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Series study of random animals in general dimensions

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We construct general-dimension series for the random animal problem up to 15th order. These represent an improvement of five terms in four dimensions and above and one term in three dimensions. These series are analyzed, together with existing series in two dimensions, and series for the related Yang-Lee edge problem, to obtain accurate estimates of critical parameters, in particular, the correction to scaling exponent. There appears to be excellent agreement between the two models for both dominant and correction exponents.

I. INTRODUCTION

The "lattice animal" problem[1] deals with the statistics of finite clusters, in general, dimension-diluted lattices. The number $A(n)$, of clusters with n bonds, that can be embedded in a d-dimensional lattice per site, is believed to take the form

$$A(n) \sim An^{-\theta}K_c^{-n}(1+Bn^{-\Delta_1}+\cdots)\,, \qquad (1.1)$$

where θ and Δ_1 are universal exponents, while A, K_c, and B are nonuniversal, lattice-dependent quantities. If one assigns to each bond a fugacity K, then the generating function $F(K)$, is given by

$$F(K)=\sum_n A(n)K^n\,. \qquad (1.2)$$

This may be viewed as the free energy of the problem[2] and exhibits critical behavior

$$[F(K)]_{\text{sing}} \sim A'(K_c-K)^{\theta-1}[1+B'(K_c-K)^{\Delta_1}+\cdots]\,, \qquad (1.3)$$

where K_c is the critical fugacity, which is lattice dependent, as are A' and B'. Similar definitions can be made for the site animal problem when one deals with the question of how many animals with a given number of sites can be embedded in a d-dimensional lattice. From universality one expects the same exponents for both problems.

There has been a great deal of interest in the animal problem during the last decade. This problem describes the statistics of dilute branched polymers.[2] Drouffe, Parisi, and Sourlas[3] also pointed out that there is a connection between this problem and phase transitions in lattice gauge theories. This connection implies that this model has physical relevance also in high dimensions.

Several analytical results have been obtained for the animal problem. The problem is trivial in one dimension (1D), where $A(n)\equiv 1$. On a Cayley tree, however, it is nontrivial, and Fisher and Essam[4] found $\theta=\frac{5}{2}$, with no nonanalytic corrections. Lubensky and Isaacson[2] obtained a field theory, with an upper critical dimension of 8, above which the Cayley tree result should hold. Below 8 dimensions they derived an ϵ expansion for the exponent θ. From their theory, it follows that the exponent θ corresponds to approaching the critical point through a constant field line and not at constant order parameter.[2,5] From Eq. (1.2) one can see that the mean-square number of bonds in a cluster, χ^B, is given as

$$\chi^B(K)=\partial^2F/\partial K^2 \approx A''(K_c-K)^{\theta-3}[1+B'(K_c-K)^{\Delta_1}]\,. \qquad (1.4)$$

Since K plays the role of a temperaturelike variable, $\chi^B(K)$ may be identified as the specific heat at constant H, where H is the field conjugate to the order parameter, which in this case, is the density of free ends. The subscript H indicates an exponent for the case when the critical exponent is approached along a path of constant H. From the field theory it is found that the same exponent, $\theta-3$, describes the dominant divergence in $\chi^S(K)$, the mean-square number of sites in a cluster, as well as that of $\chi^H \equiv \partial^2F/\partial H^2 \approx (K_c-K)^{-\gamma_H}$, so that $\gamma_H=3-\theta$. The distinction between exponents at constant H or constant order parameter is important here because unlike ordinary magnetic systems, the line $H=0$ does not correspond to zero, (i.e., constant) order parameter. A big step forward was taken by Parisi and Sourlas[6] who showed that in the critical region the dominant Feynman diagrams in the field-theoretic formulation of animals in d dimensions are the same as for the critical behavior of the Yang-Lee (YL) edge singularity[7-12] in $d-2$ dimensions. The YL edge problem deals with the singularity which occurs at the edge of the distribution of zeros of the partition function in the complex magnetic-field plane for classical n-vector models. From solving the latter problem in 0 and 1 dimensions, they[6] obtained the exact results that $\theta=1$ and $\frac{3}{2}$ in 2 and 3 dimensions, respectively. Moreover, this identification allows one to use the ϵ expansion[8,10] in $6-\epsilon$ dimensions for the YL problem, up to third order in ϵ, to obtain an ϵ expansion for θ in $8-d$ dimensions for the animal problem, which gives[2]

$$\theta=\frac{5}{2}-\frac{\epsilon}{12}-\frac{79}{3888}\epsilon^2+\left[\frac{\zeta(3)}{81}-\frac{10445}{1259712}\right]\epsilon^3, \quad (1.5)$$

where $\zeta(3)=1.202\ldots$ is the zeta function. Since the relation between the two problems was obtained by noting that the two field theories have the same most relevant terms, there are no similar relations between the correction to scaling exponents. Therefore, the ϵ expansion for Δ_1 in the YL problem,[11]

$$\Delta_1=\frac{\epsilon}{4}\{1-\tfrac{107}{162}\epsilon+[\tfrac{20}{27}\zeta(3)+\tfrac{8129}{13122}]\epsilon^2\}, \quad (1.6)$$

although in agreement to first order in ϵ with the ϵ expansion for Δ_1 in the animal problem,[2] may not apply there to higher order in ϵ. Another mapping, between the animal problem in d dimensions and the directed animal problem[12] in $d-1$ dimensions, was derived in a similar way,[13] and was used by Dhar[14(a)] to obtain $\theta=\frac{11}{6}$ in 4D. Dhar[14(a)] also obtained a mapping from the directed site animal problem to the "hard square" problem[14(b)] on the square lattice with negative activity. This mapping enables determination of directed site animals up to 42 terms and accurate numerical evaluations of θ.[14(c)]

In order to facilitate comparison between these results and our calculations and with other critical phenomena studies, we shall present a brief summary of some of the above discussed field-theoretic results and the resultant scaling relations in a slightly different notation to that of the original results.[5] The most distinctive feature of the animal problem is that it has only a single independent exponent, unlike the usual critical phenomena where ν and η (or λ_T and λ_H) are independent exponents. If we take ν_Q to be the correlation-length exponent, where subscript Q indicates an exponent at constant order parameter, Q, then the other exponents can be expressed in terms of ν_Q via $\gamma_Q=-2+(d-2)\nu_Q=-\alpha_Q$ and if we arbitrarily define β_Q to be unity, one has the usual relation $\alpha_Q+2\beta_Q+\gamma_Q=2$, and hyperscaling with reduced dimensionality: $2\beta_Q+\gamma_Q=(d-2)\nu_Q$. Note that these relations imply that $\gamma_Q=-2$ at $d=2$. As we mentioned, the exponent $\gamma_H=3-\theta$ we will study via our series work corresponds to approaching the critical point along a path of constant H. The exponents at constant H are related to those at constant Q by Fisher renormalization (Ref. 5) $\beta_H=(1+\gamma_Q)^{-1}$, $\nu_H=\nu_Q/(1+\gamma_Q)$, and $\alpha_H=\gamma_H=\gamma_Q/(1+\gamma_Q)$, so that again $\alpha_H+2\beta_H+\gamma_H=2$ and $2\beta_H+\gamma_H=(d-2)\nu_H$. For $d>2$, γ_H determines ν_H via $\nu_H=(2-\gamma_H)/(d-2)$. Since we will investigate γ_H numerically, it is of interest to note the implied result for γ_Q, namely $\gamma_Q=\gamma_H/(1-\gamma_H)$. Thus, γ_Q diverges as γ_H passes through unity, and in fact, for $\gamma_H>1$, γ_Q, if it has any meaning, becomes negative, reaching the value $\gamma_Q=-2$ at $d=2$. The exact dimension where $\gamma_H=1$ is, therefore, of some interest and it can be estimated from the Flory approximation[5] for ν_H which leads to the results $\nu_H=5/[2(d+2)]$, $\beta_H=(3d-14)/[2(d+2)]$, and $\gamma_H=(18-d)/[2(d+2)]$, so that $\gamma_H=1$ at $d=\frac{14}{3}$. The approximation, although it disagrees with the ϵ expansion, is remarkable in that it give exact results for $d=2$, 3, 4, and 8. The approximation gives $\theta=(7d-6)/[2(d+2)]$.

In addition to the above-mentioned exact and field-theoretical results, several numerical calculations have been made for the animal problem. An analytic expansion for the critical point, K_c, of hypercubic lattices, in powers of $1/\sigma$, where $\sigma=2d-1$, has been obtained.[15,16] We expect that this expansion will be reliable for higher dimensions. Real-space renormalization-group[17] and Monte Carlo methods[18] have also been applied, but the most accurate extant numerical results have been obtained by series expansions[15,19-29] and finite-size scaling renormalization methods.[26,30] A recent new approach by Dhar and Lam[31] combined Monte Carlo and series ideas, but does not appear to lead to particularly accurate numerical values.[32]

A comprehensive summary of previous analytic and numerical analysis for the animal and YL edge problems is given in Tables I, II, and III. We quote critical exponents in terms of $\gamma\equiv\gamma_H=3-\theta$, which is the variable in which our analysis is made.

The literature values in Table I are, to the best of our knowledge, the most recent estimates for each lattice with each technique. From a glance at Tables I and II, we observe that the different dominant exponent estimates for the animal problem appear to be well converged for $d=2$ and 3. However, many of the extant animal estimates for $d>3$ have large error bounds on the dominant exponent. The Monte Carlo series estimates of Lam[32] for $d>3$ are quite different from the series and ϵ-expansion values. There were also several discrepancies in the K_c estimates between different approaches, for higher dimensions, where the series were relatively short (see Table III). From the old ten-term series[15] even determination of the upper critical dimension was unsuccessful, and the K_c estimates had large error bars. Note also that the Flory-type estimate for γ, on 2–4 dimensions, disagrees with the ϵ expansion, and cannot be trusted in high dimensions, though it might serve as a rough estimate. Better estimates for θ (or any other leading singular exponent) are clearly needed. Moreover, extant estimates for the correction exponent Δ_1 are not accurate enough in all dimensions. Even in two dimensions, where long series are available, and there is a general agreement on K_c and γ values, there are disagreements between different workers concerning Δ_1.

These disagreements will be discussed in length below. In higher dimensions there are fewer estimates, but more problems. Breuer[33] has given ϵ-expansion estimates for Δ_1 from an expansion to order ϵ^2 [we quote[11] to order ϵ^3 in Eq. (1.6)]. We did make extrapolations from this expansion to order ϵ^3 but found them to be extremely ill converged. Some approximants gave negative exponents and therefore we decided that this expansion may be problematic and decline to quote the results. However, such problems do suggest that one cannot know if the estimates from order ϵ^2 are any improvement over the simple first-order estimate $\Delta_1=\epsilon/4$. Breuer[33] also gives ratio estimates for Δ_1 from the directed site animal series, some of which agree with his ϵ-expansion results quite nicely. However, on a closer look it appears that some of the series Δ_1 estimates are apparently biased from older critical temperature estimates and from γ estimates for ϵ^2

TABLE I. Literature values of animal critical exponents in two and three dimensions. s, site; b, bond; sq, square; tr, triangle; HC, honeycomb; D, diamond; ROG, radius of gyration; MC, Monte Carlo.

Reference	γ	Δ_1	ν	Method/model
		$d=2$		
6	2			Exact
18	2.0 ± 0.2	0.75[a]		MC
20	2.00 ± 0.01	0.4 ± 0.3		Series/sq(s)
	2.00 ± 0.02	0.7 ± 0.2		Series/tr(s)
		1.3 ± 0.2		Series/tr(b)
		1.0 ± 0.2		Series/sq(s)[a]
		0.75 ± 0.1		Series/tr(s)[a]
		1.1 ± 0.5		Series/HC(b)[a]
		1.1 ± 0.3		Series/sq(b)[a]
22	2.00 ± 0.02	0.8 ± 0.15		Series/sq(s)
23		0.86 ± 0.05		Series/tr(s)[a]
26		0.83	0.6412 ± 0.035	Finite size
30		1.15 ± 0.2	0.64075 ± 0.00015	Transfer matrix
27		0.87 ± 0.07	0.655 ± 0.010	Series/ROG
		0.87 ± 0.06	0.640 ± 0.004	Series/ROG
28		0.93 ± 0.15		Series/sq,tr(b)[a]
29		0.50 ± 0.05		Series/ROG[a]
		$d=3$		
6	$\frac{3}{2}$			Exact
15	1.49 ± 0.02			ϵ exp
20	1.45 ± 0.05	<1.8		Series/sc(s)
	1.5 ± 0.1	0.8 ± 0.3		Series/fcc(s)
	1.55 ± 0.1	1.4 ± 0.1		Series/bcc(b)
	1.6 ± 0.1	>1.2		Series/fcc(b)
	1.55	1.4 ± 0.4		Series/D(s)[b]
	1.50	1.0 ± 0.3		Series/sc(b)[b]
	1.53	0.6 ± 0.5		Series/bcc(s)[b]
	1.53	0.65 ± 0.2		Series/fcc(s)[b]
	1.55	1.25 ± 0.15		Series/sc(b)[b]
	1.58	1.2 ± 0.1		Series/bcc(b)[b]
	1.58	1.3 ± 0.2		Series/fcc(b)[b]
27		0.64 ± 0.06		Series/ROG[a]
29		0.45 ± 0.1		Series/ROG[a]

[a] Biased by the exact value of the leading exponent.
[b] Stability analysis.

series, and therefore, their apparent convergence may be spurious. We think that the ±0.005 error that Breuer places on his γ estimates from the ϵ^2 series is clearly too small since it excludes the ϵ^3 order estimates from Ref. 10 in some cases. Thus, the ±0.1 that he places on his final Δ_1 estimates must also be questioned, especially since some of his $d=6$ estimates from different methods differ by more ±0.1. Breuer[33] concludes that longer series are needed in order to determine correction exponents for these problems. Since we have considerably longer series in the higher dimensions, we can see that a comprehensive analysis of correction terms from these series in all dimensions is highly desirable.

At the upper critical dimension, the critical behavior (1.3) becomes

$$[F(K)]_{\rm sing}\sim A'(K_c-K)^{\theta-1}|\ln(K_c-K)|^{z}, \qquad (1.7)$$

where $\theta=\frac{5}{2}$ is the mean-field value and z is believed to be $\frac{1}{3}$ from field theory.[2] This form can be used to extract the upper critical dimension, or alternatively to evaluate K_c and z if we set the upper critical dimension equal to 8.

In this paper, we report on extensive studies of series for the animal problem. We constructed the series for the free energy [Eq. (1.2)] up to 15 terms in general dimension, which is an improvement of one term in 3D and 5 terms in 4D and above. This was done using a method of constructing series from diagrams with no free ends.[16,34] The coefficients of these series are listed in Table IV. We analyzed the new series and the existing series in 2D using improved methods, and obtained reliable estimates for K_c, θ, and Δ_1 in Eq. (1.3) and K_c in Eq. (1.7). Our results are summarized in Tables V and VI and an overall summary of the best lattice animal exponents is given in Table VII.

The outline of this paper is as follows: In Sec. II we give details of the series construction, while Sec. III contains a detailed report of our analysis. We concentrated

TABLE II. Literature values of critical exponents for the isotropic animal problem, the Yang-Lee edge problem, and the directed animal problem in four to seven dimensions.

Reference	$\gamma(d=4)$	$\gamma(d=5)$	$\gamma(d=6)$	$\gamma(d=7)$	Method/model
14(a)	$\frac{7}{6}$	...	...	...	Exact[a]
14(c)	1.1663	...	...	...	Series[a]
5	$\frac{7}{6}$	$\frac{13}{20}$	$\frac{3}{4}$	$\frac{11}{18}$	Flory[b]
10	1.15 ± 0.01	0.915 ± 0.006	0.736 ± 0.002	0.600 ± 0.001	ϵ expansion (third order)[c]
9	1.1645 ± 0.002	0.914 ± 0.015	...	...	Series[c]
15	1.1 ± 0.07	0.8 ± 0.1	0.7 ± 0.2	0.6 ± 0.2	Series[b]
32	1.05 ± 0.225	0.75 ± 0.20	0.55 ± 0.25	0.325 ± 0.125	MC series[b]
33	...	0.926 ± 0.005	0.744 ± 0.005	0.603 ± 0.005	ϵ expansion (second order, biased)[c]
Reference	$\Delta_1(d=4)$	$\Delta_1(d=5)$	$\Delta_1(d=6)$	$\Delta_1(d=7)$	Method/model
14(a)	$\frac{5}{6}$	...	...	...	Exact[a]
33	...	0.6500	0.4501	0.2336	ϵ expansion (second order, biased)[c]
33	0.72 ± 0.10	0.48 ± 0.10	0.32 ± 0.10	0.15 ± 0.10	Series, I[a]
33	≈ 0.65	≈ 0.5	≈ 0.4	≈ 0.35	Series, II[a,d]
33	≈ 0.8	≈ 0.6	≈ 0.45	≈ 0.35	Series, III[a,d]

[a] Directed animals in $d-1$.

[b] Isotropic animals.

[c] Yang-Lee edge problem in $d-2$.

[d] Approximate readings from graphs.

our efforts on the clarification of K_c estimates for $4\leq d\leq 8$ and γ values for $5\leq d\leq 7$ and on the estimation of Δ_1 for $2\leq d\leq 7$. In addition to the new animal series, we have reanalyzed several extant animal series and the Yang-Lee series of Ref. 9 for purposes of comparison. Section IV contains our conclusions.

II. CONSTRUCTION OF THE SERIES

In order to generate the series for the free energy [Eq. (1.2)], on the d-dimensional hypercubic lattice, we use the scheme proposed by Harris[16] in which the sum over all animals is replaced by a sum over only diagrams with no free ends, i.e., diagrams with no valence one sites. This is done by implementing Eq. (78) of that reference. The free energy is then written as

$$F(K)=F_{\mathrm{CT}}(K)+\sum_{\Gamma}\delta F(\Gamma,K)\,, \tag{2.1}$$

where $F_{\mathrm{CT}}(K)$ is the free energy for the Cayley tree having the same coordination number, $2d$, as the d-dimensional hypercubic lattice and the contribution to the free energy from the diagram Γ, denoted $\delta F(\Gamma,K)$, is

$$\delta F(\Gamma,K)=\lim_{n\to 0}\left[\frac{1}{n}\prod_{i\in\Gamma}\mathrm{Tr}h_{i0}^{z-z_i(\Gamma)}\times\prod_{\langle ij\rangle\in\Gamma}\left(K\sum_{\alpha}S_i^{\alpha}S_j^{\alpha}+1-h_{i0}h_{j0}\right)\right], \tag{2.2}$$

where here and below sums over the replica index α run from 1 to n. Also, $\langle ij\rangle\in\Gamma$ indicates that the product is over all bonds $\langle ij\rangle$ in Γ, z is the coordination number of the pure lattice, $z_i(\Gamma)$ is the number of bonds of Γ which intersect site i, S_i^{α} are operators which obey the trace rules that

$$\mathrm{Tr}(S_i^{\alpha})^q=\delta_{q,0}+\delta_{q,2} \tag{2.3}$$

and all other traces [(e.g., $\mathrm{Tr}(S_i^{\alpha}S_i^{\beta})$, with $\alpha\neq\beta$] vanish. The quantity h_{i0} is constructed so as to vanish if Γ has any free ends. To order n one has

$$h_{i0}=1+an+b\sum_{\alpha=1}^{n}S_i^{\alpha}\,, \tag{2.4}$$

where b satisfies the self-consistent equation

$$b=K(1+b)^{z-1}\,. \tag{2.5}$$

For diagrams with more than one bond we may drop the term an in Eq. (2.4). The simplest way to evaluate (2.2) is to substitute (2.4) for h_{i0} inside the product in (2.2) over $\langle ij\rangle$. We then write the factor in this product as

$$K\sum_{\alpha}S_i^{\alpha}S_j^{\alpha}+1-h_{i0}h_{jo}=K\sum_{\alpha}S_i^{\alpha}S_j^{\alpha}-b\sum_{\alpha}S_i^{\alpha}-b\sum_{\alpha}S_j^{\alpha}-b^2\sum_{\alpha,\beta}S_i^{\alpha}S_j^{\beta} \tag{2.6a}$$

$$=K\sum_{\alpha}S_i^{\alpha}S_j^{\alpha}+V\,. \tag{2.6b}$$

To evaluate (2.2), we substitute (2.6b) into (2.2) and expand in powers of V, identifying the first term in (2.6b) with the *presence* of a bond and V with the *absence* of a bond.

Thus, (2.2) is given in terms of the $2^{b(\Gamma)}$ possible choices of occupying or not occupying a bond of Γ, where $b(\Gamma)$ is the number of bonds in Γ. To each such covering

TABLE III. Estimates of the critical threshold, $\lambda_c = 1/K_c$, for animals in $2 \le d \le 8$. (See Table I for abbreviations.)

d	Lattice	λ_c	Reference
2	sq(s)	4.065 ± 0.005	20
		4.063 ± 0.02	20
		4.0625 ± 0.001	22
		4.06256 ± 0.00017	26
	sq(b)	5.21 ± 0.006	5
		5.25 ± 0.1	20
		5.208 ± 0.004	20
	tr(s)	5.183 ± 0.01	20
		5.18336 ± 0.00027	26
	tr(b)	8.62 ± 0.02	20
	HC(b)	3.367 ± 0.002	20
3	sc(s)	8.33 ± 0.02	20
		8.368 ± 0.04	30
	sc(b)	10.62 ± 0.08	15
		10.63 ± 0.05	10
	fcc(s)	13.94 ± 0.05	20
	fcc(b)	23.9 ± 0.1	20
	bcc(s)	11.18 ± 0.11	20
	bcc(b)	15.3 ± 0.01	20
	D(s)	5.55 ± 0.01	20
4	s	13.48 ± 0.02	32
(hypercubic)	b	16.3 ± 0.4	15
		15.8	16
5	s	18.924 ± 0.29	32
(hypercubic)	b	22.1 ± 0.8	15
		21.9	16
6	s	24.475 ± 0.5	32
(hypercubic)	b	27.75 ± 0.1	15
		27.6	16
7	s	30.3 ± 0.3	32
(hypercubic)	b	33.25 ± 1.0	15
		33.27	16

γ of Γ we can easily associate a factor. In so doing, we consider diagrams with more than two bonds, for which the covering γ with no bonds does not contribute. First of all, note that any free sum over α gives a factor of n. A diagram must have at least one such sum, and it is this term linear in n which survives the $n \to 0$ limit in (2.2). Terms proportional to higher powers of n drop out. One can verify that to give a contribution linear in n a covering γ must be (a) connected (so as not to have more than one replica sum), and (b) any site in Γ must either be a site in γ or a nearest neighbor to a site in γ (with reference, of course, to bonds in Γ and not in the original lattice). Such adjacent sites are allowed, because in V we may take the term which refers to either site of a bond. As long as one site in the bond belongs to γ, we get a linear in n contribution from γ. For each such allowable covering γ of Γ we classify sites and bonds (of γ) as follows. Sites which are in γ are denoted $i \in \gamma$. The number of bonds in γ is denoted $b(\gamma)$. Bonds not in γ must have one end at least in γ and can therefore be classified as either "external" in which case one end of the bond is in γ and the other not, or they are "internal," i.e., the bond, although not occupied, connects two sites in γ. For a given γ the numbers of internal and external bonds are denoted $p_{\text{int}}(\gamma)$ and $p_{\text{ext}}(\gamma)$, respectively. Then we have

$$\delta F(\Gamma,K) = \sum_{\gamma \in \Gamma} K^{b(\gamma)} (-b)^{p_{\text{ext}}(\gamma)} (-2b - b^2)^{p_{\text{int}}(\gamma)} \times \prod_{i \in \gamma} (1+b)^{z - z_i(\Gamma)} . \tag{2.7}$$

To illustrate this, we show in Fig. 1 the possible coverings, γ, of a square and give the various diagrammatic quantities associated thereto. For the square the final results using (2.5) can be written as

$$\delta F(\Gamma,K) = [b/(1+b)]^4 [1 - 4(1+b)^{z-1}] , \tag{2.8}$$

where b is given in terms of K via (2.5). One can also check that if Eq. (2.7) is applied to a diagram Γ_3 (like a chain of three bonds) which does have free ends $\delta F(\Gamma_3, K) = 0$. It is relatively easy to implement (2.7) for any given diagram with no free ends. In Eq. (2.7), note that since every bond Γ must intersect γ, one has that $b(\Gamma) = b(\gamma) + p_{\text{ext}}(\gamma) + p_{\text{int}}(\gamma)$. Thus, $\delta F(\gamma, K)$ is a series in powers of K, whose leading term is of order $K^{b(\Gamma)}$. The

TABLE IV. Series coefficients of the free-energy equation (1.2), $F(K,d)=\sum'_{m,n}a(m,n)K^m d^n$.

k	l	$a(k,l)$	k	l	$a(k,l)$
1	1	0.100 000 000 000 000 000 000 000[+01]	2	1	−0.100 000 000 000 000 000 000 000[+01]
2	2	0.200 000 000 000 000 000 000 000[+01]	3	1	0.166 666 666 666 666 666 666 667[+01]
3	2	−0.600 000 000 000 000 000 000 000[+01]	3	3	0.533 333 333 333 333 333 333 333[+01]
4	1	−0.200 000 000 000 000 000 000 000[+01]	4	2	0.163 333 333 333 333 333 333 333[+02]
4	3	−0.300 000 000 000 000 000 000 000[+02]	4	4	0.166 666 666 666 666 666 666 667[+02]
5	1	−0.560 000 000 000 000 000 000 000[+01]	5	2	−0.250 000 000 000 000 000 000 000[+02]
5	3	0.118 000 000 000 000 000 000 000[+03]	5	4	−0.144 000 000 000 000 000 000 000[+03]
5	5	0.576 000 000 000 000 000 000 000[+02]	6	1	0.423 333 333 333 333 333 333 333[+02]
6	2	−0.754 777 777 777 777 777 777 778[+02]	6	3	−0.253 833 333 333 333 333 333 333[+03]
6	4	0.760 555 555 555 555 555 555 556[+03]	6	5	−0.686 000 000 000 000 000 000 000[+03]
6	6	0.213 422 222 222 222 222 222 222[+03]	7	1	−0.317 142 857 142 857 142 857 143[+02]
7	2	0.552 466 666 666 666 666 666 667[+03]	7	3	−0.445 377 777 777 777 777 777 778[+03]
7	4	−0.225 466 666 666 666 666 666 667[+04]	7	5	0.462 488 888 888 888 888 888 889[+04]
7	6	−0.327 680 000 000 000 000 000 000[+04]	7	7	0.832 203 174 603 174 603 174 603[+03]
8	1	−0.560 250 000 000 000 000 000 000[+03]	8	2	−0.154 702 380 952 380 952 380 952[+02]
8	3	0.453 898 333 333 333 333 333 333[+04]	8	4	−0.720 425 000 000 000 000 000 000[+03]
8	5	−0.180 393 333 333 333 333 333 333[+05]	8	6	0.271 696 666 666 666 666 666 667[+05]
8	7	−0.157 464 000 000 000 000 000 000[+05]	8	8	0.337 422 857 142 857 142 857 143[+04]
9	1	−0.270 577 777 777 777 777 777 778[+04]	9	2	−0.226 849 047 619 047 619 047 619[+04]
9	3	−0.358 315 855 379 188 712 522 046[+04]	9	4	0.306 893 333 333 333 333 333 333[+05]
9	5	0.174 482 962 962 962 962 962 963[+05]	9	6	−0.133 589 866 666 666 666 666 667[+06]
9	7	0.156 091 792 592 592 592 592 593[+06]	9	8	−0.761 904 761 904 761 904 761 905[+05]
9	9	0.141 093 474 426 807 760 141 093[+05]	10	1	0.507 106 000 000 000 000 000 000[+05]
10	2	−0.142 072 187 460 317 460 317 460[+06]	10	3	0.937 136 067 460 317 460 317 460[+05]
10	4	−0.969 636 874 779 541 446 208 113[+05]	10	5	0.174 399 980 555 555 555 555 556[+06]
10	6	0.282 519 027 037 037 037 037 037[+06]	10	7	−0.934 803 977 777 777 777 777 778[+06]
10	8	0.883 192 569 312 169 312 169 312[+06]	10	9	−0.371 184 209 523 809 523 809 524[+06]
10	10	0.604 892 785 890 652 557 319 224[+05]	11	1	0.117 271 181 818 181 818 181 818[+06]
11	2	0.744 393 352 380 952 380 952 381[+06]	11	3	−0.219 370 920 063 492 063 492 063[+07]
11	4	0.181 645 889 841 269 841 269 841[+07]	11	5	−0.129 701 401 269 841 269 841 270[+07]
11	6	0.731 053 022 222 222 222 222 222[+06]	11	7	0.296 518 967 111 111 111 111 111[+07]
11	8	−0.627 173 793 015 873 015 873 016[+07]	11	9	0.494 339 296 507 936 507 936 508[+07]
11	10	−0.182 002 834 285 714 285 714 286[+07]	11	11	0.264 731 395 324 675 324 675 325[+06]
12	1	−0.368 696 900 000 000 000 000 000[+07]	12	2	0.101 434 066 717 051 467 051 467[+08]
12	3	0.104 202 663 773 809 523 809 524[+07]	12	4	−0.211 969 437 211 037 624 926 514[+08]
12	5	0.206 356 581 118 386 243 386 243[+08]	12	6	−0.125 493 513 405 276 308 054 086[+08]
12	7	0.528 025 890 000 000 000 000 000[+06]	12	8	0.261 878 505 326 102 292 768 959[+08]
12	9	0.407 557 116 730 158 730 158 730[+08]	12	10	0.274 505 183 970 605 526 161 082[+08]
12	11	−0.897 735 396 656 084 656 084 656[+07]	12	12	0.117 884 446 025 546 469 990 914[+07]
13	1	−0.163 399 048 461 538 461 538 462[+08]	13	2	−0.477 016 044 422 799 422 799 423[+08]
13	3	0.174 189 679 132 708 032 708 033[+09]	13	4	−0.643 950 146 258 553 791 887 125[+08]
13	5	−0.164 271 147 834 591 416 813 639[+09]	13	6	0.187 515 471 600 000 000 000 000[+09]
13	7	−0.993 430 415 748 971 193 415 638[+08]	13	8	−0.334 511 671 839 153 439 153 439[+08]
13	9	0.209 831 402 980 176 366 843 034[+09]	13	10	−0.258 369 217 331 922 398 589 065[+09]
13	11	0.151 529 232 858 788 947 677 837[+09]	13	12	−0.445 221 210 160 269 360 269 360[+08]
13	13	0.532 743 328 396 903 508 014 619[+07]	14	1	0.403 034 995 714 285 714 285 714[+09]
14	2	−0.137 455 818 969 968 761 397 333[+10]	14	3	0.653 325 052 806 284 271 284 271[+09]
14	4	0.145 379 324 653 851 891 935 225[+10]	14	5	−0.897 608 291 695 216 049 382 716[+09]
14	6	−0.117 593 685 539 005 731 922 399[+10]	14	7	0.150 903 692 427 279 100 529 101[+10]
14	8	−0.672 513 085 077 902 494 331 066[+09]	14	9	−0.504 843 066 255 291 005 291 005[+09]
14	10	0.157 728 396 091 128 747 795 414[+10]	14	11	−0.160 617 663 363 506 172 839 506[+10]
14	12	0.832 673 136 673 028 699 695 366[+09]	14	13	−0.221 895 292 207 792 207 792 208[+09]
14	14	0.243 840 980 448 123 305 266 162[+08]	15	1	0.262 057 485 080 000 000 000 000[+10]
15	2	0.458 427 605 735 690 309 690 310[+10]	15	3	−0.261 060 304 132 440 220 484 982[+11]
15	4	0.220 799 945 936 209 764 309 764[+11]	15	5	0.532 529 960 625 964 958 224 217[+10]
15	6	−0.707 774 542 132 315 696 649 030[+10]	15	7	−0.850 042 746 362 549 402 312 365[+10]
15	8	0.112 435 367 278 408 112 874 780[+11]	15	9	−0.388 272 420 227 795 526 441 029[+10]
15	10	−0.534 707 784 681 481 481 481 481[+10]	15	11	0.113 275 807 532 832 482 853 224[+11]
15	12	−0.982 884 308 320 697 771 364 438[+10]	15	13	0.455 962 264 350 406 641 369 604[+10]
15	14	−0.111 088 901 897 374 132 040 799[+10]	15	15	0.112 852 217 800 507 054 771 605[+09]

TABLE V. Results of our analysis.

Series	λ_c	γ(M1)	Δ_1(M1)	γ(M2)	Δ_1(M2)	Figure
Two dimensions						
Square site[a]	4.0625[b]	2.00 ± 0.01	0.85 ± 0.15	2.00 ± 0.01	0.85 ± 0.10	2(a),2(b)
	4.0627	1.98 ± 0.02	0.70 ± 0.10	1.99 ± 0.01		2(d)
Square bond[c]	5.2085[b]			1.995 ± 0.005	0.87 ± 0.05	2(c)
	5.2070	1.98 ± 0.01	0.75 ± 0.10	1.99 ± 0.01	0.85 ± 0.05	
Triangular site[d]	5.183			2.00 ± 0.01	0.90 ± 0.10	
Bond weighted tri site[e,f]	5.183	1.98 ± 0.02	0.80 ± 0.20	2.00 ± 0.01	0.87 ± 0.05	
Three dimensions						
sc bond[g]	10.615[b]	1.50 ± 0.02	1.10 ± 0.01	1.50 ± 0.01	1.30 ± 0.20	3(a)
	10.616			1.49 ± 0.01	1.00 ± 0.10	
	10.6175			1.47 ± 0.01	0.90 ± 0.20	
Four dimensions						
YL($d=2$)[f,h]	0.088964[b]	1.166 ± 0.01	1.00 ± 0.02	1.166 ± 0.02	0.75 ± 0.20	4(a)
	0.088962	1.160 ± 0.01	0.80 ± 0.20	1.160 ± 0.02	0.75 ± 0.15	
	0.088950	1.160 ± 0.01	0.75 ± 0.01	1.15 ± 0.02	0.80 ± 0.10	
Hypercubic bond[g]	16.327			1.10 ± 0.02	0.75 ± 0.50	
	16.325	1.10 ± 0.02	0.60 ± 0.01	1.12 ± 0.02	0.80 ± 0.50	
	16.322			1.13 ± 0.02	0.85 ± 0.15	
	16.320[b]			1.14 ± 0.02	0.85 ± 0.15	4(b)
	16.3175	1.15 ± 0.02	0.80 ± 0.10	1.15 ± 0.02	0.90 ± 0.15	4(c)
D1[g]	1.0			1.115 ± 0.25	0.65 ± 0.05	
D2[g]	1.0			0.96 ± 0.06	0.65 ± 0.15	
Five dimensions						
YL($d=2$)[h,f]	0.052026	0.92 ± 0.02	0.80 ± 0.10	0.92 ± 0.02	0.70 ± 0.15	
	0.052022[b]	0.91 ± 0.01	0.70 ± 0.10	0.91 ± 0.01	0.70 ± 0.10	5(a),5(b)
	0.052018	0.89 ± 0.02	0.60 ± 0.20	0.89 ± 0.01	0.60 ± 0.20	
	0.052016	0.88 ± 0.02	0.60 ± 0.20	0.88 ± 0.01	0.60 ± 0.20	
Hypercubic bond[g]	22.060			0.85 ± 0.02	0.80 ± 0.20	
	22.040			0.89 ± 0.02	0.75 ± 0.10	
	22.042			0.90 ± 0.02	0.70 ± 0.15	
	22.043[b]			0.90 ± 0.02	0.65 ± 0.20	5(c)
	22.044			0.90 ± 0.02	0.65 ± 0.20	
	22.100			0.90 ± 0.10		
D1[g]	1.0			0.90 ± 0.06	0.80 ± 0.10	
Six dimensions						
Hypercubic bond[g]	27.720			0.66 ± 0.02	0.50 ± 0.15	
	27.710			0.69 ± 0.02	0.50 ± 0.15	6(a)
	27.707[b]			0.70 ± 0.02	0.50 ± 0.20	
					or 0.30 ± 0.10	
					and 0.55 ± 0.10	
	27.690			0.76 ± 0.02	0.80 ± 0.10	
D1[g]	1.0			0.70 ± 0.025	0.50 ± 0.05	
Seven dimensions						
Hypercubic bond[g]	33.315			0.58 ± 0.02	0.50 ± 0.15	
	33.312			0.57 ± 0.02	0.45 ± 0.20	
	33.310[b]			0.59 ± 0.02	0.40 ± 0.20	6(b)
					or 0.30 ± 0.05	
					and 0.50 ± 0.10	
	33.305			0.62 ± 0.02	0.50 ± 0.15	
D2[g]	1.0			0.62 ± 0.06	0.60 ± 0.10	

[a] Reference 21.

[b] Best λ_c.

[c] Reference 15.

[d] References 19 and 23.

[e] Reference 28.

[f] The exponent $\gamma+1$ was actually calculated for these series.

[g] Extended hypercubic series from Table I.

[h] Reference 9.

TABLE VI. z estimates for eight dimensions from 15-term hypercubic series.

λ_c	$\partial^2\chi/\partial p^2$	$\partial^3\chi/\partial p^3$	$\partial^4\chi/\partial p^4$	$\partial^5\chi/\partial p^5$	$\partial^6\chi/\partial p^6$
38.950	0.0–1.5				
38.895	0.7–1.5				
38.885			0.1–0.85	−0.06–2.7	
38.880			0.2–0.75	0.1–0.35	0.06–0.36
38.875	1.2–1.6		0.35–0.75	0.22–0.40	
38.873[a]			0.27–0.40		
38.870			0.4–0.75		
38.865	1.25–1.75	0.6–1.125	0.4–0.75	0.44–0.505	
38.845	1.4–1.9		0.8		

[a] Best λ_c.

sum over all such diagrams is done using the tabulation of Harris and Meir[34] which extends to all such diagrams with less than 16 bonds. In Table IV, we list the coefficients of the series up to order K^{15} in general dimensions.

III. ANALYSIS

A. Methods

For comparison with other literature analyses, our analysis was made in terms of $\lambda = K^{-1}$ for the animal series and in terms of K for the YL series. The animal series are assumed to take the form $F(\lambda) = F_{\text{reg}}(\lambda) + F_{\text{sing}}(\lambda)$, with $F_{\text{reg}}(\lambda)$ the regular part, and

$$F_{\text{sing}}(\lambda) = B(\lambda-\lambda_c)^{(\theta-1)}\left[1 + C_{1F}(\lambda-\lambda_c)^{\Delta_1} + C_{2F}(\lambda-\lambda_c)^{\Delta_2} + \cdots\right] \tag{3.1a}$$

for $d \neq 8$ and

$$F_{\text{sing}}(\lambda) = B'(\lambda-\lambda_c)^{(\theta-1)}\,|\ln(\lambda-\lambda_c)|^{z} \tag{3.1b}$$

at $d = 8$.

We note that the greek letter θ has been used to denote both the dominant exponent and the correction exponents Δ_1 and z in references on the animal problem. Our usage is as defined above. In this "free energy," we had reason to suspect that F_{reg}, the nondivergent analytic background, is quite considerable. This was borne out by preliminary $d\ln$ Padé analyses and thus we have chosen to study the second derivative χ of F in depth. We have

$$\chi \sim (\lambda-\lambda_c)^{-\gamma}[1 + C_{1\chi}(\lambda-\lambda_c)^{\Delta_1} + C_{2\chi}(\lambda-\lambda_c)^{\Delta_2}] \tag{3.1c}$$

for $d \neq 8$ and

$$\chi \sim (\lambda-\lambda_c)^{-\gamma}\,|\ln(\lambda-\lambda_c)|^{z} \tag{3.1d}$$

for $d = 8$, with $\gamma = 3 - \theta$.

Taking derivatives strengthens the singular part of the series under study but, of course, loses some information. For $d \neq 8$ two derivatives appeared to be the best all round choice. For no or one derivative $d\ln$ Padé analysis[35] gave dominant exponents of zero (i.e., no divergence), but for two or more derivatives the critical behavior was no longer sensitive to the number of derivatives. Convergence was best for two derivatives. For $d = 8$ more derivatives were required and this will be discussed in detail below.

An analysis of the systematic errors caused by analytic backgrounds in Padé based analysis has been made by Adler.[36]

For series of the form of Eq. (3.1c), the simple $d\ln$ Padé method, although easy to apply, does not take into account nonanalytic confluent corrections to scaling, since it assumes $\Delta_1 = 1$. Thus, not only no estimates for the correction exponent Δ_1 can be extracted, but the values of the leading critical exponent γ and the singular point λ_c are also shifted. In order to account for the corrections we

TABLE VII. Summary of our results for lattice animals.

Dimension	$\gamma_H = \alpha_H$	$\nu_H = (2-\gamma_H)/(d-2)$	Δ_1
2	2[a]	0.641 ± 0.005[b]	0.85 ± 0.10[c]
3	$\frac{3}{2}$[a]	$\frac{1}{2}$	1.3 ± 0.2
4	$\frac{7}{6}$[a]	$\frac{5}{12}$	0.8 ± 0.2
5	0.90 ± 0.03	0.367 ± 0.011	0.65 ± 0.15
6	0.70 ± 0.04	0.325 ± 0.010	0.5 ± 0.2
7	0.59 ± 0.03	0.282 ± 0.006	0.4 ± 0.2
8	$\frac{1}{2}$[d]	$\frac{1}{4}$	. . .

[a] Exact.

[b] Selected literature average; see Table I.

[c] Literature average biased by our results; see discussion in Sec. III.

[d] Field theory which gives logarithmic corrections, $z = \frac{1}{3}$.

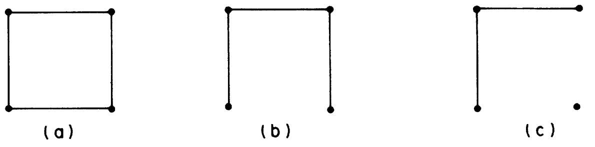

FIG. 1. Possible coverings γ_a, γ_b, and γ_c, when Γ is a square. The contribution to $\delta F(\Gamma,k)$ from γ_a is $K^4(1+b)^{4(z-2)}$, from the 4 coverings equivalent to γ_b is $-4K^3b(2+b)(1+b)^{4(z-2)}$, and from the 4 equivalent to γ_c is $4K^2b^2\,(1+b)^{3(z-2)}$.

have used two methods of analysis which have been introduced previously.[37]

In the first method, denoted below as M1, we study the logarithmic derivative of

$$B(\lambda)=\gamma\chi+(\lambda-\lambda_c)\frac{d\chi}{d\lambda}\,, \tag{3.2}$$

which has a pole at λ_c with residue $\gamma-\Delta_1$. For a given value of λ_c we have graphs of Δ_1 vs input γ, for all Padé approximants, and we choose the triplet λ_c, γ, Δ_1 where all Padé's converge to the same point. In the second method, denoted below as M2, we first transform the series in λ into series in the variable y, where

$$y=1-\left[\frac{\lambda}{\lambda_c}-1\right]^{\Delta_1}, \tag{3.3}$$

and then take Padé approximants to

$$G(y)=\Delta_1(y-1)\frac{d}{dy}\ln\chi\,, \tag{3.4}$$

which should converge to $-\gamma$.

Here we plot graphs of γ versus the input Δ_1 for different values of λ_c and choose again the triplet λ_c, γ, Δ_1 where all Padé's converge to the same point. Both these methods have proven very useful for many problems, but do require the simultaneous determination of three critical quantities.

A different type of analysis method relies on the fact[38,39] that when one divides term by term two series, both of which diverge at the same point, the resulting series diverges at $\lambda=1$. In this way one does not have to rely on estimates for the critical point. In the animal problem, there is only one independent exponent, and thus we have only one independent series. We obtain series with $\lambda_c=1$ by either dividing the logarithmic derivative of the series term by term, by the original series (denoted by D1 below) or, alternatively, dividing the square of the series term by term, by the series itself (D2). The procedure of dividing series term by term introduces a strong analytic correction.[39] Thus, as long as this term is stronger than the nonanalytic corrections, one can use methods M1 and M2, with $\Delta_1=1$. There is no way, however, to estimate which correction will dominate, and thus this method may fail when the nonanalytic correction is too strong.

These methods can be used when the critical behavior is like Eq. (3.1c). In the upper critical dimension, the critical behavior changes to that of Eq. (3.1d). Here we used the method of Adler and Privman,[40] where one takes Padé approximants to

$$L(\lambda)=(\lambda_c-\lambda)\ln(\lambda_c-\lambda)\left[\frac{\chi'}{\chi}+\frac{\gamma}{\lambda-\lambda_c}\right]. \tag{3.5}$$

B. Two and three dimensions

In two and three dimensions our main aim is to settle the question of the correction exponents.

In 2D one can see from Table I that most recent estimates for Δ_1 lie in the range 0.7–0.9, except Lam's value of 0.50 ± 0.05. Privman[26] found in the finite-size series a correction of order L^{-1}, where L is the size. If this correction represents a nonanalytic term, it gives rise to $(\lambda_c-\lambda)^{1/\nu}$, with $\nu=0.65$. We reanalyzed four existing series, including the square site series of Redelmeir[21] which contains 24 terms. Figures 2(a) and 2(b) illustrate resulting plots of γ vs Δ_1 for $\lambda_c=4.0625$, for M1 and M2 methods, respectively. This value of λ_c corresponds to one of Guttman's[20] choices. Both methods give the same γ and Δ_1 estimates, but M2 gives slightly smaller error bounds on Δ_1. From M2 we see a second correction $\Delta_2\sim1.2$. Plots for the square bond series[15] are given in Fig. 2(c) for the M2 method at $\lambda_c=5.2085$. We sampled λ_c at intervals of 0.0005 and this value appeared to give the tightest convergence. While the best convergence at this λ_c is clearly near $\Delta_1\sim0.87$, we see that there is some possibility of a confluence near $\Delta_1\sim0.4$. For $\lambda_c=5.207$ (not pictured), this region has moved to $\Delta_1\sim0.6$ and we conjecture that this region may correspond to either or both Lam's estimates of $\Delta_1\sim0.5$ or Privman's[26] lower estimate of 0.65. We then attempted to look for improved convergence of this former region. This improved convergence was seen best in the square site series at $\lambda_c=4.0627$ [Fig. 2(d)], where we obtained a weak convergence for $\Delta_1\sim0.4$ and stronger convergences near $\Delta_1=0.7$, 0.9, and 1.2. M1 gives $\Delta_1=0.7\pm0.1$. The value of 0.7 is close to that of Privman.[26] At $\lambda_c=4.0627$, however, if we take $\Delta_1\sim0.7$ we get $\gamma<2.0$ and we therefore conclude that optimal convergence for this series is closer to $\lambda_c=4.0625$. The triangular site series[19,23] also appears to have a strong convergence at $\Delta_1\sim0.9$ and a weak convergence for $\Delta_1\sim0.6$. Similar results were obtained for the bond-weighted triangular site series,[28] and are displayed in Table V.

Analyzing the series D1 and D2 gives rise to $\gamma=2.00\pm0.05$ with $\Delta_1=1.00$. For the other series, we quote $\gamma=2.00\pm0.03$ as an overall estimate and we see a correction of the order $\Delta_1\sim0.85$. There is some evidence of a correction below 0.8 in these series, but it appears to have a small amplitude.

For three dimensions, the situation is somewhat different. Here there is no accepted literature value for Δ_1 and the few estimates that exist vary from $\Delta_1=0.45$ to $\Delta_1=1.44$ with large error bars. For the newly extended 15 terms, simple cubic bond animal series optimal convergence is seen at $\lambda_c=10.615$ (using the M2 method, Fig. 3), with convergence region near $\Delta_1\sim1.3$, which appears not to have been observed in the past and a weaker region near $\Delta_1\sim0.6$, which appears to correspond to the value of

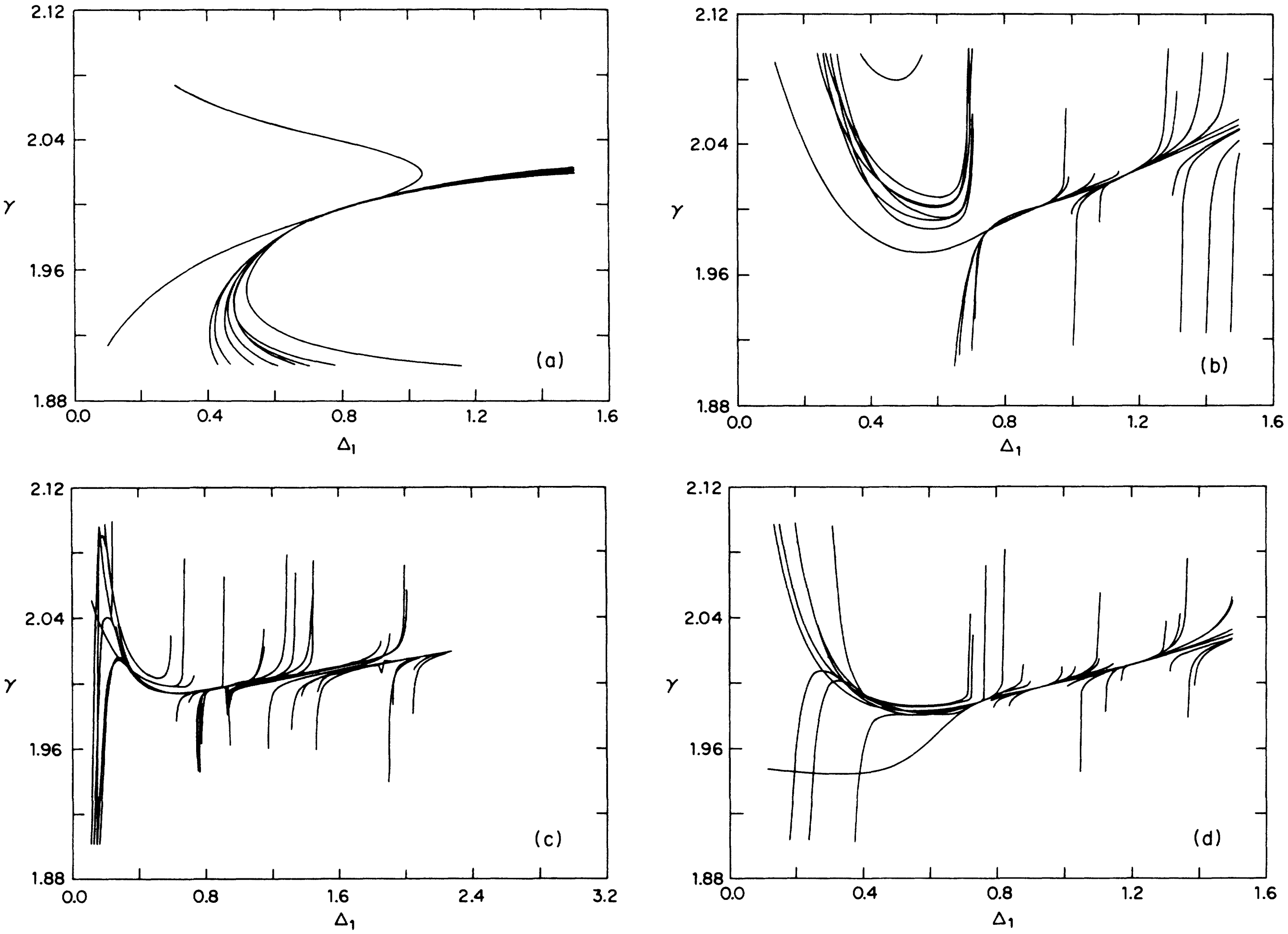

FIG. 2. Graphs of different central and near diagonal Padé approximants to γ and Δ_1 in two dimensions at (a) $\lambda_c = 4.0625$ [M1, square site animal series (Ref. 21)]; (b) $\lambda_c = 4.0625$ [M2, square site animal series (Ref. 21)]; (c) $\lambda_c = 5.2085$ [M2, square bond animal series (Ref. 15)]; and (d) $\lambda_c = 4.0627$ [M2, square site animal series (Ref. 21)].

Ref. 27. As we raise λ_c the regions coalesce, but γ falls well below the exact $\frac{3}{2}$ value. We see no clear convergence near $\Delta_1 \sim 0.45$, which is the central estimate of Lam,[29] but could agree with the upper limit of his range. The lower Δ_1 correction definitely has a smaller amplitude, as in two dimensions. We cannot exclude the possibility that these lower Δ_1 convergences are "resonances" of the higher value (see Ref. 41 for details of this phenomenon).

Note that the corresponding YL problems in 0 and 1 dimensions have only analytic corrections.[6] Although our analyses cannot entirely exclude these corrections, they do tend to suggest that the corrections in the YL model are not the same as in the animal problem in 2 and 3 dimensions. Analyzing D1 and D2 series gives rise to $\gamma = 1.5 \pm 0.05$.

C. From four to seven dimensions

In four and five dimensions, we undertook parallel but independent analysis of the (hypercubic $d-2$ dimensions) YL and animal series. Kurtze and Fisher[9] found "little positive indication of the presence" of confluent singularities in the YL series but suspected that they may be playing a role in the observed slow convergence. We began our analysis of the YL series by looking at the central Kurtze-Fisher[9] K_c choices and found that in two dimensions their choice of $K_c = 0.088\,963 \pm 2$ was very reasonable. In Fig. 4(a) we show the results of our M2 analysis at $K_c = 0.088\,964$, where for $\gamma + 1$, which is closer to the exact 2.166, we find $\Delta_1 = 0.75 \pm 0.20$. The spread of the Δ_1 estimates is quite large (for $K_c = 0.088\,95$ M1 gives nice convergence to $\Delta_1 = 0.7$), but it would appear to be distinguishable from 1.0 for most of the λ_c's that gave reasonable convergence. We prefer $K_c = 0.088\,964$ as the central K_c choice. For four-dimensional animals, the convergence is not as good as for the YL. Different γ estimates for the various λ_c choices are illustrated in Table V. In Fig. 4(b) we show the best M2 convergence at $\lambda_c = 16.3200$, which gives a central γ estimate below the exact result. For lower λ_c choices (which correspond to higher γ values), the M2 convergence downgrades. However, for M1 we show in Fig. 4(c) the situation at

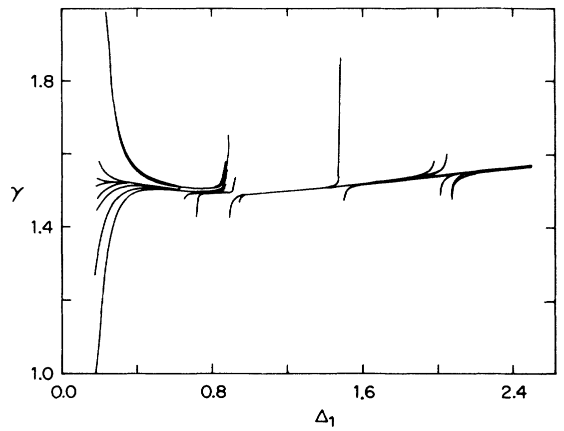

FIG. 3. Graph of different central and near diagonal Padé approximants to γ and Δ_1 in three dimensions at $\lambda_c = 10.615$ (M2, simple cubic bond animal series, this paper).

$\lambda_c = 16.3175$, where $\gamma = 1.15 \pm 0.02$. The quality of the convergence, in the M1 method, is similar throughout the range presented in Table V. This analysis leads to the estimate $\Delta_1 = 0.8 \pm 0.2$ for 4D animals which, as can be seen from Table V, is very close to the YL correction. We suggest $\lambda_c = 16.32 \pm 0.01$ for the bond animal problem on the hypercubic lattice.

In 5D the situation is very similar (see Fig. 5). We find best convergence for $K_c = 0.05202 \pm 0.00001$ for the YL series in 3D, with the exponent shown in Table V. The difference between these results and those of Ref. 9 should be attributed to the relatively larger effect of the corrections in this case. The Δ_1 estimates for both problems again are very close and fall within the range $\Delta_1 = 0.65 \pm 0.15$. Our result for the leading exponent is $\gamma = 0.905 \pm 0.015$, and $\lambda_c = 22.043 \pm 0.002$. No results are given for M1 in 5D because all approximants fall on a single line, within our λ_c range, with no clear intersection points. The results of M2 all lie on this line.

In six and seven dimensions we found λ_c results in good agreement with the $1/\sigma$ expansion.[16] We suggest $\lambda_c = 27.71 \pm 0.02$ (6D) and $\lambda_c = 33.31 \pm 0.02$ (7D). The estimates for the leading singular exponent yield $\gamma = 0.70 \pm 0.02$ (6D) and $\gamma = 0.59 \pm 0.02$ (7D). The morphology of the M2 graphs in both dimensions is similar. It is not completely clear whether there are one or two intersection regions in the range of 0.5 ± 0.2 (6D) and 0.4 ± 0.2 (7D). We show the one-region case in Fig. 6(a) for $\lambda_c = 27.71$ in 6D and the two-region case in Fig. 6(b) for $\lambda_c = 33.31$ in 7D. We suggest $\Delta_1 = 0.50 \pm 0.20$ in 6D and $\Delta_1 = 0.40 \pm 0.20$ in 7D.

Analyzing the series D1 and D2, we observe that the strongest confluence is no longer analytic in high dimensions. For example, at $d = 20$ we find $\Delta_1 = 0.5$. Thus, one cannot just read the value of γ from the intersection with $\Delta_1 = 1$, as was done in lower dimensions, but further analysis is needed. We show in Table V results of this analysis, which are consistent with the results of the analysis of the original series. In general, the analysis of the D1 series led to better results.

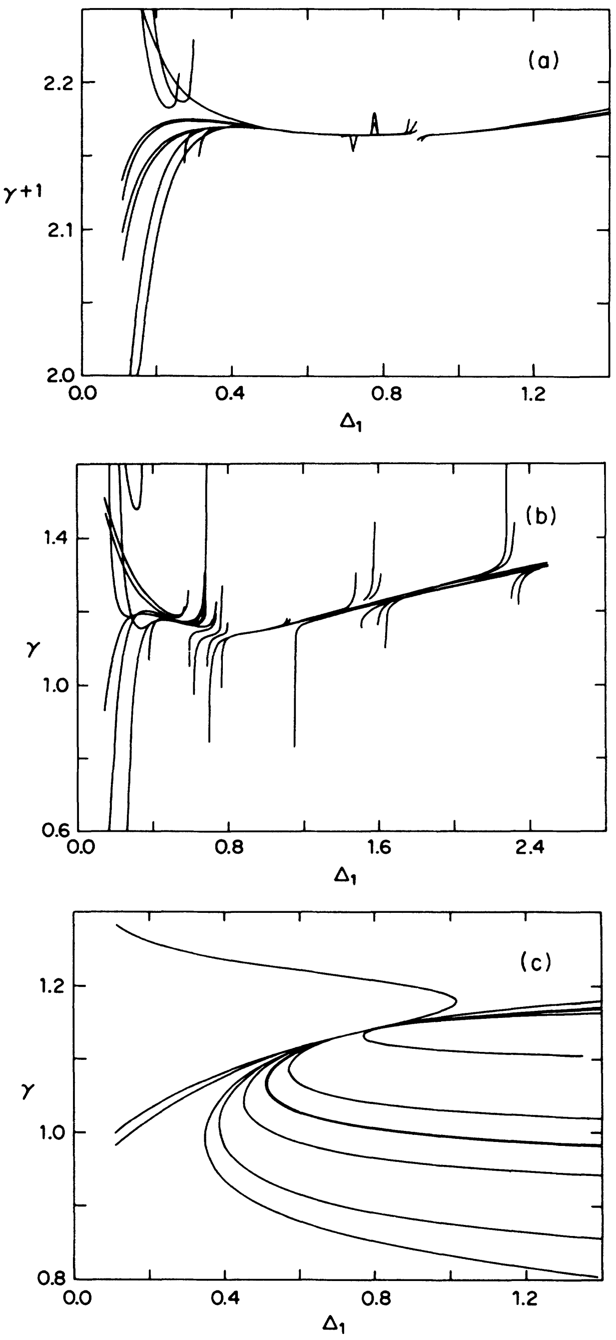

FIG. 4. Graphs of different central and near diagonal Padé approximants to γ and Δ_1 in 4 dimensions at (a) $K_c = 0.088964$ (M2, YL series in $d = 2$[9]); (b) $\lambda_c = 16.3206$ (M2, hypercubic animal series, this paper); and (c) $\lambda_c = 16.3175$ (M1, hypercubic animal series, this paper).

D. Eight dimensions

According to the field theory[2] [see also Ref. 42, Eq. (4.28)] the free energy, at the upper critical dimension, eight in this case, behaves as Eq. (1.6), with $\theta = \frac{5}{2}$, which is the mean-field result, and $z = \frac{1}{3}$. In order to check this prediction we estimated z using a method developed by Adler and Privman.[37] Results of the analysis for some

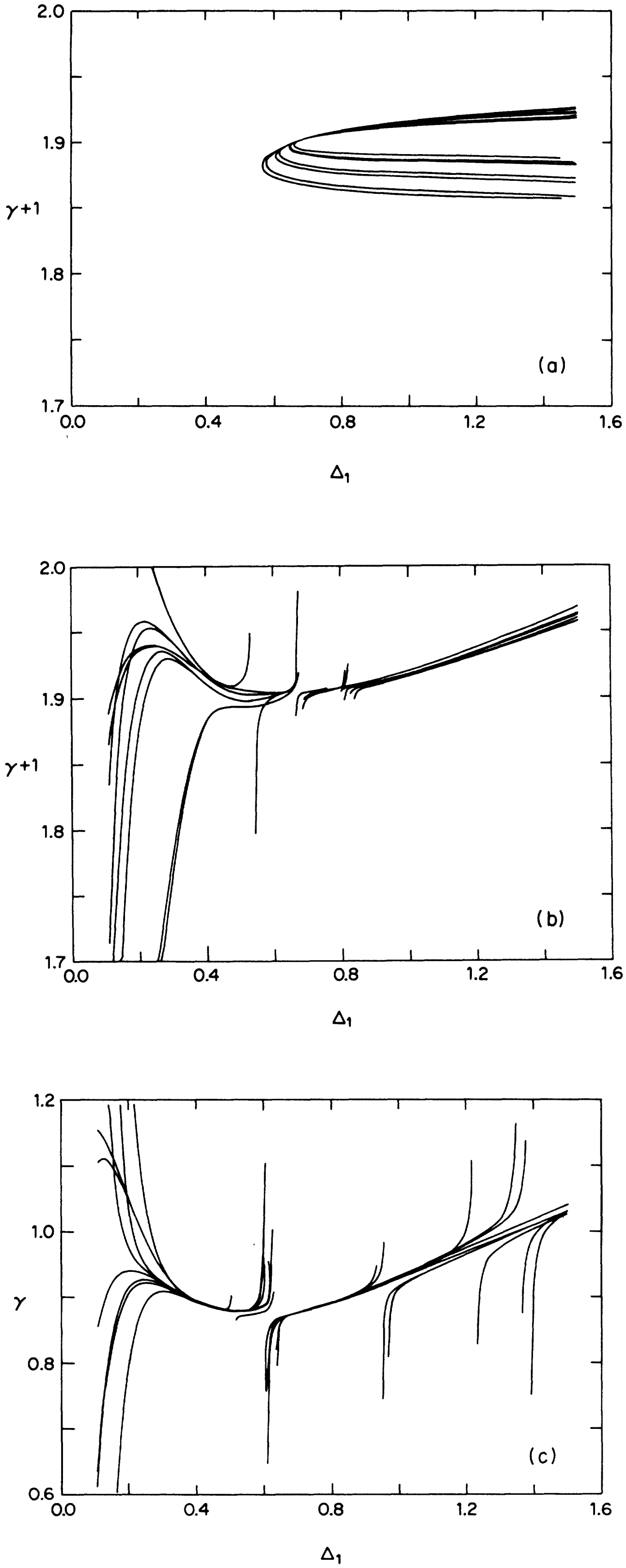

FIG. 5. Graphs of different central and near diagonal Padé approximants to γ and Δ_1 in five dimensions at (a) $K_c = 0.052022$ (M1, YL series in $d = 3^9$); (b) $K_c = 0.052022$ (M2, YL series in $d = 3^9$); and (c) $\lambda_c = 22.043$ (M2, hypercubic animal series, this paper).

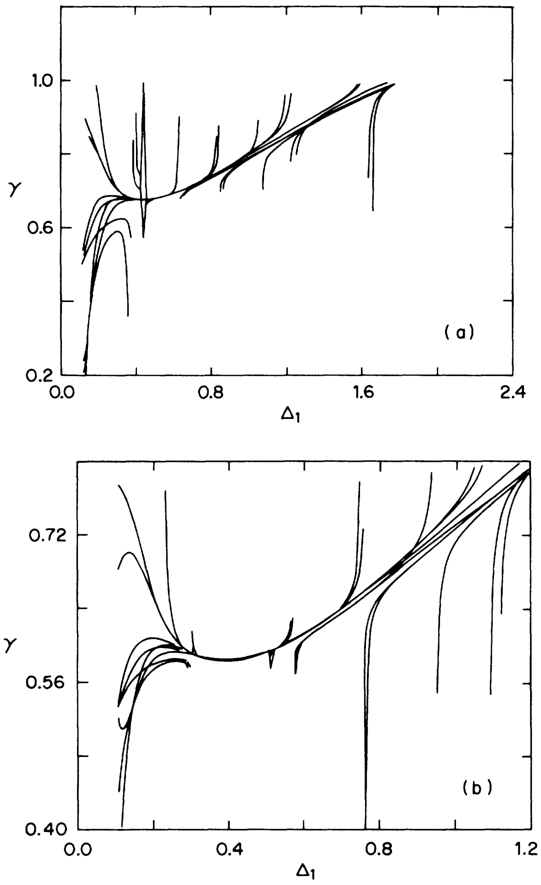

FIG. 6. Graphs of different central and near diagonal Padé approximants to γ and Δ_1 in 6 and 7 dimensions for hypercubic animal series (a) $d = 6$, $\lambda_c = 27.71$ (M2); and (b) $d = 7$, $\lambda_c = 33.31$ (M2).

different initial λ_c estimates are given in Table VI. We analyzed several derivatives of the free energy. In Table VI we list the estimates of z corresponding to the dominant exponent given by mean-field theory. We see that taking one to three derivatives lowers the z estimate, but a further derivative does not have a strong effect. We interpret this as meaning that after three derivatives we have minimized the effect of the analytic background. The upper range of the z estimates for χ'' has two Padé approximants which are considerably above the others and appear to be defective. In Fig. 7, we show a graph of z vs $\gamma+3$ which is a result of the analysis of χ''', for $\lambda_c = 38.873$. We find that the best convergence is near $\lambda_c = 38.875$ and suggest $\lambda_c = 38.875 \pm 0.020$ in excellent agreement with the $1/\sigma$ estimate[16] of $\lambda_c = 38.865 \pm 0.542$. This is significantly better agreement with the $1/\sigma$ value than that of Ref. 15 who found $\lambda_c = 39.0 \pm 2.0$.

IV. CONCLUSIONS

In this paper, we have constructed general dimensional series for the number of lattice animals embedded in a d-dimensional hypercubic lattice. These series extend exist-

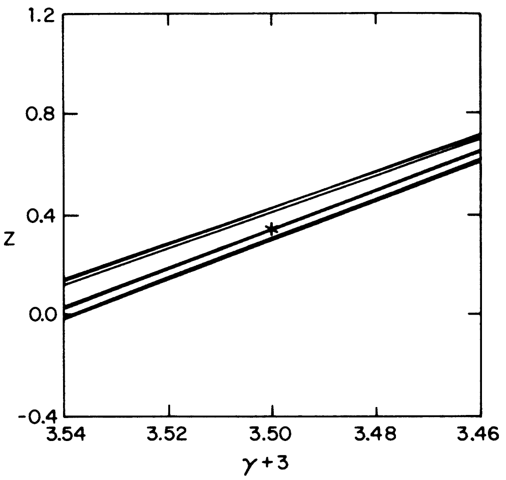

FIG. 7. Graph of different central and near diagonal Padé approximants to z as a function of γ for $d=8$ at $\lambda_c=38.873$.

ing tabulations by five terms in 4D and above and enabled us to extract reliable estimates of the singular quantities even in high dimensions.

Our λ_c results for the higher dimensions are in excellent agreement with the $1/\sigma$ expansion.[16] The γ results for the lower dimensions are in excellent agreement with the exact and other series calculations and those for the higher dimensions agree well with the ϵ expansion values for the Yang-Lee problem[10] and are considerably more precise than the older series values. Our values are $\gamma=0.59\pm0.03$, $\gamma=0.70\pm0.04$, $\gamma=0.90\pm0.03$, $\gamma=1.15\pm0.03$, $\gamma=1.50\pm0.02$, and $\gamma=2.00\pm0.02$ for seven, six, five, four, three, and two dimensions, respectively. These values exclude the Flory-type estimate[5] at $d=5$ and confirm the deviation of the ϵ expansion result from the Flory value at $d=6$.

We were also able to determine the correction to scaling exponents in all dimensions. The values of these exponents $\Delta_1=0.4\pm0.2$, 0.5 ± 0.2, 0.65 ± 0.15, 0.8 ± 0.2, and 1.3 ± 0.2, in seven, six, five, four, and three dimensions, respectively, are very close to the leading ϵ-expansion[2] value of $\epsilon/4$. The values are fairly close to Breuer's[33] method III results, but like this method III values are considerably higher than his other estimates in the higher dimensions. We also determined the correction to scaling exponent for the Yang-Lee edge problem in two and three dimensions and found values that are very close to those of the animals problem in two more dimensions. However, in low dimensions where the YL problem is exactly solved, the agreement is not that good. This may be because the YL corrections are analytic and the situation may resemble that of the 2D Ising model,[43] where the exactly solved spin-$\frac{1}{2}$ model has only analytic connections, but other members of the same universality class have nonanalytic corrections. It is also possible that the two problems differ beyond the leading singularity. The fact that the leading diagrams and hence the leading singularity are equivalent does not prevent one of the problems from having additional corrections terms. This is an intriguing possibility. Although our numerical analysis suggests that the values of the corrections for the animal and YL problems are very close, it is not enough to determine whether the two problems have identically the same correction to scaling exponents or not. The best way to resolve this question would be to carry the ϵ expansion for the animals problem to ϵ^2 and compare with the available ϵ expansion[11] for the correction exponent for the YL problem, and we plan to do so in the future.

Our results are consistent with the predictions of the field theory[2] as regards the behavior in eight dimensions, which is the upper critical dimension. Our analysis enables accurate determination of λ_c at $d=8$ and together with the new λ_c estimates for $d=6$ and 7 demonstrates that excellent agreement exists between series and $1/\sigma$ estimates for λ_c. Close agreement with the ϵ-expansion exponents is also found.

A summary of the best current results for lattice animals is given in Table VII. Exact results are cited wherever possible. Where these are not available literature averages have been cited for two dimensions. For higher dimensions we have quoted our results in the absence of exact values since these include the third-order ϵ-expansion results in every case.

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