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Surface exponent in percolation and central-force percolation: A test for splay rigidity

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We study two related problems: one in the usual percolation and the other in central-force percolation; namely, the probability that a site sitting on the border of a semi-infinite domain belongs to either the infinite cluster in usual percolation or the infinitely rigid cluster in central-force percolation. We study the critical exponents describing the critical behavior of these probabilities by a numerical simulation using a transfer-matrix technique. Our results are consistent with the hypothesis that both critical phenomena belong to the same universality class. In addition, our results suggest that the splay-rigid phase threshold is *different* from the rigidity threshold in central-force percolation.

The theory of percolation is now a well-understood subject; most fundamental questions, either about geometric or transport properties, have been answered.[1] On the other hand, the case of central-force percolation[2] is a much more debated question. In particular, a very important point has been raised recently:[3] Does central-force percolation (CFP) belong or not to the same universality class as usual percolation? This issue is far from being simple as the threshold of CFP cannot easily be defined by means of simple geometric arguments,[4] thus leading to a situation quite analogous to that of the experimentalist seeking to determine critical exponents. As the propagation of long-range order in the central-force model is nonlocal, perhaps in an analogous way to the nearest-neighbor three-state antiferromagnetic Potts model,[5] the geometrical properties of the rigid clusters which are the quantities of interest in this problem, are very hard to handle both theoretically and numerically. As a result, the uttermost interesting question of the possible existence of a splay-rigid phase,[6] which has been suggested by Wang and Brooks-Harris to exist between the rigidity threshold and a new specific threshold below, is still a completely open question. We present in this Rapid Communication evidence that indeed such a phase does exist.

In previous work, we have studied[7,8] the transport properties of CFP: the elastic modulus for the random dilution case and the elastic compliance for the random reinforced case of super-rigid-elastic elements. The results of these investigations favor the hypothesis that CFP and usual connectivity percolation belong to the same universality class. In order to further strengthen this hypothesis we study (to our knowledge, for the first time) scaling exponents associated with surface criticality of CFP. This was done by calculating *with the same algorithm* as the one used to calculate the elastic modulus in CFP,[8] the probability for a site sitting on the edge of a semi-infinite lattice to belong to the infinite rigid cluster. The results we find are in excellent agreement with the above hypothesis on the equivalence of the universality classes.

We discuss first the analogous problem in the framework of usual connectivity percolation in order to present the spirit of the computation.

Let p be the probability for a bond to be present and p_c the threshold value. In an infinite domain, the probability $P_\infty(p)$ for a site to belong to the infinite cluster for p larger than p_c, has a singularity at threshold

$$P_\infty(p) \propto (p-p_c)^\beta . \tag{1}$$

For a semi-infinite domain and if we restrict to the sites sitting on the edge of this domain, the probability to belong to the infinite cluster is strongly affected by the "repulsion" effect of the cutting and the critical behavior of $P_\infty(p)$ is now $P'_\infty(p)$:

$$P'_\infty(p) \propto (p-p_c)^{\beta'} , \tag{2}$$

where β' is a surface-critical exponent, which is numerically very different from the bulk-critical exponent β. It is this quantity that we still study in the remainder of this Rapid Communication in different contexts. The quantity $P'_\infty(p)$ describes an edge singularity, and is a *universal* feature of percolation [in the same way as $P_\infty(p)$]. However, to our knowledge, the critical exponent β' cannot be related to β by any means. $P'_\infty(p)$, for the case of usual percolation, has been studied in the past by different techniques: Series expansion,[9] invariant embedding Monte Carlo simulation,[10] renormalization group,[11] and Monte Carlo simulation.[12] Very recently, using a conformal invariance technique, Vanderzande and Stella[13] have conjectured the value of $\beta' = \frac{4}{9}$.

We can obtain an estimate of β' by a transfer-matrix method using finite-size scaling. We study bond percolation on a square lattice. Let us consider a strip of moderate width w (typically from 2 to 32) and of length L (up to 10^6) and imagine that one longitudinal border of the strip is free whereas the other one is connected to a bus bar. The strip consists in a percolation lattice at threshold. A general finite-size scaling argument gives the following dependence of the probability $P'(w)$ for a site on the free border, and far away from the two ends of the strip, to be connected by a continuous path to the bus bar

$$P'(w) \propto (p-p_c)^{\beta'} \Phi(w/\zeta) , \tag{3}$$

where ζ is the correlation length that is known to diverge at threshold as $\zeta \propto (p-p_c)^{-\nu}$, and Φ is a scaling function. For $w \gg \zeta$, Φ tends to a constant and, for $w \ll \zeta$ (always valid if $p = p_c$), $P'(w)$ should not depend on $p - p_c$; therefore,

$$\Phi(x) \propto x^{-\beta'/\nu}, \quad (4)$$

or

$$P'(w) \propto w^{-\beta'/\nu}, \quad (5)$$

where $\nu = \frac{4}{3}$ in two dimensions.

The strip is created by the junction of two independent strips along one of their transverse ends (see Fig. 1). Once the junction is performed, we record whether or not the site located on the junction row and on the free longitudinal border [A in Fig. 1(b)] is connected to the bus bar. In order to test the connectedness, we use two matrices (one for each strip) whose Boolean elements are the connectedness between any pair of sites (the bus bar may be one of these sites) on the junction border.[14] The element (i,j) of these matrices is 1 if the sites i and j are connected by a continuous path, and 0 if not. It is thus a simple series of logical operations that finally gives 1 if the site A is connected to the bottom border and 0 otherwise on the geometry of Fig. 1(b). Afterwards, we disconnect again the two strips [Fig. 1 (c)] and add two independent transverse rows to each strip along the old junction border, and update accordingly the connectedness matrices. We then reiterate the process over again. We begin to record the connectedness of the border site once the two strips have a length equal to 100 times their width so as to avoid spurious bias due to the proximity of the ends. We estimate the error bars on each value by computing the mean deviation of the probability $P'(w)$ obtained for ten pieces of the strip whose lengths are equal to one-tenth of the complete length.

Figure 2 shows a log-log plot of this surface probability as a function of the strip width w. We can extract from the slope of the curve, the critical exponent $\beta'/\nu = 0.32 \pm 0.02$ or $\beta' = 0.43 \pm 0.03$ (using $\nu = \frac{4}{3}$). The error bar on the exponent is a rather subjective appreciation on the uncertainty of the determination of the slope (see Fig. 2)

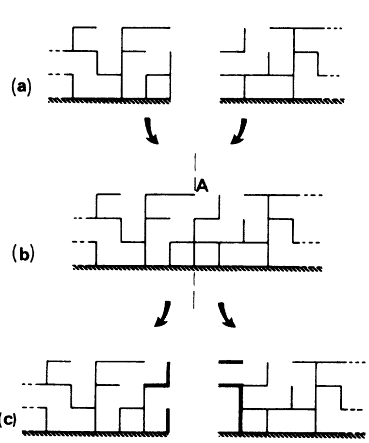

FIG. 1. Geometry of the strips used to compute the probability for a site to be connected to the lower border. Two strips are considered (a) and joined along a transverse direction (b). Then the connected of the site A is tested, and two strips are again disconnected (c). A row is added to each strip [double lines in (c)] and the process is repeated again.

which has been done without taking into account the smallest-width data which clearly displays corrections to scaling. This value agrees with the recent conjecture[13] $\beta'/\nu = \frac{1}{3}$ and is consistent with other numerical estimates of β' published so far in the case of usual percolation[9-12] (see Table I). This demonstrates the ability of the method to give reliable results.

We now turn to the problem of CFP. At threshold, there exists an infinite *rigid* cluster. Therefore, the probability to be considered is whether or not the site added is rigid. However, in this problem there can exist intermediate states where a site is allowed to move freely in one direction and is constrained in any other direction. We shall denote those sites as being *not-free*, whereas completely constrained will be referred to as *rigid*. For bonds, the situation is richer since they have three degrees of freedom, (compared to two for sites). Thus, one can encounter bonds which have from 0 to 3 constraints on their degrees of freedom. Wang and Brooks-Harris have suggested that there could exist a splay-rigid phase below the CFP threshold, i.e., there could exist an infinite set of bonds that cannot rotate, but are free to translate in any direction. Then in addition to the probability for a site on the border of a semi-infinite domain to be rigid P'_r, or to be not free P'_{nf}, one can also study the probability for a bond sitting on the edge of the semi-infinite domain to be splay-rigid P'_{st}. If the conjecture of Wang and Brook-Harris is correct then at the CFP threshold we should expect that P'_r and P'_{nf} scale with the lattice size whereas P'_{sr} is not critical and thus should saturate at a finite nonzero value.

We study this problem numerically using a transfer-matrix algorithm which is very similar in spirit to the one presented above. However, because of the nonlocality of CFP, it is not possible to use a Boolean formulation,[14] i.e., to obtain a yes/no answer to the question: Is the site rigidly connected to the border? Therefore, we were obliged to use a slightly indirect way: We considered two strips made out of a triangular lattice of springs free to rotate at their junctions. One longitudinal border is attached to a rigid bar whereas the opposite border is free. For each of

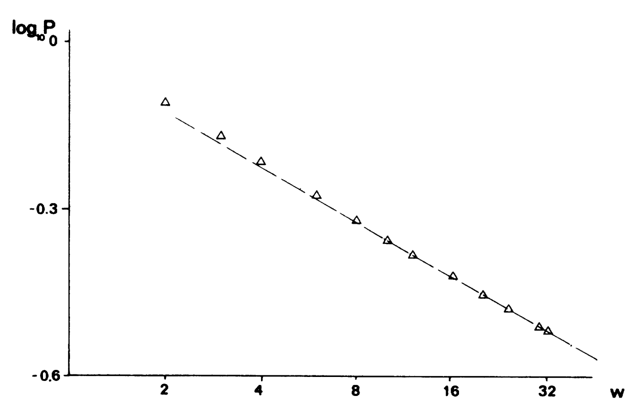

FIG. 2. A log-log plot of the probability for a site to be connected to the lower border at bond percolation threshold on a square lattice. The lower line is the best linear fit estimated on the last points where the slope is 0.32.

TABLE I. Summary of the results obtained about the estimates of β'/ν for usual percolation and central-force percolation (CFP). Our data for usual percolation agrees with the exact result and is consistent with previous numerical estimates. There are two possible definitions of β'/ν for central-force percolation: probability for a site to be rigidly connected, or not free. For the latter, our result is in agreement with the above quoted values for usual percolation, whereas the former seems to scale with a new exponent. In addition, it seems that the probability for a bond to be splay-rigid saturates to a finite nonzero value for the rigidity threshold.

Problem	Technique used	Estimate of β'/ν	Reference
Usual percolation	Series expansion	0.286 ± 0.004	9
Usual percolation	Series expansion	0.322 ± 0.02	9
Usual percolation	Invariant embedding	0.298 ± 0.004	10
Usual percolation	Renormalization group	0.36	11
Usual percolation	Monte Carlo	0.326 ± 0.01	12
Usual percolation	Conformal invariance	$\frac{1}{3}$	13
Usual percolation	Transfer matrix	0.32 ± 0.02	This work
CFP (not free)	Transfer matrix	0.31 ± 0.03	This work
CFP (rigid)	Transfer matrix	0.51 ± 0.03	This work

these strips, we compute the rigidity matrix $R_{ij}^{\alpha\beta}$, which relates the forces F_i^α to be applied onto the site i in the direction α so as to obtain a unit displacement of the site j in the direction β. i and j are two sites on the transverse side of the strip (eventually on the rigid bar or on the free border). As in the previous case, we connect the two strips and calculate the rigidity matrix $\mathbf{R}^{\alpha\beta}$, relative to the site A located on the junction line and on the free border. This matrix $\mathbf{R}$ relates to the force $\mathbf{F}$ to be exerted on A so as to obtain a displacement $\mathbf{U}$: $\mathbf{F}=\mathbf{R}\cdot\mathbf{U}$. Now, if the determinant of $\mathbf{R}$ is nonzero then the site A is rigidly connected to the rigid bar; if the trace of $\mathbf{R}$ is zero then, on the contrary, A is completely free (since the trace is the sum of the two positive eigenvalues, in this case $\mathbf{R}$, is null). Finally if the trace of $\mathbf{R}$ is nonzero, but the determinant is null then it means that one eigenvalue of $\mathbf{R}$ is zero and the other is not. The site A is allowed to move freely, with no force applied, in the eigendirection associated to the eigenvalue 0, but not in any other direction. It is "not free."

Of course, because of round-off errors, neither the trace nor the determinant can be zero in most cases. Therefore, we recorded a joined histogram of the trace and the determinant of all $\mathbf{R}$ matrices. The example of Fig. 3(a) shows that, indeed, we can easily distinguish between the three different cases. We used an estimate of the threshold $p=0.642$ obtained previously using different numerical methods (conjugate gradient,[7] transfer matrix[8]). Figure 4 shows the log-log plot of both P_r' and $P_{\rm nf}'$ as a function of the strip width, w. We obtained the estimates of the exponents

$$\beta'/\nu=0.31\pm0.03 \quad \text{for } P_{\rm nf}', \tag{6}$$

$$\beta'/\nu=0.51\pm0.03 \quad \text{for } P_r'. \tag{7}$$

The scaling of $P_{\rm nf}'$ seems indistinguishable from that of P' in usual percolation (see Table I). This suggests that the appropriate order parameter of this problem, so as to compare it to the case of usual percolation, is the probability for a site to be not free. With this definition, the critical exponents seem identical in both problems.

P_r' scales also with a well-defined exponent (see Fig. 4), but the value of this exponent is clearly different from the previous ones (see Table I). We were not able to relate this apparently new exponent with other ones encountered in the framework of percolation.

One major point in the issue of identifying the universality class of CFP is the determination of the rigidity threshold. The latter had been estimated to be of order 0.65 in some studies.[3,4] Using this threshold, it is possible to obtain numerically a seemingly good power-law relation between the elastic modulus and the size of the system as shown in particular in Ref. 8 (for $p=0.653$). This

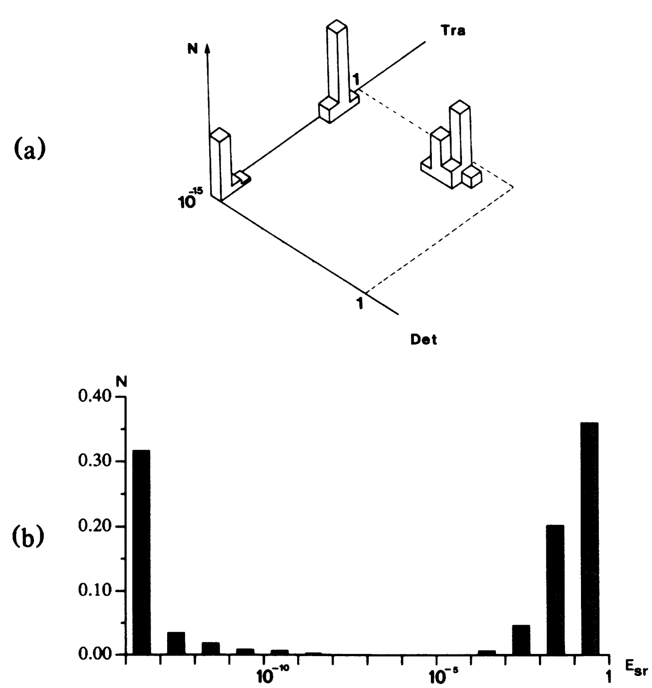

FIG. 3. (a) An example of a joined histogram of the logarithm of the trace and determinant of the rigidity matrix of the border site A in central-force percolation. One clearly distinguishes the free, not-free, and rigid sites. (b) An example of a histogram of the splay rigidity modulus of a bond close to the free border of the strip.

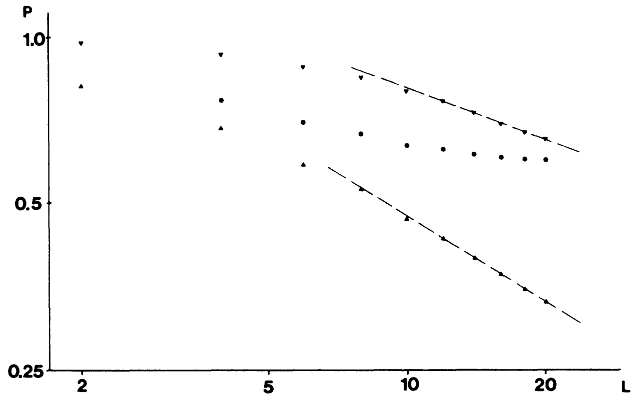

FIG. 4. A log-log plot of the probability for a site to be rigid (▲) or to be not free (▼) as a function of the strip width and for a bond to be splay rigid (●). The length of each side of the strip is 5×10^4 which amount to a total length of 10^5. We are at threshold $p=0.642$. The probability of a bond to be splay rigid seems to saturate whereas the other two probabilities scales with an exponent reported in Table I (the slope is indicated by a line).

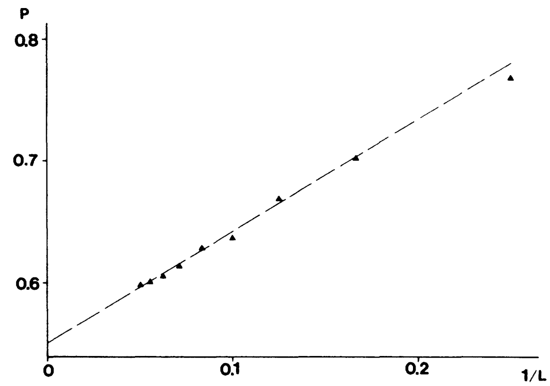

FIG. 5. Plot of the probability for a bond to be splay rigid as a function of the inverse of the strip width. The probability converges to the nonzero value 0.55.

computation had been performed with an algorithm very close in spirit to the one used here. In the present study, for this value of the threshold, $p=0.653$, we observe that the probabilities $P'(w)$ did not scale with the width of the strip, whereas they should have followed a power-law relation with w Eq. (5) if this was the actual rigidity threshold. If, instead of evaluating the threshold by the scaling of the elastic modulus versus the size of the system, one relies on other critical properties such as the mass of the force-carrying part of the lattice[7] or the elastic modulus of the super-rigid-elastic problem,[8] then the best estimate of the threshold is 0.642 ± 0.002. Indeed, for this value of p, the two probabilities P'_{nf} and P'_r scale with the width of the strip, which confirm our previous estimate of the threshold.

Still using the same program it is possible to check for the existence or lack of existence of a splay-rigid phase below threshold. We considered the last bond on the junction line near the free border. We computed the elastic modulus for an applied pure torque; i.e., we calculated the torque to be applied so as to achieve a unit rotation. We can therefore record the probability P'_{sr} by analyzing the histogram of this modulus along the strip, in order to distinguish between what is floppy from what is rigid [see Fig. 3(b)]. A log-log plot of this latter probability is displayed on Fig. 4. Clearly the probability P'_{sr} seems to saturate to a finite value. More precisely, the plot of this probability versus the inverse of the strip width can be well fitted by a straight line (see Fig. 5). This procedure indicates that P'_{sr} tends to the value 0.55 ± 0.05. Therefore, our results suggest that P'_{sr} is *not critical* for $p=0.642$. And thus, it agrees with the suggestion of Wang and Brooks-Harris[6] that there exists a second threshold specific of splay rigidity, lower than the rigidity threshold.

In summary, our determinations of the exponents seem to corroborate three conclusions: The exponent obtained for the property of being not free in CFP is indistinguishable from its value in usual percolation; the exponent obtained for the property of being completely rigid in CFP seems to be a new critical exponent unrelated to others; the splay rigidity threshold is distinct from the rigidity threshld in CFP.

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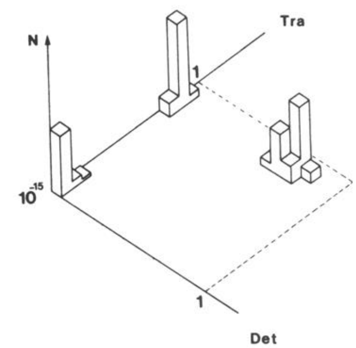
(a)



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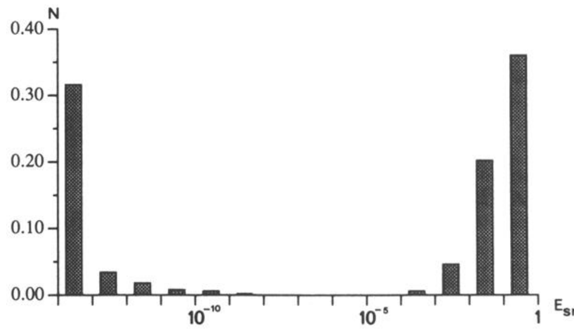

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