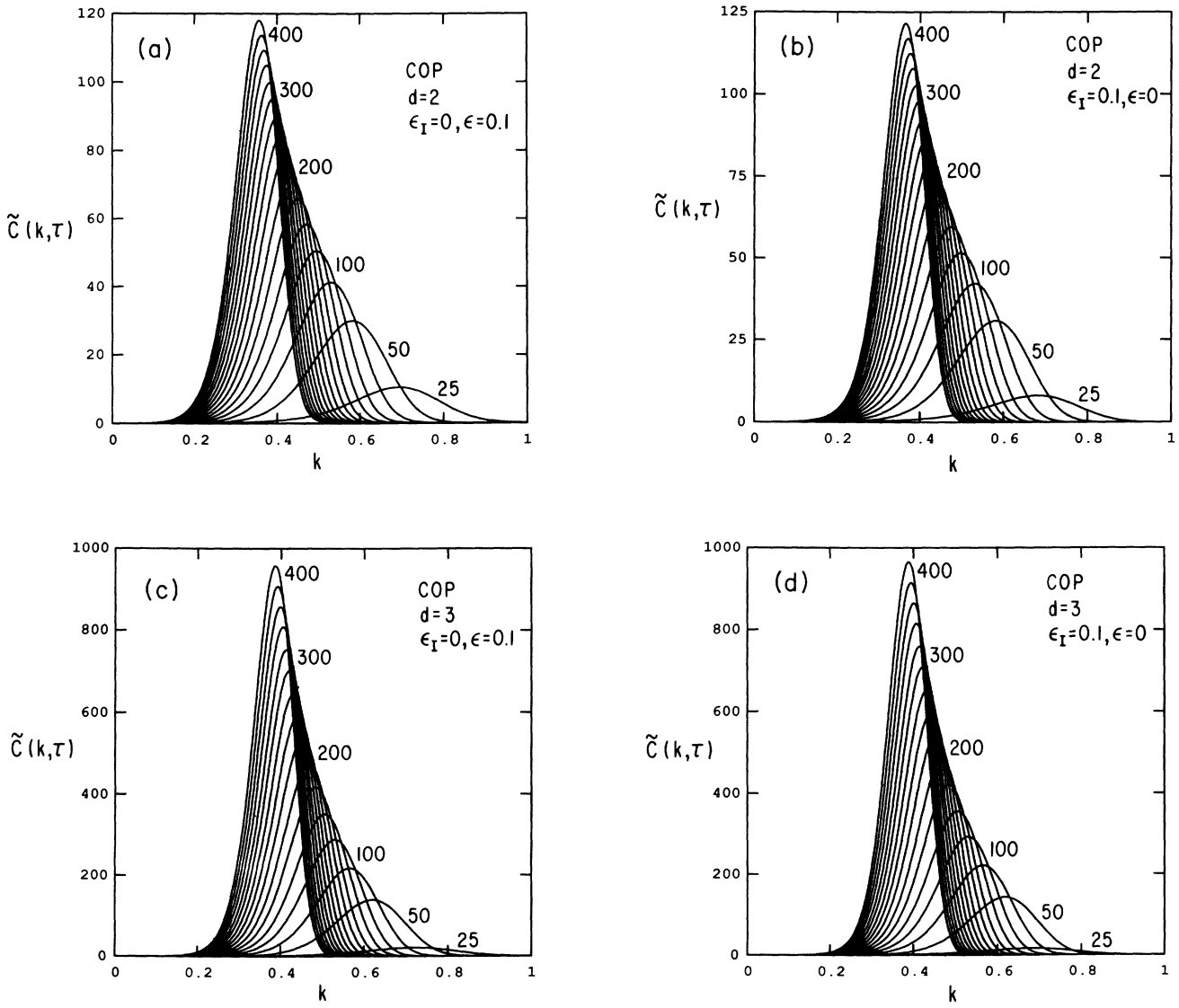


FIG. 3. Time evolution of $\tilde{C}(k,\tau)$ in time steps of 25 units for COP.

The curves in Fig. 2 contain a large amount of information. The overall gross feature for very early times is that all of the moments grow rapidly $[\tilde{S}(\tau)\sim e^{\tau/4}/\sqrt{\tau}]$ and the variables $\sigma(1)$, $\zeta(1)$, and $a(1\bar{1})m(\bar{1})$ are all initially unstable. However one finds as $\tilde{S}$ increases that the $W(t)$ functions associated with the correlation functions $\tilde{C}_m$, $\tilde{C}^0_\zeta$, $\tilde{C}_a$, $\tilde{C}_{aa}$ increase from their shared initial negative value of -1 (in dimensionless units) toward zero. When $\tilde{S}(\tau)\sim\frac{1}{3}$, one finds that the mass terms for $\tilde{C}^0_\zeta$ and $\tilde{C}_{aa}$ pass through zero and their subsequent positive value causes $\tilde{S}_\zeta$ and $\tilde{S}_{aa}$ to reach maximum values and begin exponential decay toward their final equilibrium values:

$$\tilde{S}_\zeta(\infty)=\int\frac{d^dk}{(2\pi)^d}\frac{\epsilon}{k^2+2} \tag{4.16}$$

and $\tilde{S}_{aa}=0$. As time evolves and $\tilde{S}(\tau)$ approaches $\frac{1}{2}$ the same change of sign occurs in the $S_a(\tau)$ mass term, causing the subsequent decay of this moment to zero. During the latest stages $\tilde{S}$ approaches 1 and the mass associated with the peak goes to zero as described below (3.32). Since the area under the central peak tends to the magnetization squared for long times:

$$M^2=\int\frac{d^dq}{(2\pi)^d}C_m(\mathbf{q})=S(\infty) \tag{4.17}$$

we obtain, using (4.6),

$$M_0^2=\frac{-r}{u} \tag{4.18}$$

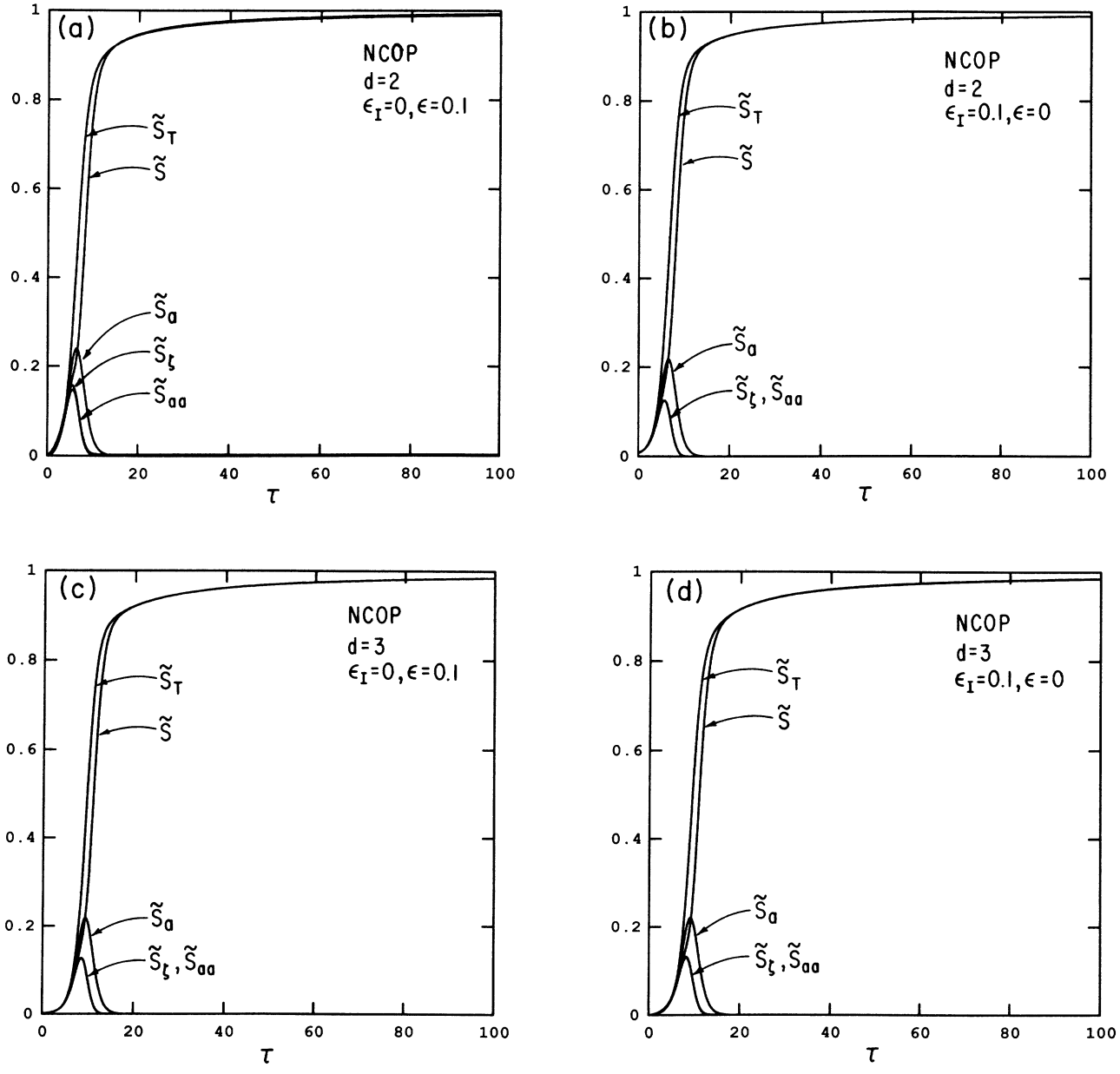


FIG. 4. Moments for NCOP: $\tilde{S}_{aa}$, $\tilde{S}_\zeta$, $\tilde{S}_a$, $\tilde{S}$, $\tilde{S}_T$. $\tilde{S}_{aa}$ and $\tilde{S}_\zeta$ are distinguishable only in panel (a).

as expected.

Note there is no major difference between quenches to zero temperature and nonzero initial conditions and quenches to nonzero temperature with zero initial conditions, thus, the expectation that initial condition averaging and noise averaging are equivalent appears warranted. Finally we observe that dimensionality seems to play a minor role, except for a change in the time scale.

Once the various moments shown in Fig. 2 are obtained one can proceed to evaluate the singlet distribution function given by (3.38). Figure 1 shows $\rho(y)$ multiplied by 2π, in terms of the dimensionless time defined by (4.2) and y measured in units of M_0. The parameters characterizing the system are $\epsilon=0.1$ and $\tilde{g}(\mathbf{k})=0$.

In Fig. 3 we show $\tilde{C}(\mathbf{k},\tau)$ as a function of k for several values of τ in the intermediate time regime. The four panels in the figure correspond to the same four curves as in Fig. 2 and similar comments apply. The evolution of the wave vector $k_p(\tau)$, corresponding to the maximum value of $\tilde{C}(\mathbf{k},\tau)$, toward smaller values can be clearly seen. Its behavior with time is analyzed in more detail below.

We turn, in Figs. 4 and 5, to the case where the order parameter is not conserved. The moment $\tilde{S}_T(\tau)$ and its constituents are shown in Fig. 4. Except for a large change in the time scale reflecting a much faster equilibration, the qualitative behavior is as in the conserved case. In Fig. 5 results are shown for $\tilde{C}(\mathbf{k},\tau)$ for a NCOP. Here we have the very obvious difference, compared to the COP case, that the ordering peak now grows at $k=0$.

We turn now to an examination of asymptotic behaviors. For very short times and with initial conditions such that $\tilde{S}(\tau)\ll 1$, one can easily determine all of the correlation functions and their moments. One obtains,

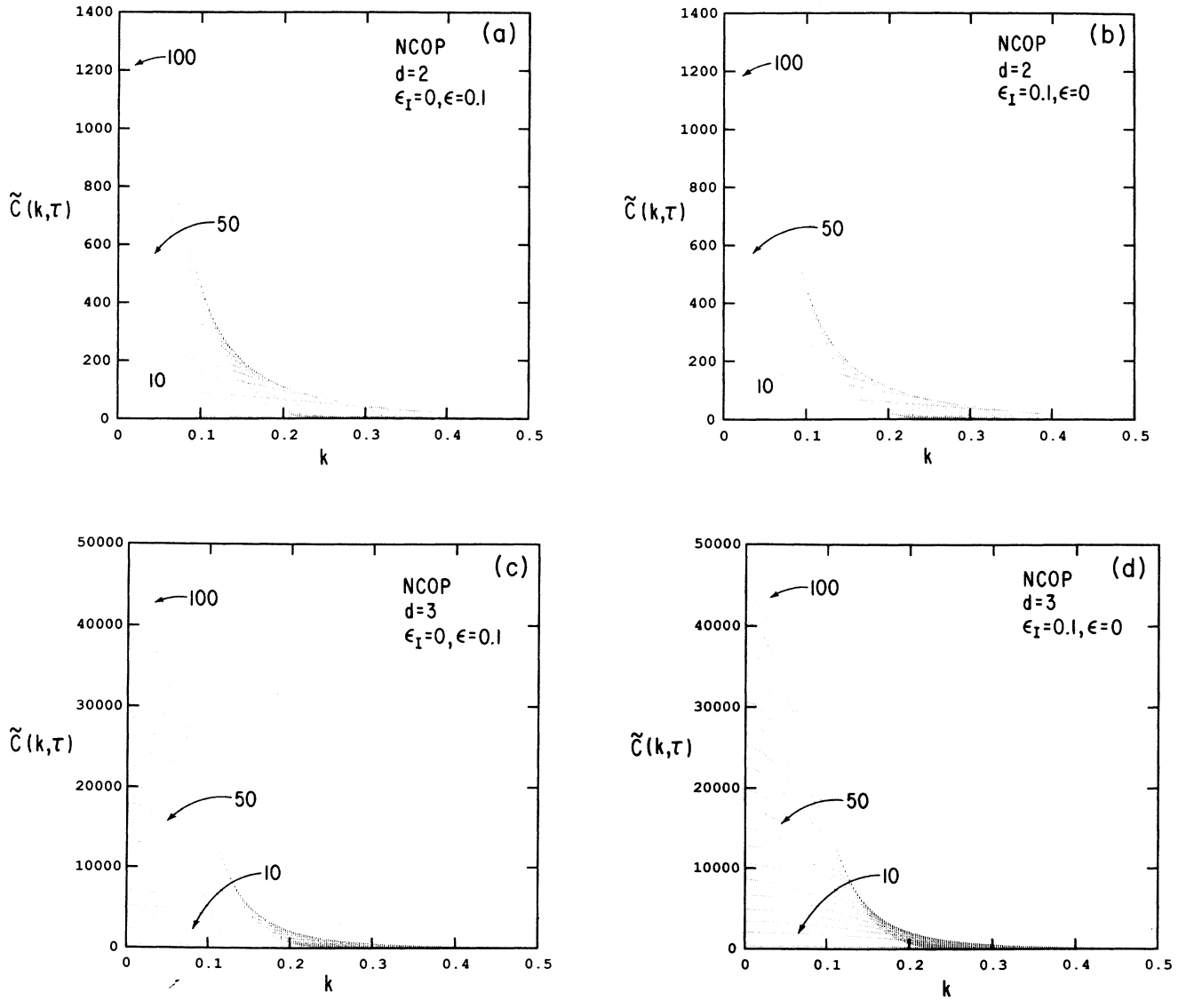


FIG. 5. Time evolution of $\tilde{C}(k,\tau)$ in time steps of 5 units for NCOP.

for example, for a COP:

$$\tilde{C}_m(\mathbf{k},\tau)=e^{-\tau k^2(k^2-1)}[\tilde{g}(\mathbf{k})+\epsilon k^2(1-e^{-k^2\tau})] , \quad (4.19)$$

and

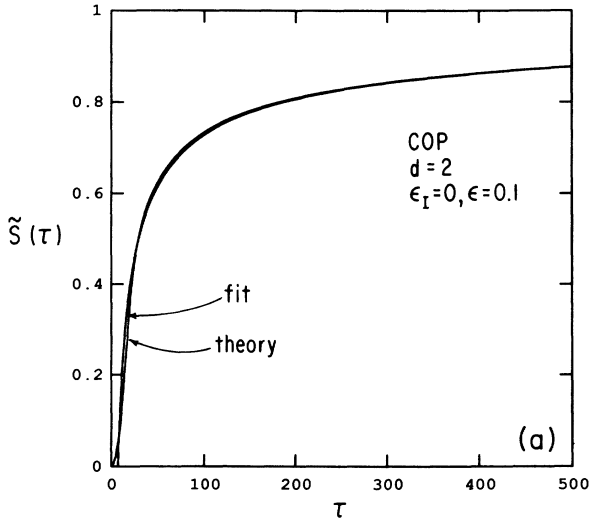
$$\tilde{C}^{0}_{\zeta}(\mathbf{k},\tau)=e^{-\tau k^2(k^2-1)}\left[\tilde{g}(\mathbf{k})+\frac{\epsilon}{k^2-1}(e^{\tau k^2(k^2-1)}-1)\right] . \quad (4.20)$$

The moments are seen to grow as $e^{\tau/4}/\sqrt{\tau}$ as pointed out above. This exponential growth is clearly associated with the Cahn-Hilliard mechanism.

We turn next to the long-time behavior. We consider first the behavior of the moments at long times. In this regime $\tilde{S}_T(\tau)=\tilde{S}(\tau)+\tilde{S}(\tau)+\tilde{S}_{\zeta}(\infty)$. Since the latter is a constant at long times, the time dependences of $\tilde{S}_T(\tau)$ and $\tilde{S}(\tau)$ are identical. In Fig. 6 we show the long-time behavior of $\tilde{S}(\tau)$ for a COP. The maximum time depicted is nearly one order of magnitude larger than the longest time attained in numerical simulations. We find that as $\tau\to\infty$:

$$\tilde{S}(\tau)=1-\frac{A_s}{\tau^{1/2}}+\frac{B_s}{\tau^{3/4}}+O\left[\frac{1}{\tau}\right] . \quad (4.21)$$

The parameters A_s and B_s depend weakly on ϵ and ϵ_I. For the parameters characterizing Fig. 6(a) ($\epsilon=0.1,\epsilon_I=0$), $A_s=2.77$ and $B_s=0.15$. We have studied $A_s(\epsilon,\epsilon_I=0)$ and $A_s(\epsilon=0,\epsilon_I)$ from a fit to $\tilde{S}(\tau)$ for times $\tau>20$, and found that $A_s=a+b\ln\tilde{\epsilon}$, where $\tilde{\epsilon}$ is either ϵ or ϵ_I, whichever is nonvanishing, and $a=2.4$, $b=-0.19$ for $\epsilon_I=0$ while $a=3.7$ and $b=-0.14$ for



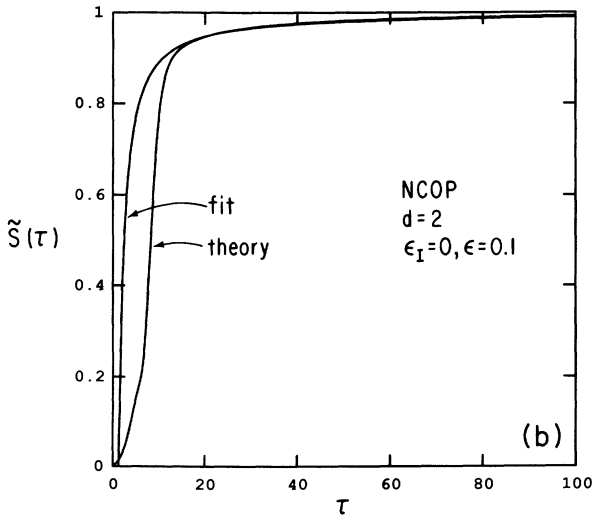
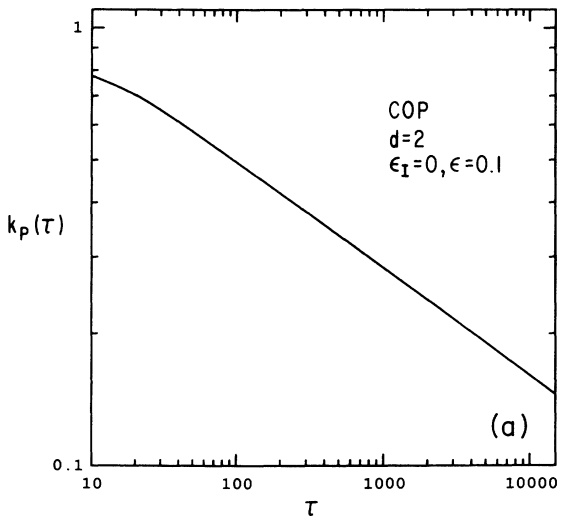


FIG. 6. Best fit for $\tilde{S}(\tau)$ vs τ in $d=2$. (a) COP, $\tilde{S}(\tau)=1-2.77\tau^{-1/2}+0.15\tau^{-3/4}$; (b) NCOP, $\tilde{S}(\tau)=1-0.97\tau^{-1}+0.40\tau^{-3/2}$.



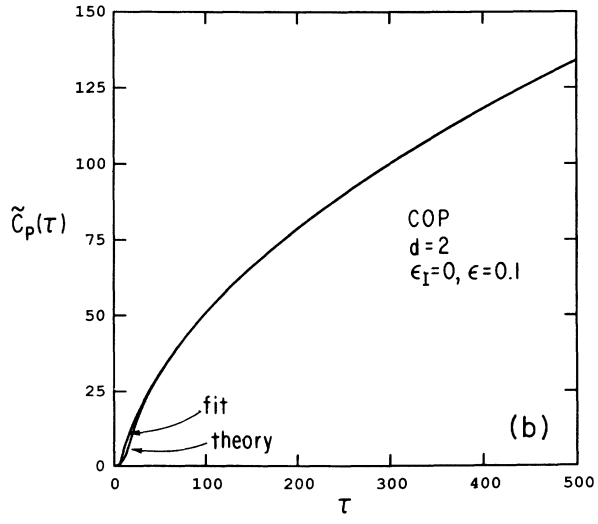


FIG. 7. Quench with COP from $\epsilon_I=0$ to $\epsilon=0.1$ in $d=2$. (a) k_p vs τ. Data coincide with best fit $k_p^{-1}=0.025+0.62\tau^{1/4}$; (b) $\tilde{C}_p(\tau)$ vs τ. Data coincide with best fit $\tilde{C}_p(\tau)=6.83\tau^{1/2}-0.69\tau^{1/4}-15.4$.

$\epsilon=0$. The intermediate time behavior where $\tau<20$, but beyond the initial exponential region is hard to characterize quantitatively.

When the order parameter is not conserved we find that $\tilde{S}(\tau)$ reaches its asymptotic value according to:

$$\tilde{S}(\tau)=1-\frac{A_s}{\tau}+\frac{B_s}{\tau^{3/2}}+O\left[\frac{1}{\tau^2}\right], \tag{4.22}$$

where, for example, for two dimensions, $\epsilon=0.1$ and $\epsilon_I=0$, the values $A_s=0.97$, $B_s=-0.40$ give a good fit over the time interval $20<\tau<300$. It is not surprising that the asymptotic limit is reached earlier and with a faster power law in the absence of a conservation law.

We now turn to the long-time behavior of the structure factor itself. To characterize the domain growth, in the conserved case we use the position $k_p(\tau)$ of the peak in $\tilde{C}(\mathbf{k},\tau)$. We also define the peak height, $\tilde{C}_p(\tau)=\tilde{C}[k_p(\tau),\tau]$. For the case $\epsilon=0.1$ and zero initial conditions, we find at $d=2$ the results shown in Fig. 7. In all cases considered we find that $k_p^{-1}(\tau)$ and $\tilde{C}_p(\tau)$ are given asymptotically by:

$$k_p^{-1}(\tau)=L_0+A\tau^{1/4}, \tag{4.23}$$

$$\tilde{C}_p(\tau)=A_p\tau^{1/2}+B_p\tau^{1/4}+C_p, \tag{4.24}$$

where L_0, A, A_p, B_p, C_p depend weakly on ϵ and ϵ_I. For the case shown in Fig. 7 we find, in the range $2000<\tau<15\,000$, $L_0=0.025$, $A=0.62$ and for the peak height $A_p=6.83$, $B_p=-0.69$, $C_p=-15.4$ in the range $2000<\tau<15\,000$. Analyzing Fig. 7(a) and defining the effective exponent,

$$n_{\text{eff}}(\tau)=\frac{d\ln k_p(\tau)}{d\ln\tau}, \tag{4.25}$$

one can see that $n_{\text{eff}}\to\frac{1}{4}$ monotonically from below.

In three dimensions (4.23) remains valid while (4.24) is replaced by

$$\tilde{C}_p(\tau)=A_p\tau^{3/4}+B_p\tau^{1/2}+C_p\tau^{1/4}+d_p. \tag{4.26}$$

This is in agreement with the expectation that the dimensionality should not affect the growth exponent n, while $\tilde{C}_p\sim\tau^{nd}$ implying the scaling law $\tilde{C}_p k_p^d=A_p/A^d$. We show in Fig. 8 the results for $k_p(t)$ obtained in three dimensions for $\epsilon=0$, $\epsilon_I=0.1$. In this case, a fit in the range $10^3<\tau<10^4$ yields $L_0=0.29$, $A=0.51$, $A_p=11.1$, $B_p=13.7$, $C_p=-81.6$, and $D_p=65.6$.

The development of asymptotic behavior occurs much earlier when the order parameter is not conserved. We then find in this case that the width of the peak at $k=0$ decreases asymptotically as:

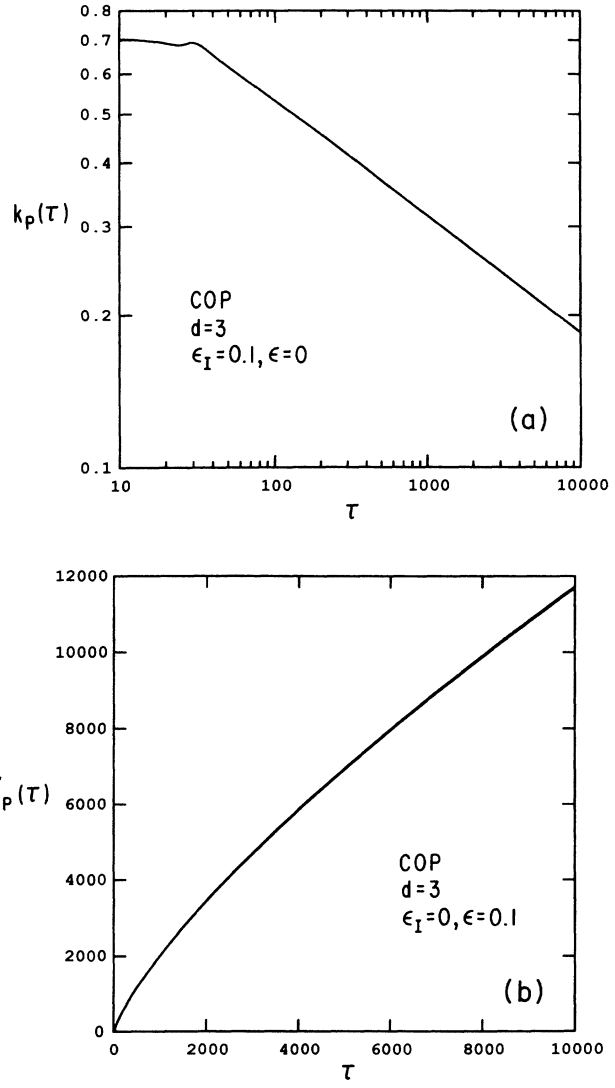


FIG. 8. Quench with COP from $\epsilon_I=0$ to $\epsilon=0.1$ in $d=3$. (a) k_p vs τ. Data coincide with best fit $k_p^{-1}=0.29+0.51\tau^{1/4}$; (b) $\tilde{C}_p(\tau)$ vs τ. Data coincide with best fit $\tilde{C}_p(\tau)=11.1\tau^{3/4}+13.2\tau^{1/2}-33.5\tau^{1/4}+47.1$.

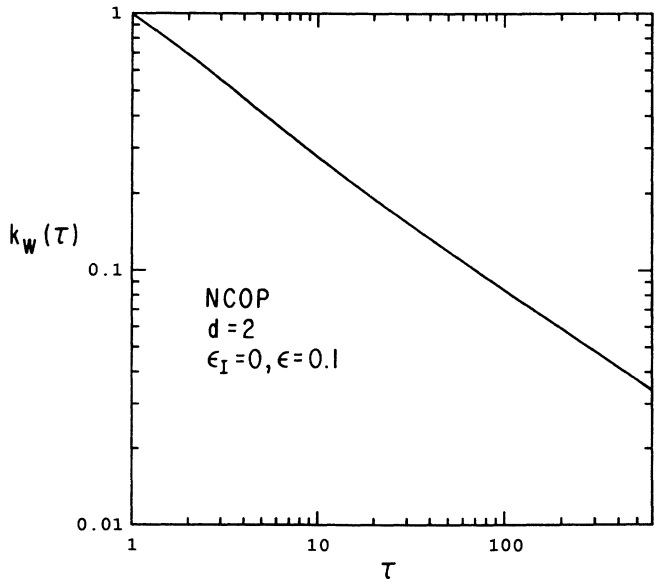


FIG. 9. $k_W(\tau)$ vs τ. Data coincide with best fit $k_W^{-1}(\tau)=1.20\tau^{1/2}-0.13$.

$$k_W^{-1}(\tau)=A_W\tau^{1/2}+B_W \tag{4.27}$$

which is in agreement with the Lifshitz-Cahn-Allen law, as expected. A ln-ln plot of k_W versus τ for the case $\epsilon=0.1$, $\epsilon_I=0$ is given in Fig. 9. The fit to (4.27) with $A_W=1.21$ and $B_W=-0.13$ cannot be distinguished from the solution for k_W.

In the long-time regime we have also verified that scaling is satisfied. In the asymptotic region the structure factor for a COP can be written as

$$\tilde{C}(\mathbf{k},\tau)=\tilde{C}_p(\tau)F[\mathbf{k}/k_p(\tau)] \tag{4.28}$$

with $F(1)=1$. The scaling function $F(x)$ is plotted in Fig. 10 for $d=2$ and $d=3$ (narrower peak) in the case of a quench with $\epsilon_I=0$, $\epsilon=0.1$.

We have also obtained the shape function for the nonconserved case where we define $F(x)$ by:

$$\tilde{C}(\mathbf{k},\tau)=\tilde{C}(0,\tau)F[\mathbf{k}/k_W(\tau)]\ . \tag{4.29}$$

In this case the shape function is Gaussian for x not too large, as it is demonstrated by plotting (Fig. 11) $\ln F(x)$ versus x^2. In Fig. 11 the shape functions for a quench in $d=2$ and $d=3$ (with $\epsilon_I=0$, $\epsilon=0.1$) are compared and, as in the COP case, a narrower peak corresponds to the higher dimensionality as it should be expected. We have found in all cases (COP and NCOP) that $F(x)$ decreases as $|x|^{-4}$ for large values of $|x|$ in both two and three dimensions. This is in agreement with Porod's law ($x^{-(d+1)}$) in three dimensions, but not in two. It is possible to show that the $|x|^{-4}$ result follows analytically from substituting (4.21) into (4.4), and carefully examining the long-time limit.

While the long-time result for the growth law in the nonconserved case [$L(t)\sim t^{1/2}$] is well established, the exponent for the conserved case is more controversial.

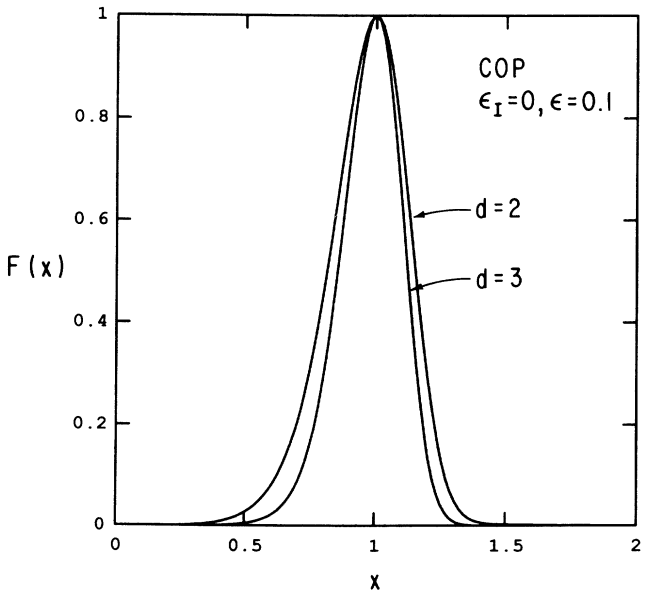


FIG. 10. Scaling function $F(x)$ vs x for COP in two and three dimensions.

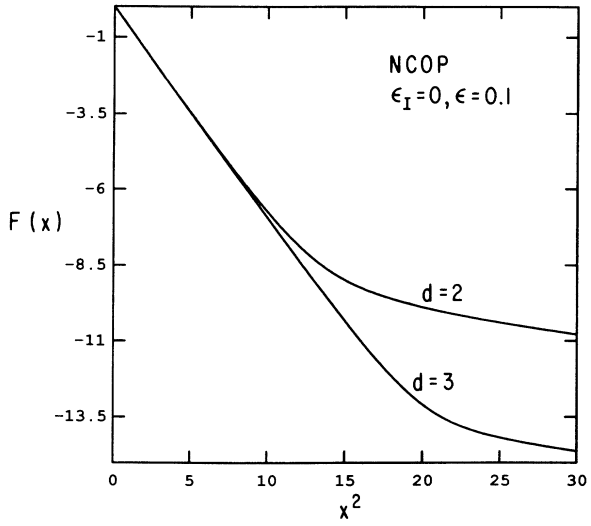


FIG. 11. Scaling function $F(x)$ vs x^2 for NCOP in two and three dimensions.

Recent numerical simulations[4] of (2.1) extend to relatively early times ($\tau<2000$) and yield results varying between $\frac{1}{4}$ and $\frac{1}{3}$. Taking into account the uncertainty due to the numerical methods, these results are compatible with the asymptotic behavior obtained here. A detailed comparison between the two methods will be made in future work.

V. CONCLUSIONS

In this paper we have addressed the long-standing open problem of constructing a theory which is capable of describing the entire time evolution of a system undergoing a nonequilibrium process from an initial disordered state to a final ordered state. We have done this in the context of a field-theoretic method for Langevin dynamics and we have developed a systematic low-temperature perturbation expansion. The lowest-order theory has been analyzed in detail producing all the expected features of the global time evolution, such as a reduced singlet probability distribution evolving from an initial Gaussian to a final bimodal distribution, the separation of the domain size $L(t)$ from the correlation length $\xi(t)$ generated by the dynamics, and the equilibration to a final ordered state without Nambu-Goldstone modes. The growth law at long time $L(t)\sim t^n$ has also been computed to lowest order, obtaining $n=\frac{1}{4}$ for COP and $n=\frac{1}{2}$ for NCOP. We find that these models are of the class-I type introduced in Ref. 9, with no activated processes at very low temperatures.

The stability of these results has been checked when first-order corrections are taken into account, and it has been found that no change occurs in the growth law, while the final equilibrium quantities, like the magnetization and the correlation length, acquire the appropriate first-order corrections in the temperature.

Also of interest is the fact that the method is not limited to purely uniform boundary conditions. It is in principle possible and in practice rather straightforward to use the basic equations of Sec. IV with nonuniform initial conditions. One can study, for example, the evolution of hot drops in a fluid[42] or the evolution of an initially hot system quenched in the presence of an ordered boundary. These problems contain the germ of the physics which must be introduced to study problems such as dendritic growth, domain wall motion, and nucleation theory. We believe that the extension of the perturbation methods discussed here to more complicated and physically relevant models is not unduly difficult.

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*Permanent address: Dipartimento di Fisica, Università di Napoli, Mostra d'Oltremare Pad. 19, Napoli, 80125 Italy.

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[27]Throughout the paper we develop the theory in the continuum formalism. The discretized theory on a lattice with spacing a is obtained by the replacements $\int d^dR \to (1/a^d)\sum_R$, where the sums run over lattice vectors, and

$$[\nabla\psi(\mathbf{R})]^2 = \frac{1}{2a^2}\sum_{\delta}[\psi(\mathbf{R})-\psi(\mathbf{R}+\delta)]^2 ,$$

$$\nabla^2\psi(\mathbf{R})=\frac{1}{a^2}\sum_{\delta}[\psi(\mathbf{R}+\delta)-\psi(\mathbf{R})]\ ,$$

where the sum runs over nearest-neighbors lattice vectors.

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[37]Note that $Q[\sigma\mid m]$ is not positive definite [since, for example, with $\sigma(l)=-\sqrt{S}(t_l), 1-m(l)/\sqrt{S}(t_l)$ is negative for large positive $m(l)$] and $P_\sigma[\sigma]$ cannot literally be identified as a probability distribution. However, as it will be clear from the computations of Sec. IV, this is not a serious drawback.

[38]It is possible to define C_N somewhat differently by including in the definition of G_∞^{-1} a mass term which vanishes faster than W_p as $t\rightarrow\infty$. We have found that the inclusion of such a term leads to insignificant changes in the short-time evolution of the system and has no effect for long times. We have, therefore, dropped any such mass term.

[39]The time and space derivatives acting on the discrete σ variable have a well-defined meaning in the theory on a space-time lattice, which is the theory computed in the numerical calculations of Sec. IV.

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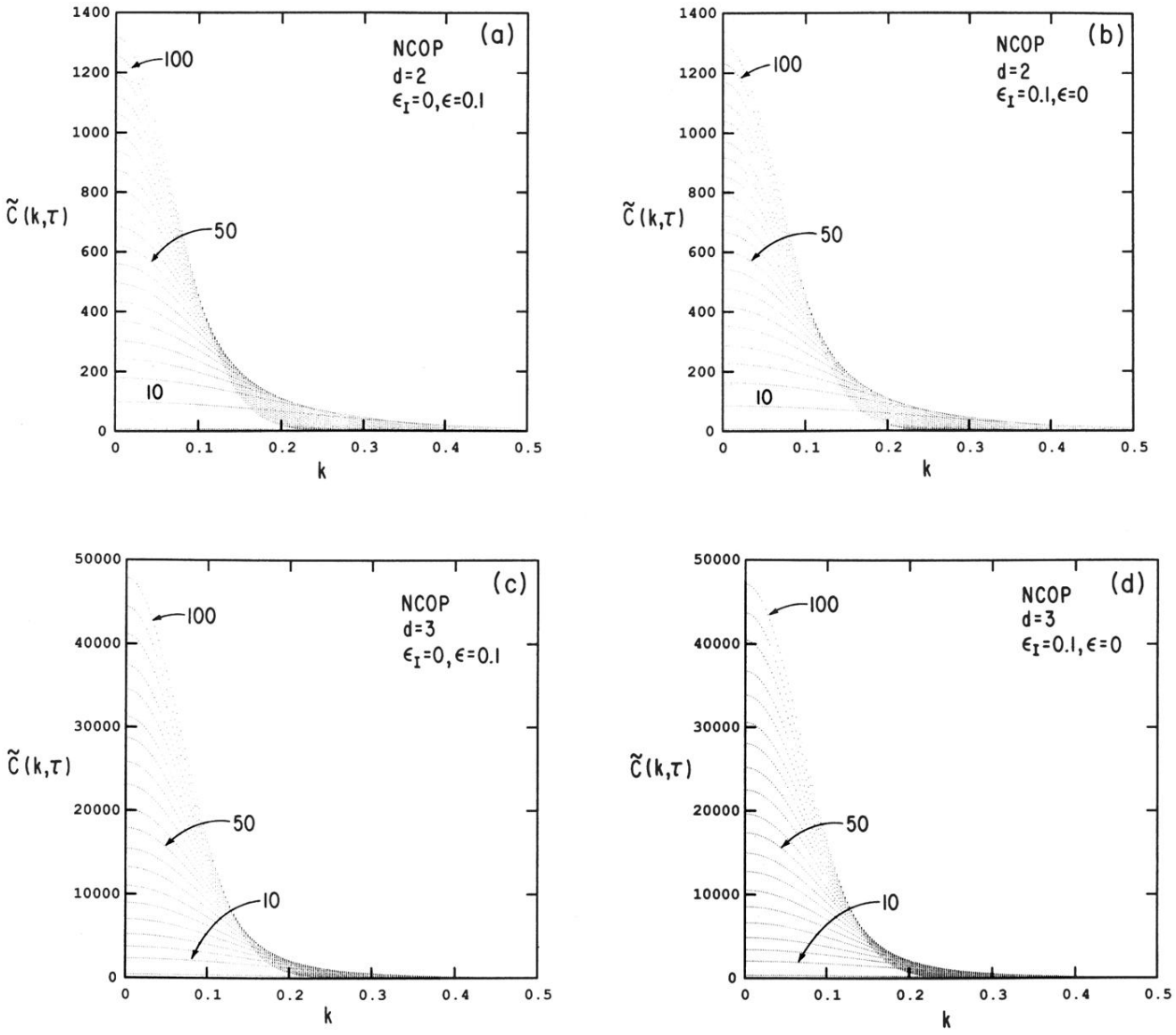

FIG. 5. Time evolution of $\tilde{C}(k,\tau)$ in time steps of 5 units for NCOP.