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Quasiperiodic anisotropic *XY* model

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The quasiperiodic anisotropic *XY* model in one dimension exhibits ordered and disordered phases with Cantor spectra which we characterize in terms of the exponent δ and the $f(\alpha)$ curve. The transition to the long-range-order phase is signaled by a nonanalyticity in δ in addition to the singular behavior of the long-range correlation function. Based on our numerical results, we conjecture that $f(\alpha)$ is a smooth function in the disordered phase, becoming discontinuous in the *ordered* phase. At a special point in the *ordered* phase, the system exhibits a pointlike spectrum with localized states.

In this paper, we describe our zero-temperature study of the one-dimensional quantum quasiperiodic (QP) anisotropic *XY* model

$$H=\sum J(n)\{\sigma^x(n)\sigma^x(n+1)+[1+g(n)]\sigma^y(n)\sigma^y(n+1)\}\,. \qquad (1)$$

Here the $\sigma^j(n)$ are Pauli matrices. The $J(n)$ is the exchange constant and $g(n)$ is the anisotropy which breaks the O(2) symmetry of the *XY* model. The system is made QP by choosing two values of either $J(n)$ or $g(n)$, in a quasiperiodic sequence.[1] In our study, the QP was characterized by the golden-mean ratio $\sigma_g=(\sqrt{5}+1)/2$.

In systems with translational invariance, the onset of long-range order (LRO) is signaled by a discontinuous jump from zero to some finite value in the order parameter which is usually the magnetization. Our study of the model (1), along with our previous study of the QP quantum Ising model,[2] shows that the onset of LRO in QP systems is also characterized by a discontinuity in the magnetization which is a modulating function of sites. However, in QP systems exhibiting a phase transition from a disordered to ordered phase, additional singularities signal the onset of transition and characterize the LRO phase. These singularities are associated with the scaling properties of the eigenvalues of the system which form a Cantor set in both disordered and ordered phases. In this paper, we study the variation in the scaling properties of the Cantor spectrum for various values of the parameters. Based on this and our previous study of the QP Ising model,[2] we conjecture that these new singularities which we identify in the exponent δ and the $f(\alpha)$ curve are generic and characterize the phase transitions in all QP systems exhibiting LRO.

By means of the Jordan-Wigner transformation, Eq. (1) is equivalent to a fermion model, quadratic in fermion degrees of freedom

$$H=\sum[c^\dagger(n)A_{nm}c(m)+(c_nB_{nm}c_m+\text{H.c.})]\,. \qquad (2)$$

Here, the $c(n)$ are anticommuting fermion operators. The matrices A and B are respectively symmetric and antisymmetric matrices with nonzero elements defined in the unit of $J(n)$, as follows: $A_{n,n+1}=1+G(n)$ and $A_{1,N}=1+G(N)$, $B_{n,n+1}=1-G(n)$ and $B_{1,N}=-[1-G(N)]$. Here, $G(n)=1+g(n)$. The N denoted the size of the spin chain. The pure model [$J(n)=J$ and $g(n)=g$] was solved exactly by Lieb, Schultz, and Mattis.[3] For any finite value of g, the system exhibits long-range-order with nonzero long-range correlation. The energy spectrum of the model is continuous with a gap which vanishes at the onset of LRO.

The quasiperiodic isotropic model [$g(n)=0, J(n)=J_1$ or J_2] was studied by Luck.[4] In this limit, the model is equivalent to the tight-binding model with B_{nm} of Eq. (2) equal to zero. Such a model does not exhibit any LRO. These tight-binding models with no LRO have been studied very extensively in the past.[5] The scaling properties of the Cantor spectrum were found to be smooth functions of the parameters.

We will study the quasiperiodic anisotropic model with nonzero values of g for which the system exhibits LRO. The main motivation for this study is to investigate the effect of QP on the phase transition from the disordered to ordered phase with LRO and to see how to characterize the scaling properties and LRO phase in systems with no translational invariance.

In our numerical study, the QP system is approximated by a sequence of periodic systems with a progressively larger unit cell of size F_n. The F_n are the Fibonacci numbers obtained by optimal rational approximations to σ_g. In our study of the LRO phase, we will keep $J(n)$ to be a constant equal to J and the $g(n)$ will form a QP sequence in g_1 and g_2. As described by Lieb, Schultz, and Mattis,[3] the eigenspectrum of the system is obtained by diagonalization of the matrix $(A+B)(A-B)$. This matrix can be viewed as the tight-binding model associated with our problem resulting in the following eigenvalue equation

$$4\{G(n-1)\psi_{n-2}+[1+G^2(n-1)]\psi_n+G(n+1)\psi_{n+2}\}=E^2\psi_n\,. \qquad (3)$$

This corresponds to the tight-binding model with nextnearest-neighbor interaction. It should be noted that for the even N case, the even-site problem completely decouples from the odd-site problem. On the other hand, for N odd, the even and odd sites remain coupled due to periodic boundary conditions.[6]

Figure 1 shows the Cantor spectrum of the model. The goal of this paper is to understand the effect of LRO on the scaling properties of the spectrum. We computed the scaling index δ which describes the scaling of the total allowed bandwidth with the size of the system.[5] This involves calculating the bandwidths of all the bands associated with F_l eigenvalues as the Bloch index k is varied in the first Brillouin zone.[5] Near the onset of long-range order, the exponent δ is found to be a linear function of g (see the inset in Fig. 2), becoming nonanalytical at the transition point. The nonanalyticity of δ in QP systems exhibiting a phase transition from disordered to ordered phase was also seen in our study of the quasiperiodic Ising model in a transverse field. We believe that it is the generic result for all QP systems exhibiting a transition of long-range order. This makes δ a very important quantity in describing the phase transition in QP systems as the onset of the transition is signaled by a nonanalyticity in δ.

The Cantor spectrum of the system is a multifractal with not one but a whole distribution of characteristic exponents. To characterize such a fractal, we use a recently proposed thermodynamical formalism.[7] We define a partition sum Γ, which partitions the set into l pieces

$$\Gamma_l=\sum_{i=1}^{l}(F_l)^{-q}/\omega_i^{\tau}\,, \tag{4}$$

where ω_i is the width of the $i^{\rm th}$ band and F_l^{-1} corresponds to its measure. The normalization condition $\Gamma=1$ defines the function τ in terms of q. The generalized Renyi dimensions[8] $D_q=\tau/(q-1)$ characterize the fractal set in terms of infinity of dimensions. This spectrum of dimensions has been linked to a spectrum of scaling indices commonly known as the $f(\alpha)$ curve by a Lengendre transformation

$$\alpha=d\tau/dq\,, \tag{5}$$

$$f(\alpha)=q\alpha-\tau\,. \tag{6}$$

The sum Γ, $q(\tau)$, and τ^{-1} are interrelated in the same way as the thermodynamic partition function, free energy, and temperature. This can be seen by rewriting Eq. (4) as

$$q=\frac{1}{\ln(F_l)}\ln\sum\exp[-\tau\ln(\omega_i)]\,. \tag{7}$$

Therefore, the $f-\alpha$ formalism is linked significantly to the thermodynamics defined from Gibb's ensemble. This analogy is particularly helpful in providing a new theoretical insight when the function Γ undergoes "phase transition." The exponent α defined above is the local scaling exponent of the integrated density of states for a given energy. The $f(\alpha)$ is the fractal dimension of the subset in the set consisting of all points with scaling index α.

For the pure model, the scaling is trivial almost everywhere with $\alpha=1$ and $f(\alpha)=1$ except at the band edges where the Van Hove singularities give $\alpha=0.5$ with $f(\alpha)=0$. Therefore, theoretically, the $f(\alpha)$ consists of only two points.[9] In a numerical study of the $f(\alpha)$ curve for QP systems, we find that this function is smooth in the disordered phase (see Fig. 3). In our model (1), this behavior was seen when $g=0$ and the exchange couplings J_1 and J_2 are in QP sequence. This also implies a smooth $q-\tau$ plot as shown in Fig. 4. The study of the $f(\alpha)$ curve in the LRO phase is hampered by some convergence problems. However, this study suggests that the function is

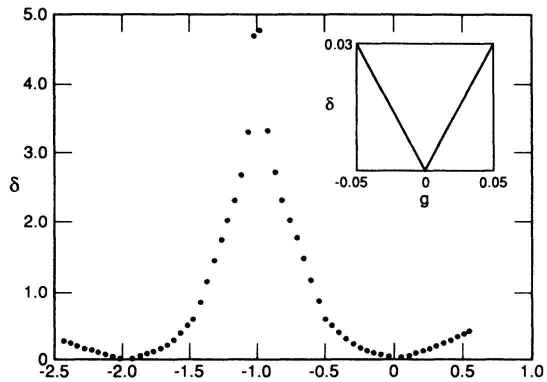

FIG. 2. A plot of exponent δ vs g_1 with $g_2=0$.

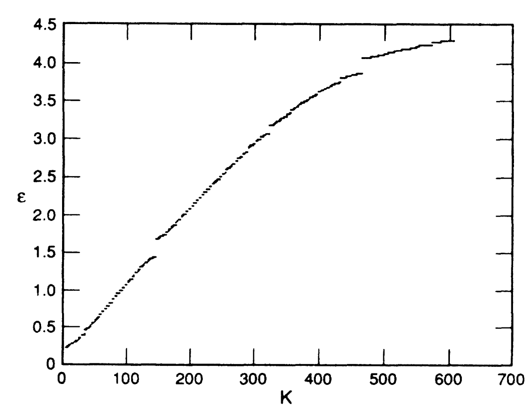

FIG. 1. The figure shows the Cantor spectrum for 610 sites and $g_1=0.2$ and $g_2=0.1$.

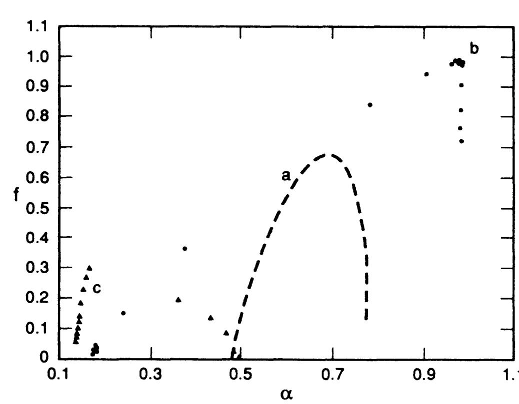

FIG. 3. The $f(\alpha)$ curves. a, disordered phase, $J_1=1$ and $J_2=0.5$, $g=0$. b, ordered phase, $g_1=0.03$ and $g_2=0$. c, ordered phase, $g_1=0.8$ and $g_2=0$.

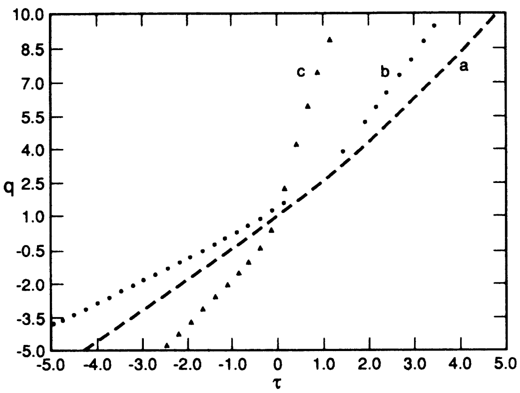

FIG. 4 The corresponding q vs τ plots.

discontinuous in the LRO phase (see Fig. 4). Based on very detailed numerics for various values of the parameter, we conjecture that the $f(\alpha)$ is discontinuous in the LRO phase.[9] This discontinuity in $f-\alpha$ is also signaled by a cusp in the $q-\tau$ plot and hence corresponds to a phase transition in the partition sum Γ. This is very important result as it characterizes the LRO phase in QP systems. We suggest that this is the generic feature of all Cantor spectra when the QP system has a LRO.

The $g_1=-1$ and $g_2=0$ is a special point where the interaction between nearest-neighbor spins is pure Ising type or pure XY type, in a QP sequence. At this point, the bandwidths associated with each eigenvalue go to zero and hence the spectrum is pointlike and the states are localized. At this point, the exponent δ goes to infinity as $(g_2-1)^{-\eta}$, where $\eta \cong 0.25$. In this limit, the $f(\alpha)$ collapses to zero with α equal to zero everywhere. It is interesting to compare the behavior of this model with that of the Harper equation[10] which exhibits critical behavior with the Cantor spectrum at a critical value $\lambda=2$. Below criticality, the system exhibits extended states while above the critical value, the states are localized. This is in contrast with our model (1), which exhibits critical states for all values of g_1 and g_2 except when g_1 and g_2 are equal to zero and g_1 and g_2, respectively, are equal to 0 and -1. At these special points the states are, respectively, extended and localized.

In summary, based on our study of two QP systems (the Ising model in a transverse field and anisotropic XY model), we conclude that the QP systems exhibiting LRO show additional singularities which appear in the scaling properties of the Cantor spectrum. We believe that these results are generic.

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[1]Let $\sigma_g=F_n/F_{n+1}$ be a rational approximate of the golden mean. Consider the sequence defined by the equation $dx(j)+1[j/\sigma]/\sigma-[(j-1)/\sigma]/\sigma$ (7) for any integer $j\geq 1$. The symbol $[X]$ stands for the integer part of X. This sequence has periodicity F_n and has only two distinct values to which we associate J_1 and J_2 or g_1 and g_2.

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[6]This requires different numerical treatment for the even and odd N cases. For N even, the first Brillouin zone is $[-\pi/N,\pi/N]$ whereas for N odd, the first Brillouin zone is $[-\pi/2N,\pi/2N]$.

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[9]Numerical computation of $f(\alpha)$ for the pure case gives few more scattered points which may be the consequence of finite size effects. This leads us to believe that even in the QP systems, some of the points are scattering points which we can identify by their low density. We therefore conjecture that some of the points in the middle of $f(\alpha)$ curve (for α between 0.4 and 0.9) in Fig. 3 in LRO phase are just the scattering points and the $f(\alpha)$ curve is actually discontinuous. In Ref. 2, we had joined all the points obtained in numerics to draw a continuous curve. A more detailed study of the system suggests, however, that some points on the curve with very low density may not exist for the infinite system thereby implying that the $f(\alpha)$ curve is discontinuous. We believe that this happens only in the LRO phase.

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