Theory of auto-oscillations in high-power ferromagnetic resonance

X. Y. Zhang* and H. Suhl
Institute for Pure and Applied Physical Sciences and Department of Physics, University of California, San Diego, La Jolla, California 92093
(Received 9 July 1987)

An account is given of the frequently observed next notable event as the microwave signal power is increased well beyond the threshold of the old spin-wave instability: relatively-low-frequency auto-oscillations. These are attributed to a Hopf bifurcation. Taking into account excitation of the entire degenerate spin-wave manifold in the three standard cases (main resonance, subsidiary absorption, parallel pumping), we use center-manifold theory to show that a small number of new modes grow out of the excited steady state of that manifold, yielding a limit cycle which we identify with the observed auto-oscillations with characteristics qualitatively in agreement with experiment.

I. INTRODUCTION

It has been known experimentally for many years that a ferromagnetic sample resonating in a microwave field with pump power exceeding a certain threshold will display an absorption rate that pulsates at a frequency 3–4 orders lower than the ferromagnetic resonant frequency.[1] This phenomenon is called "auto-oscillation," and constitutes one of several instabilities in high-power ferromagnetic resonance. In all cases reported so far, the auto-oscillation instability sets in at a power level well exceeding the one needed to induce the Suhl instability (SI), a mechanism that breaks up the spatial uniformity of the magnetization.[2] The SI causes the growth of spinwave excitations to high, nonthermal, levels and the interaction of those excited spin waves is believed to be the cause of auto-oscillation.[3,4] This is true for all three basic resonant conditions: subsidiary resonance, parallel pumping, and main resonance. In the case of main resonance, for instance, the external drive excites the uniform mode which then feeds some of its energy to other nonuniform modes energetically degenerate with it. When the power transferred to the nonuniform modes exceeds the damping losses, these modes start to grow parametrically,[2] and then feed back, additionally damping the uniform mode in such a way that a steady state is ultimately reached. This steady state will be broken up if the external drive becomes too strong or the back-and-forth power shuffling is too high. An oscillatory state is then reached. After a transformation to a rotating frame at the resonance frequency, the equations of motion of the modes have no explicit time dependence. One can then describe the sequence of instabilities in an abstract space whose coordinate axes represent the amplitudes corresponding to the different wave numbers (the uniform precession being regarded as a mode with zero wave number). At very low pump power, the state is described by a fixed point along the axis of the uniform mode (all its other coordinates being zero). As the pump power increases, that fixed point moves continuously along the uniform-mode axis up to a certain pump power, beyond which it leaves the axis again continuously. That critical power is the SI threshold, which, when exceeded, leads to a finite level of nonuniform spin-wave excitations. As the power is increased further, the fixed point traces out a curve in the abstract space until a second threshold is reached. There it stops and undergoes Hopf bifurcation into a limit cycle. In the entire process, the bifurcations are local, which makes the analysis possible.

A number of numerical calculations on the truncated spin-wave modes provide qualitatively the same picture as described above.[4] But these calculations involved only a two-mode basis. Favoring so few out of infinitely many degenerate spin-wave modes makes the result unconvincing. Adding more modes and extrapolating to infinitely many is, as some hydrodynamics experts have pointed out, at best, a dangerous practice. Moreover, experimentally one can measure the critical power at which the auto-oscillation appears and the frequency of that oscillation. A reliable theory should be able to predict these two measurable values without any *ad hoc* assumptions. The calculation on truncated modes fails in this respect because there is no *a priori* knowledge as to which modes should be kept and whether or not adding more of the eligible modes will change the oscillation frequency and the bifurcation point.

The location of the fixed point beyond the SI was determined 30 years ago[2] (for the subsidiary resonance only). The location of the fixed point for parallel pump instability (PPI) was found to a limited extent by Zakharov, *et al.*,[3] who found the sum of the squares of the fixed-point coordinates. These authors also point out the crucial role played by spin-wave–spin-wave coupling in determining the fixed point for the PPI. Furthermore, they report numerical work with a few interacting spin waves that indicates the existence of limit cycles, etc. As a preliminary to finding the characteristics of the next higher instability, we are obliged to find the actual coordinates of the fixed point, not just the sum of their squares. In a brief earlier report[5] we used a shortcut to establish the fixed-point coordinates for the main resonance: we invoked the quantum-mechanical zero-point amplitudes of the spin wave and traced their growth to a final self-consistent value as threshold is approached. We will show in this paper that this can also be achieved by a purely classical treatment which gives qualitatively the same results and, in most cases, more accurate ones since the quasiquantum approach is only valid for zero temperature. At finite temperatures, spin waves are excited to small levels even without the drive. This is taken into account in our calculation and as a result, a distribution of spin-wave amplitudes is obtained.[6] The distribution becomes sharper as the instability threshold is approached. If the limit of zero thermal excitation is taken, the sum of squares in the case of PPI agrees with Zakharov *et al.*

The stability of the fixed point in the face of increasing signal power is investigated by subjecting each of its coordinates (i.e., the various highly excited spin-wave amplitudes) to infinitesimal deviations and examining the time development of these deviations. This amounts to finding new linear modes (not to be confused with the original spin waves) and their spectrum. Fortunately the linear matrix of this infinite-dimensional system is sparse thanks to a major simplification: Interaction terms in the equations of motion that do not involve the uniform mode explicitly serve only to renormalize the coupling constants of terms that do involve that mode. This conclusion is exact in the stable regime of the steady state, and we assume it to hold also at, and slightly beyond, bifurcation. The spectrum thus calculated consists of two isolated complex conjugate pairs, corresponding to collective modes of the system, and two bands of eigenvalues corresponding to "single-particle" decay modes.

When an eigenvalue crosses the imaginary axis from left to right, instability occurs. In our case, a complex conjugate pair with a finite imaginary part crosses the imaginary axis, suggesting (but not proving) a Hopf bifurcation. By applying the center-manifold theory, we show that the system settles into a stable limit cycle, proving that we are indeed dealing with Hopf bifurcation.

The essence of the center-manifold theory is this: In a situation in which the spectrum can naturally be divided into inevitably decaying modes X and potentially nondecaying modes Y (the two being coupled by nonlinear terms), one approximately solves for X in terms of Y. One then substitutes this approximate solution into the equations for Y. The center-manifold theory states that the resulting equations of motion for the Y modes capture locally the basic characteristics of the original full equations of motion. The function that approximates X in terms of the Y modes is called the center manifold. Appendix B illustrates, more rigorously, the statement of the theory.

The remaining sections of this paper provide detailed calculations of the three stages enumerated above. The auto-oscillation frequency is found to be of the same order as damping for a given system, and increases with pump power within certain ranges. This is in good agreement with experiments. The ratio of auto-oscillation instability threshold to that of SI threshold is also calculated and can be compared with future experiments.

II. STEADY STATE

We use classical equations of motion,

$$\frac{d\mathbf{m}}{dt} = -\gamma \mathbf{m} \times \mathbf{H} + \text{damping} ,$$

where

$$\mathbf{H} = \mathbf{H}_{\text{exch}} + \mathbf{H}_{\text{dipol}} + \mathbf{H}_{\text{dc}} + \mathbf{H}_{\text{rf}}$$

is the sum of exchange, dipolar, dc, and rf external fields. Following Ref. 2, we expand $\mathbf{m}$ in spin-wave modes and then transform to a rotating frame; the mode equations then become equations of slowly varying amplitudes. The steady state is referred to as the rotating frame. For convenience, we confine the calculations to the degenerate spin-wave manifold, namely, all the spin waves under consideration have the same frequency ω_0. Therefore, a time-independent spin-wave-amplitude solution is equivalent to steady-state solutions. Although the existence of the steady state is an experimental fact and is not expected to change when other spin waves outside the degenerate manifold are included, it is not straightforward to make correspondence of the time-independent spin-wave-amplitude state and the system's steady state, because

$$m^{+} = \sum_k a_k e^{-ikx + i\omega_k t}$$

and the observables are related to $|m^{+}|^2$. This is one of the drawbacks of the mode-expansion method. Also, for simplicity, the anisotropy field is omitted. In most of the ferrite materials, anisotropy is negligible compared to the dc saturation field. Even when it is included, it does not create new terms but modifies the coefficients of some existing terms. This may be important for further instabilities, e.g., route to chaos, etc., but is not a concern for the present purpose. Here, we assume the dc field is along the easy axis. Other cases will be discussed in the end.

For the three different resonant conditions, only the lowest-order nonlinear terms are kept. All the detuning terms are omitted. Eigenmode B_k is used which is related to a_k by a canonical transformation.[2]

We begin with the main resonance. The equations linking the spin-wave amplitudes B_k with the uniform precession amplitude B_0 are

$$\dot{B}_k = i\rho_{k0} B_0^2 B_{-k}^{*} - \eta_k B_k + \sum_{k'} \rho_{kk'} B_{k'} B_{-k'} B_{-k}^{*} + \xi_k , \tag{1a}$$

$$\dot{B}_0 = i \sum_k \rho_{k0} B_k B_{-k} B_0^{*} - \eta_0 B_0 - i\omega_s + \xi_0 . \tag{1b}$$

η_k and ξ_k are, respectively, the damping constant and the thermal noise amplitude at wave number k. We assume that $\eta_k = \eta =$ constant and that on the average

$$\overline{\xi_k \xi_{k'}^{*}} = \delta_{kk'} |\xi_k|^2 ,$$

$$\overline{\xi_k \xi_{k'}} = 0 .$$

B_0 is the amplitude of the uniform mode driven directly by external rf field ω_s. We solve for the steady state of (1) by iteration. First, the $\rho_{kk'}$ terms are neglected, based on the fact that spin-wave excitations B_k are small and we have

$$B_k = \frac{-i\xi_k \eta_0 - \xi^*_{-k}\rho_{k0}B_0^2}{\eta^2 - \rho_{k0}^2 |B_0|^4}$$

and

$$B_k B_{-k} = \frac{2i\,|\xi_k|^2 \rho_{k0}\eta B_0^2}{(\eta^2 - \rho_{k0}^2 |B_0|^4)^2} \; . \qquad (2)$$

Equation (2) shows that as the SI threshold is approached, i.e., $\rho_{k0}|B_0|^2 \to \eta$, spin-wave amplitudes start to grow and the distribution becomes sharper. Substituting (2) in the terms containing $\rho_{kk'}$ in Eq. (1) and regrouping like terms, we find a renormalized ρ_{k0},

$$\rho'_{k0} = \rho_{k0} + 2i \sum_{k'} \frac{\rho_{kk'}|\xi_{k'}|^2 \rho_{k'0}}{(\eta^2 - \rho_{k'0}^2 |B_0|^4)^2} \; .$$

Therefore, after taking into account the nonuniform spin-wave interactions, expression (2) is still valid except ρ_{k0} should be replaced by ρ'_{k0} and this leads to an iteration scheme, whose fixed point ρ_{k0} is given by

$$\bar{\rho}_{k0} = \rho_{k0} + 2i \sum_{k'} \frac{\rho_{kk'}|\xi_{k'}|^2 \bar{\rho}_{k'0}}{(\eta^2 - |\bar{\rho}_{k'0}|^2 |B_0|^4)^2} \; . \qquad (3)$$

Now the coupling constant $\bar{\rho}_{k0}$ is a function of $|B_0|^2$. It monotonically decreases with increasing $|B_0|^2$ such that $|\bar{\rho}_{k'0}|\ |B_0|^2$ remains less than η. This is obvious if we replace $\bar{\rho}_{k0}$ in (3) by a k-independent variable $\bar{\rho}$,

$$|\bar{\rho}|^2 = \frac{\rho_{k0}^2}{1+\Sigma^2} \; ,$$

where

$$\Sigma = 2 \sum_k \frac{\rho_{kk'}|\xi_{k'}|^2}{(\eta^2 - |\bar{\rho}|^2 |B_0|^4)^2} \; .$$

This important result shows that the expressions

$$B_k B_{-k} = \frac{2i\,|\xi_k|^2 \bar{\rho}_{k0}\eta B_0^2}{(\eta^2 - |\bar{\rho}_{k0}|^2 |B_0|^4)^2} \qquad (4a)$$

and

$$|B_k|^2 = \frac{|\xi_k|^2(\eta^2 + |\bar{\rho}_{k0}|^2 |B_0|^4)}{(\eta^2 - |\bar{\rho}_{k0}|^2 |B_0|^4)^2} \qquad (4b)$$

will never have vanishing denominators. It also shows that the interaction with other spin waves stabilizes the parametric excitation by the uniform mode. Note that this main resonance stabilization contrasts sharply with subsidiary resonance stabilization which depends on ρ_{k0} only. The sharp drop in χ'', the imaginary part of susceptibility observed above threshold, is adequately accounted for by ρ_{k0} coupling alone. The $\rho_{kk'}$ stabilization, on the other hand, explains the observed slow decline of the adsorption at the main resonance.

We also note that the pair of spin waves with opposite wave number which are excited by the uniform mode have their phase locked to the uniform mode through (4),

$$\varphi_k + \varphi_{-k} = 2\varphi_0 + \text{const} \; .$$

This has been concluded independently by several authors.[7,3] Furthermore, it should be pointed out that although the spin-wave amplitudes appear to be proportional to the thermal drive in (4) as threshold is approached, the denominator also becomes proportional to that drive, resulting in a finite, nonthermal value for the ratio. If one wishes to dispense with the thermal drive, one finds that the spin-wave excitation is zero up to the threshold, but then becomes finite because the denominator of (4) also vanishes. It is then still possible to determine the sums $\sum_k |B_k|^2$ and $\sum_k B_k B_{-k}$, but not the individual terms.[8] We feel, however, that inclusion of the thermal field (or at least some zero-point motion) corresponds more nearly to the physical situation, even though it has little or no effect on the values of observables. A case in point is the equation for $|B_0|$: substituting (4) in (1), we obtain for the steady state at the main resonance

$$|B_0|^2 = \frac{\omega_s^2(\eta^2|\Sigma|^2 + 2\eta\,\mathrm{Im}\Sigma B_0^2)}{(\eta^2 - |\Sigma|^2)} \; , \qquad (5)$$

where

$$\Sigma = \sum_k B_k B_{-k} \; .$$

Equation (5) contains no critical dependence on the thermal drive. The linear spectrum, as will be seen in the next section, is also not affected by the thermal drive.

For the case of parallel pumping, the equations are,

$$\dot{B}_k = -\eta B_k = i\omega_s V_k B^*_{-k} + \sum_k \rho_{kk'} B_{k'} B_{-k'} B^*_{-k} + \xi_k \; , \qquad (6)$$

where V_k is the coupling between external drive and the kth mode. By the same token, because of the spin-wave interactions, V_k is renormalized to $\bar{V}_k$,

$$\bar{V}_k = V_k - 2i \sum_{k'} \frac{\bar{V}_{k'}\eta\,|\xi_{k'}|^2 \rho_{kk'}}{(\eta^2 - |V_{k'}|^2 \omega_s^2)^2} \; , \qquad (7)$$

therefore,

$$B_k B_{-k} = \frac{2i\eta \bar{V}_k |\xi_k|^2 \omega_s}{(\eta^2 - |\bar{V}_k|^2 \omega_s^2)^2} \qquad (8)$$

is the steady-state solution. The qualitative feature of $\bar{V}_k$'s dependence on ω_s can be sketched from (7). $|\bar{V}_k| \sim V_k$ if $\omega_s < \eta/V_k$ and $|\bar{V}_k| \lesssim \eta/\omega_s$ if $\omega_s > \eta/V_k$. This guarantees the nonsingular behavior of (8), again because of the spin-wave interactions. However, in this case it is more significant than in the case of main resonance. For even without renormalization of ρ_{k0}, the amplitude in the main resonance can still be finite provided $|B_0|^2$ sticks below η/ρ_{k0}, which it will do because of the feedback equation (5). For parallel pumping, ω_s is controlled by the experimenter, at least in the situation we are considering. So it can assume any value it wishes and a nonmodified V_k inevitably leads to divergence. This was pointed out earlier in a different way by Zakharov *et al.* The steady state of subsidiary resonance is already available in Ref. 2.

III. LINEAR SPECTRUM

Next, we consider the spectrum of small excitations of the steady state discussed in Sec. II. We only allow the spin-wave amplitudes to vary and keep the drive fixed. This may not be the case in some experimental conditions, e.g., when a high-Q microwave cavity is employed and the sample strongly reacts on the drive.[9] In some of those cases, nonlinear circuit coupling alone can give rise to various interesting phenomena, but here we are only interested in the intrinsic nonlinear behavior solely caused by spin-wave couplings.

We begin with the main resonance. As is seen from the steady-state calculation, the effect of the interaction of waves with nonzero wave numbers is to modify the coupling constant ρ_{k0} of uniform to nonuniform modes. We thus assume that nonuniform spin-wave interaction terms can be dropped in the variation. The linearized equation is then,

$$\begin{aligned}\delta\dot{B}_k &= i\bar{\rho}_{k0}B_0^2\delta B^*_{-k} - \eta\delta B_k \\ &\quad + i\left[\frac{\partial\bar{\rho}_{k0}}{\partial B_0}B_0^2 + 2B_0\bar{\rho}_{ko}\right]B^*_{-k}\delta B_0 , \\ \delta\dot{B}_0 &= i\sum_k \rho_{k0}B_kB_{-k}\delta B_0^* - \eta\delta B_0 \\ &\quad + i\sum_k \rho_{k0}B_kB_0^*\delta B_{-k} + i\sum_k \rho_{k0}B_{-k}B_0^*\delta B_k .\end{aligned} \tag{9}$$

Careful examination of $\bar{\rho}_{k0}$ as a function of $|B_0|^2$ in (3) shows that at and above the SI threshold, $|\bar{\rho}_{k0}|\ |B_0|^2 \lesssim \eta$. So a rough approximation of $(\partial\bar{\rho}_{k0}/\partial B_0)B_0^2$ will be $2\eta B_0^2/B_0^3$ which is $2\eta B_0/(\eta/\bar{\rho}_{k0})$ or $2B_0\bar{\rho}_{k0}$. Therefore, it can be incorporated in the second term from the same bracket. We simply neglect it and let ρ_k denote $\bar{\rho}_{k0}$. Assuming a $e^{\lambda t}$ time dependence, we then arrive at the characteristic equation,

$$\begin{aligned}(\lambda+\eta)&\left[1+4|B_0|^2\sum_k \frac{\rho_k^2|B_k|^2}{D_k}\right] \\ &= \pm 2|B_0|^2\left[\sum_k \rho_k C_k + 4|B_0|^4\sum_k \frac{\rho_k^3 C_k}{D_k}\right] ,\end{aligned} \tag{10}$$

where

$$D_k = (\eta+\lambda)^2 - \rho_k^2|B_0|^4 ,$$

$$C_k = \frac{\rho_k\eta|\xi_k|^2}{(\eta^2-\rho_k^2|B_0|^4)^2} .$$

We first note that (10) has two bands of single-particle decay modes

$$\lambda = -\eta \pm \rho_k|B_0|^2 \mp \delta_k ,$$

where δ_k is a small shift from the original single-particle spectrum, which can be determined from (10). To seek collective mode solutions, we can convert the sums to integration; this leads to a complex transcendental equation that has to be solved numerically. As will be shown in Sec. IV, what we need is an exact analytical expression for the eigenvalues that cross the imaginary axis from left to right, and a distribution of the rest of the eigenvalues but not their exact values. This can be achieved if we assume ρ_k to be k independent. For simplicity, we let $\rho_k=1$. The characteristic equation becomes

$$(\lambda+\eta)\left[1+\frac{4|B_0|^2|B|^2N}{D}\right] = \pm N|B|^2\left[1+\frac{4|B_0|^4}{D}\right] . \tag{11}$$

In obtaining this we have assumed $\sum_k |B_k|^2 = N|B|^2$ and $|B_kB_{-k}| = |B|^2$. The single-particle decay modes collapse from two bands to two points, $\lambda = -\eta \pm |B_0|^2$. Since $D=(\eta+\lambda)^2 - |B_0|^4$, we regroup (11),

$$4|B_0|^2|B|^2N\frac{[(\lambda+\eta)\mp|B_0|^2]}{(\lambda+\eta)^2-|B_0|^4} = \pm N|B|^2 - (\lambda+\eta) , \tag{12}$$

the quartic equation then reduces to two quadratic equations,

$$(\eta+\lambda)^2 \pm (\eta+\lambda)(|B_0|^2 - N|B|^2) + 3|B_0|^2N|B|^2 = 0 \tag{13}$$

with solutions

$$\lambda = -\eta \pm \frac{|B|^2N - |B_0|^2}{2} \pm \frac{i}{2}(14N|B|^2|B_0|^2 - |B_0|^4 - N^2|B|^4)^{1/2} . \tag{14}$$

These are the collective modes of the system that consist of two pairs of complex conjugates. When ω_s is increased to a point that

$$\frac{|B|^2N - |B_0|^2}{2} = \eta ,$$

one pair crosses the imaginary axis with a value of $\lambda = \pm 2i\sqrt{2}\eta$. This is the auto-oscillation frequency for the main resonance case.

The collective modes can also be derived if we only keep two modes, a uniform one and a nonuniform one. This is no surprise because our derivation assumes that all the nonuniform modes are equal and are coupled only to the uniform mode. The collective mode is strictly a manifestation of interaction between the uniform spin wave and nonuniform spin waves. In Appendix A, we demonstrate exactly how an infinite number of spin waves, once assumed to have equal amplitudes, reduce to two spin waves and show that the two bands that collapse to two points has a degeneracy of $N-1$.[10] Since the characteristic equation (10) is nonsingular, turning on the k dependence of ρ_k only spreads the two points to two bands but keeps the number of decay modes $2(N-1)$ intact. This means an exact solution with ρ_k should also have only four eigenvalues that correspond to the collective modes, because with N nonuniform spin waves and one uniform spin wave, the total number of degrees of freedom is $2(N+1)$, leaving $2(N+1)-2(N-1)=4$ modes. In addition, (10) is a real coefficient equation. In fact, one of the pairs that crosses the imaginary axis can be calculated directly from (10) right at the point of crossing as shown in Ref. 5. Thus we are certain that the results from the simplified calculation ($\rho_k=1$), qualitatively represents the true spectrum. It is important to note that since $\rho_k|B_0|^2$ is fixed below η, when ω_s increases, only two pairs of collective modes move and they move in opposite directions, the rest of the single-particle decay modes stand still. This can be seen from numerical interactions of (10). We can estimate the power dependence of auto-oscillation frequency and the bifurcation point through (14). Substituting the steady-state solution

$$N|B|^2=\frac{\omega_s}{\eta^{1/2}}-\eta \quad \text{and} \quad |B_0|\sim\eta$$

(see Appendix B), we have the auto-oscillation frequency

$$f=\left[\left[\frac{\omega_s}{\eta^{1/2}}-\eta\right]\left[15\eta-\frac{\omega_s}{\eta^{1/2}}\right]\right]^{1/2},$$

where $\omega_s=4\eta\eta^{1/2}$ is the auto-oscillation threshold. Beyond it, f increases with ω_s until $\omega_s=8\eta\eta^{1/2}$, and then starts to decrease. It might well be that before ω_s reaches $8\eta\eta^{1/2}$, a second instability occurs and the limit cycle no longer exists, so that experimentally one only observes a monatonic increase of f with ω_s.[11] Since the SI threshold is at $\omega_s=\omega_s^0=\eta\eta^{1/2}$ and

$$\ln\frac{\rho'}{\rho_0}=\ln\left[\frac{\omega_s'}{\omega_s^0}\right]^2=\ln\left[\frac{4\eta\eta^{1/2}}{\eta\eta^{1/2}}\right]^2=\ln 16\ ,$$

auto-oscillation should be observed at the power about 10 dB above the SI, i.e., 10 dB above the onset of premature saturation of absorption.

For the case of parallel pumping, the linear spectrum calculation is more difficult. Taking into account the mode-mode coupling only to the extent that it modifies the coupling constant to the drive will not yield any collective modes. Keeping these interactions to their full extent seems to render the problem intractable. To compromise, we keep the coupling of each (and every) individual spin wave to only one mode: namely to the one that has the strongest coupling constant to the drive. Denoting the wave number of that mode by k_0, the linearized equations are found to be

$$\delta\dot{B}_k=-\eta\delta B_k+i\omega_s\bar{V}_k\delta B^*_{-k}+\rho_{kk_0}B_{k_0}b_{-k_0}\delta B^*_{-k}$$
$$+\rho_{kk_0}B^*_{-k}(B_{k_0}\delta B_{-k_0}+B_{-k_0}\delta B_{k_0})\ ,$$
$$\delta\dot{B}_{k_0}=-\eta\delta B_{k_0}+\omega_s\bar{V}_{k_0}\delta B^*_{-k_0}+\sum_k\rho_{k_0k}B_kB_{-k}\delta B^*_{-k_0}$$
$$+\sum_k\rho_{k_0k}B^*_{-k_0}(B_k\delta B_{-k}+B_{-k}\delta B_k)\ ,$$

where the "bare" coupling constant V_{k_0} is the largest among all V_k's. The characteristic equation is,

$$A_k=\pm|T_k|\ ,$$

where

$$A_k=(\lambda+\eta)\left[1+4|B_0|^2\sum_k\frac{\rho_{k_0}^2|B_k|^2}{[(\lambda+\eta)^2-|\bar{V}_k|^2\omega_s^2]^2}\frac{\bar{V}_k}{\bar{V}_0}\right],$$

$$T_k=i\omega_s\bar{V}_0\left[1+4|B_0|^2\sum_k\frac{\rho_{k_0}^2|B_k|^2}{[(\lambda+\eta)^2-|\bar{V}_k|^2\omega_s^2]^2}\frac{\bar{V}_k}{\bar{V}_0}\right].$$

For notational simplicity, B_0 denotes B_{k_0} and ρ_{k_0} denotes ρ_{kk_0}. Because of the argument made in Sec. II that the phase of excited spin waves are locked to the drive, we have written $B_0B_0B_k^*B^*_{-k}$ as $|B_0|^2|B_k|^2$ in the above derivation. We are seeking only those eigenvalues that cross the imaginary axis. The same method was used in Ref. 5. When $\eta^2\sim|\bar{V}_k|^2\omega_s^2$, the eigenvalue equation reduces to

$$(\lambda+\eta)^2\left[1+\frac{4|B_0|^2\Sigma_1}{[\lambda(\lambda+2\eta)]}\right]^2=\omega_s^2|\bar{V}_0|^2\left[1+16|B_0|^4\frac{|\Sigma|^2}{\lambda^2(\lambda+2\eta)^2}+\frac{4|B_0|^2(\Sigma+\Sigma^*)}{\lambda(2\eta+\lambda)}\right],$$

where

$$\Sigma_1=\sum_k\rho_{k0}^2|B_k|^2\ ,$$

$$\Sigma=\sum_k\rho_{k0}^2|B_k|^2\frac{\bar{V}_k}{\bar{V}_0}\ .$$

Since $\Sigma_1^2\approx|\Sigma|^2$, we then have

$$\lambda(2\eta+\lambda)=-4|B_0|^2\Sigma_1\pm i2\sqrt{2}|B_0|\eta\sqrt{\delta}\ , \tag{15}$$

where

$$\delta=\Sigma_1-\frac{\Sigma+\Sigma^*}{2}\ .$$

Equation (15) has a solution

$$\lambda=\pm 2i|B_0|\sqrt{\Sigma_1}$$

provided

$$\Sigma+\Sigma^*=-2\Sigma \ . \tag{16}$$

This will be satisfied if the drive ω_s is such that

$$\mathrm{Re}\left[\frac{\overline{V}_k(\omega_s)}{\overline{V}_0(\omega_s)}\right]=-1 \ .$$

It certainly has a solution if we look back at the steady-state calculation in Sec. II. Also, this is only a sufficient condition for (11) to hold. We now evaluate the oscillation frequency $2|B_0|\sqrt{\Sigma_1}$. Since

$$\omega_s\overline{V}_k=\omega_s V_k+\sum_k \rho_{k0}B_kB_{-k}+\cdots \sum_k \rho_{k0}B_kB_{-k}\sim\omega_s\overline{V}_k\sim\eta \ ;$$

and $\rho|B_0|^2\sim\eta$, therefore, $2|B_0|\sqrt{\Sigma_1}\sim 2\eta$. Again, the frequency is of the same order as the damping rate.[4,12]

Finally we calculate the subsidiary resonance case. The linearized equations are

$$\delta\dot{B}_0=-\eta\delta B_0-\sum_k \rho_{k0}(B_k\delta B_{-k}+B_k\delta B_k)+i(\omega-\omega_H)\delta B_0 \ ,$$

$$\delta B_k=-\eta\delta B_k+i\rho_{k0}B^*_{-k}\delta B_0+i\rho_k B_0\delta B^*_{-k} \ ,$$

where ω and ω_h are the driving and resonant frequencies, respectively. The characteristic equation is,

$$(\lambda+\eta)^2+\left[2\sum_k\frac{\rho_k^2|B_k|^2(\lambda+\eta)}{D}-\Delta\omega\right]^2=4|B_0|^2\left[\sum_k\frac{\rho_k^3|B_k|^2}{D}\right]^2 ,$$

where

$$D=(\eta+\lambda)^2-\rho_k^2|B_0|^2 \ .$$

In the case of coincidence, $\Delta\omega=0$, we have $\lambda=\pm i(\eta/\sqrt{2})$ provided $\eta^2=(\frac{2}{3})\sum_k\rho_k^2|B_k|^2$. The latter can always be satisfied in a certain power range. In deriving the above result, it is kept in mind that $\rho_k^2|B_0|^2\sim\eta^2$. Under an ordinary subsidiary resonance condition, $\Delta\omega\neq 0$, η can be replaced by $(\eta^2+\Delta\omega^2)^{1/2}$ if $\Delta\omega$ is small. The auto-oscillation frequency then becomes $f=(1/\sqrt{2})(\eta^2+(\omega-\omega_H)^2)^{1/2}$. This dc field dependence of the frequency in subsidiary resonance was found in one of the first observations of auto-oscillation.[1]

IV. CENTER MANIFOLD

As shown in the last section, the steady state attained above the old SI threshold is in turn destroyed at a second threshold, with a pair of complex conjugate eigenvalues of the linearized equations moving across the imaginary axis. The imaginary part of these eigenvalues is identified with the auto-oscillation frequency if it is taken for granted that beyond the new threshold, the system settles down in a stable limit cycle. We will prove in this section that the limit cycle does exist. Since the calculations involved are extremely lengthy, we only show it for the main resonance case. Parallel pumping and subsidiary resonance should follow similarly.

The equations for the deviations from the steady state at the instability point of Eq. (1) with nonlinear terms retained are

$$\begin{aligned}\delta\dot{B}_k&=i\rho_{k0}B_0^2\delta B^*_{-k}-\eta\delta B_k\\&\quad+i2B_0\rho_{k0}B^*_{-k}\delta B_0 2i\rho_{k0}B_0\delta B_0\delta B^*_{-k}\\&\quad+i\rho_{k0}B^*_{-k}\delta B_0^2+i\rho_{k0}\delta B_0^2\delta B^*_{-k} \ ,\\ \delta\dot{B}_0&=i\sum_k\rho_{k0}B_kB_{-k}\delta B_0^*-\eta\delta B_0\\&\quad+i\sum_k B_0^*\rho_{k0}(B_k\delta B_{-k}+B_k\delta B_k)\\&\quad+i\sum_k\delta B_0^*\rho_{k0}(B_k\delta B_{-k}+B_k\delta B_k)\\&\quad+i\sum_k\rho_{k0}\delta B_k\delta B_{-k}\delta B_0 \ ,\end{aligned}\tag{17}$$

where the nonuniform spin-wave interactions are neglected. In principle, nonlinear terms in (17), if they have the right sign, will restore stability of the linearly unstable mode. However, the unstable mode in our system is a linear combination of the δB_k's, showing the collective nature of the unstable modes. Thus it is impossible to identify whether the nonlinear terms do have the "right" sign or not from Eq. (17) as it stands. It is necessary to transform (17) to a new normal-mode base. With $\alpha(t)$ denoting the amplitude of the unstable-mode time dependence, the equations then have the form (λ complex with positive real part):

$$\dot{\alpha}(t)=\lambda\alpha(t)+\text{nonlinear terms} \ . \tag{18}$$

However, the nonlinear terms in (18) contain various other decaying modes, and the difficulty of solving them is equivalent to solving the whole set of equations. Fortunately, center manifold theory can be applied to resolve this problem. It provides a standard recipe for systematically approximating the decay modes in terms of the unstable modes and then substituting them in (18), which then assumes a standard form:

$$\dot{\alpha}(t)=\lambda\alpha(t)+\beta|\alpha|^2\alpha+\text{higher orders in } \alpha \ . \tag{19}$$

The function that expresses the decay modes in terms of the unstable mode is called the center manifold. A mathematical introduction to the theory can be found in Ref. 13. In order to make the line of calculation clear, we first use an abstract notation. Our method of computing the center manifold follows closely to that of Crawford's treatment on plasma instabilities and Hopf bifurcation.[14]

Let v be a vector in a configuration space with its bases being the complex spin-wave amplitudes,

$$v=(\delta B_1,\delta B_1^*,\delta B_2,\delta B_2^*,\ldots,\delta B_0,\delta B_0^*) \ .$$

Equation (17) can then be written as,

$$\frac{dv}{dt}=Lv+N(v) \ , \tag{20}$$

where L and N are linear and nonlinear operators, respectively. Denote by ψ and $\tilde{\psi}$ the eigenvectors corresponding to the unstable modes λ and λ^* with their time dependence $\alpha(t)$ and $\alpha^*(t)$ factored out, we have

$$L\psi=\lambda\psi\ ,\qquad L\tilde{\psi}=\lambda^*\tilde{\psi}\ . \tag{21}$$

We then expand $v(t)$ in terms of those modes,

$$V_c=\alpha(t)\psi+\alpha^*(t)\tilde{\psi}+S_c(\alpha,\alpha^*,\psi,\tilde{\psi})\ , \tag{22}$$

where V_c is the center manifold. For the present purpose, we approximate S_c in the lowest nontrivial order of α,

$$S_c=h_1\alpha^2+h_1^*\alpha^{*2}+h_2\,|\,\alpha\,|^2\ , \tag{23}$$

where h_1, h_1^*, and h_2 are to be determined, and note that, by definition (22), S_c is biorthogonal to $\psi,\tilde{\psi}$.

Since L is not a self-adjoint operator, we need to construct eigenvectors corresponding to $L^\dagger$, as well as L, in order to use the biorthogonality property to separate our different modes, and we have

$$L^\dagger\xi=\lambda^*\xi\ ,$$

$$L^\dagger\tilde{\xi}=\lambda\tilde{\xi}\ ,$$

where

$$\langle\xi,\tilde{\psi}\rangle=0,\quad\langle\tilde{\xi},\psi\rangle=0$$

and also

$$\langle\xi,h_1\rangle=\langle\xi,h_1^*\rangle=\langle\xi,h_2\rangle=0\ ,$$

$$\langle\tilde{\xi},h_1\rangle=\langle\tilde{\xi},h_1^*\rangle=\langle\tilde{\xi},h_2\rangle=0\ .$$

The latter can be easily verified once h_1, h_1^*, and h_2 are obtained.

Substituting (22) in (20), we get

$$\dot{\alpha}\psi+\dot{\alpha}^*\tilde{\psi}+\dot{S}_c=\lambda\alpha\psi+\lambda^*\alpha^*\tilde{\psi}+LS_c+N(\alpha\psi+\alpha^*\tilde{\psi}+S_c)\ . \tag{24}$$

Taking the inner product with ξ, (24) becomes

$$\dot{\alpha}=\lambda\alpha+\frac{\langle\xi,N\rangle}{\langle\xi,\psi\rangle}\ . \tag{25}$$

The next step is to find the coefficient of $|\,\alpha\,|^2\alpha$ in $\langle\xi,N\rangle$. We first compute S_c by placing (25) and its complex conjugate [as well as (23)] back into (24),

$$2\lambda\alpha^2(t)h_1+2\lambda^*\alpha^{*2}(t)h_1^*+(\lambda+\lambda^*)\,|\,\alpha\,|^2h_2=L(h_1\alpha^2+h_1^*\alpha^{*2}+h_2\,|\,\alpha\,|^2)-\frac{\langle\xi,N\rangle}{\langle\xi,\psi\rangle}\psi-\frac{\langle\tilde{\xi},N\rangle}{\langle\tilde{\xi},\tilde{\psi}\rangle}\tilde{\psi}+N\ , \tag{26}$$

and comparing the coefficients of α^2, α^{*2}, and $|\,\alpha\,|^2$ on both sides of (26), we obtain

$$h_1=(L-2\lambda)^{-1}\ ,\qquad h_1^*=(L-2\lambda^*)^{-1}\ ,\qquad h_2=[L-(\lambda+\lambda^*)]^{-1}\ , \tag{27}$$

where h_1 is the coefficient of α^2 in Y_C, h^*, the coefficient of $(\alpha^*)^2$ in Y_C, and h_2 the coefficient of $|\,\alpha\,|^2$ in Y_C and

$$Y_C=\left[\frac{\langle\xi,N\rangle}{\langle\xi,\psi\rangle}\psi+\frac{\langle\tilde{\xi},N\rangle}{\langle\tilde{\xi},\tilde{\psi}\rangle}\tilde{\psi}-N\right]\ .$$

With the help of the above formulas, we now proceed to compute β, the coefficient of the cubic term in the normal form. It is seen in (27) that in order to calculate h, the linear matrix L has to be inverted. This is analytically feasible only if we assume that all the nonuniform spin waves are identical. The eigenvectors ψ and $\tilde{\psi}$ can be obtained from the linear part of (17):

$$(\lambda+\eta)\delta B_k-iB_0^2\delta B_{-k}^*=2iB_0B_{-k}^*\delta B_0\ , \tag{28a}$$

$$(\lambda+\eta)\delta B_0-i\sum_k B_kB_{-k}\delta B_0^*=i\sum_k B_0^*\rho_{k0}(B_k\delta B_{-k}+B_{-k}\delta B_k)\ . \tag{28b}$$

Thus δB_k and δB_k^* are expressed in terms of δB_0, δB_0^*:

$$\delta B_k=\frac{2i[(\lambda+\eta)B_0B_{-k}^*\delta B_0+\rho_k\,|\,B_0\,|^2B_kB_0\delta B_0^*]}{(\lambda+\eta)^2-|\,B_0\,|^4}\ ,\qquad \delta B_{-k}^*=\delta B_k^*\ , \tag{29}$$

but with λ unchanged. Substituting (29) in (28b), we find

$$\delta B_0=\frac{iB^2}{|\,B\,|^2}\delta B_0^*\ ,$$

where, as indicated earlier, we set all $B_k=B$. Consequently, the eigenvector has the form

$$\psi_\lambda=\begin{bmatrix}\dfrac{-B_0B}{2N\,|\,B\,|^2\,|\,B_0\,|^2}f(\lambda)\\ \dfrac{-iB_0^*B}{2N\,|\,B\,|^2\,|\,B_0\,|^2}f(\lambda)\\ \vdots\\ \dfrac{iB^2}{|\,B\,|^2}\\ 1\end{bmatrix}\ , \tag{30}$$

where $\psi=\psi_\lambda$ and $\tilde{\psi}=\psi_{\lambda^*}$, $f(\lambda)=N\,|\,B\,|^2-(\eta+\lambda)$, and the dots are the reoccurrence of the first two elements in the vector. In deriving ψ_λ, we have used the identity obtained in the characteristic equation

$$\frac{\eta+\lambda+|B_0|^2}{(\lambda+\eta)^2-|B_0|^4}=\frac{N|B|^2-(\eta+\lambda)}{4|B_0|^2N|B|^2} .$$

In the same way, the adjoint eigenvector is obtained

$$\xi_{\lambda^*}=\begin{bmatrix} \dfrac{B_0 B}{2N|B|^2|B_0|^2}f(\lambda^*) \\ \dfrac{iB_0^* B}{2N|B|^2|B_0|^2}f(\lambda^*) \\ \vdots \\ \dfrac{iB^2}{|B|^2} \\ 1 \end{bmatrix}, \tag{31}$$

where $\xi=\xi_{\lambda^*}$ and $\tilde{\xi}=\xi_\lambda$.

The linear operator L has the form,

$$\begin{bmatrix} A & & & B \\ & A & & B \\ & & \ddots & \vdots \\ & & A & B \\ C & C & \cdots \; C & D \end{bmatrix},$$

where

$$\tilde{A}=\begin{bmatrix} -\eta & iB_0^2 \\ -iB_0^{*2} & -\eta \end{bmatrix}, \quad \tilde{B}=\begin{bmatrix} 2iB^*B_0 & 0 \\ 0 & -2iBB_0^* \end{bmatrix},$$

$$\tilde{C}=\begin{bmatrix} 2iB_0^*B & 0 \\ 0 & -2iB_0B^* \end{bmatrix}, \quad \tilde{D}=\begin{bmatrix} -\eta & iNB^2 \\ -iNB^{*2} & -\eta \end{bmatrix},$$

there are a total of N A, B, and C submatrices in L. L is inverted by diagonalizing with row eliminations and by applying the same operators to a unit matrix. The inverted L has its diagonal elements all being $A^{-1}(E-W)$ except the last one, T. The last column has its elements all being $-A^{-1}BT$ and the last row TB^*A^{-1} with the exceptions of the last element. The rest of the elements in L^{-1} are all $-A^{-1}W$, where

$$A^{-1}=\frac{1}{\eta^2-|B_0|^4}\begin{bmatrix} -\eta & -iB_0^2 \\ iB_0^{*2} & -\eta \end{bmatrix},$$

$$W=\frac{4\eta|B|^2N}{\eta^2-|B_0|^4}\,\frac{-\eta+N\Delta+N|B|^2(1+\Delta)}{\det T_0} \times\begin{bmatrix} -|B_0|^2 & -iB_0^2 \\ iB_0^{*2} & |B_0|^2 \end{bmatrix},$$

$$\Delta=\frac{4|B|^2\eta^2}{\eta^2-|B_0|^4},$$

$$T_0=\begin{bmatrix} -\eta+N\Delta & iB^2N(1+\Delta) \\ -iB^{*2}N(1+\Delta) & -\eta+N\Delta \end{bmatrix},$$

$$T=\frac{1}{\det T_0}\begin{bmatrix} -\eta+N\Delta & -iB^2(1+\Delta)N \\ iB^{*2}(1+\Delta)N & -\eta+N\Delta \end{bmatrix} .$$

The last 4×4 block of L^{-1} has the form

$$\begin{bmatrix} A^{-1}(E-W) & -A^{-1}W & -A^{-1}W & -A^{-1}BT \\ -A^{-1}W & A^{-1}(E-W) & -A^{-1}W & -A^{-1}BT \\ -A^{-1}W & -A^{-1}W & A^{-1}(E-W) & -A^{-1}BT \\ TB^*A^{-1} & TB^*A^{1} & TB^*A^{-1} & T \end{bmatrix}.$$

Detailed calculations are shown in Appendix B. The common factor $\eta^2-|B_0|^4$ is about the order of thermal drive for the finite temperature case and quantum fluctuation for the zero temperature case as is argued in the steady-state calculation. We denote it by Γ. Equation (4a) shows $\Gamma\sim\eta|\bar{\xi}|$. $(L-2\lambda)^{-1}$ and $(L-2\lambda^*)^{-1}$ can be obtained simply by replacing η in L^{-1} with $\eta+2\lambda$ and $\eta+2\lambda^*$, respectively.

As a last step to calculate h, we want to obtain the projection of N on α^2, α^{*2}, and $|\alpha|^2$. This is done by substituting $v=\alpha\psi+\alpha^*\tilde{\psi}+O(\alpha^2)$ in the nonlinear terms of (17),

$$N_k=\alpha^2(2iB_0\psi_k^*\psi_0+iB^*\psi_0^2)+\alpha^{*2}(2iB_0\tilde{\psi}_0\tilde{\psi}_k^*+iB^*\psi_0^2)+|\alpha|^2[2iB_0(\psi_0\tilde{\psi}_k^*+\tilde{\psi}_0\psi_k^*)+2iB_{-k}^*\psi_0\tilde{\psi}_0] ,$$

$$N_0=\alpha^2\left[iB_0^*\sum_k\psi_k^2+2i\sum_k B\psi_0\psi_k\right]+\alpha^{*2}\left[iB_0^*\sum_k\tilde{\psi}_k^2+2i\tilde{\psi}_0\sum_k\tilde{\psi}_kB\right],$$

$$+|\alpha|^2\left[2iB_0^*\sum_k\psi_k\tilde{\psi}_k+2i\left[\psi_0\sum_k B\tilde{\psi}_k+\tilde{\psi}_0\sum_k B\psi_k\right]\right] .$$

The other two terms in (26), $\langle\xi,N\rangle$ and $\langle\tilde{\xi},N\rangle$, turned out to vanish because they are proportional to $B_0+B_0^*$ and $B_0^*=-B_0$ from steady-state calculations. This is an artifact of the simplification of $B_k=B$ and $\rho_k=1$. Fortunately, this term, when kept, contributes little and negatively to the real part of β, leaving the final result unchanged. We then have $-h_1$ equals the coefficient of α^2 in $(L-2\lambda)^{-1}N$ and $-h_2$ equals the coefficient of $|\alpha|^2$ in $L^{-1}N$. The results are (see Appendix B),

$$h_1=\begin{bmatrix} h_1^1 \\ \vdots \\ h_1^1 \\ h_1^0 \end{bmatrix},$$

where

$$h_1^1 = C_1^1 \begin{bmatrix} 1 \\ i \end{bmatrix} ,$$

$$h_1^0 = C_1^0 \begin{bmatrix} 1 \\ -1 \end{bmatrix} ,$$

$$C_1^1 = \frac{2}{3} \frac{\eta B}{(2\lambda+\eta)^2 - |B_0|^4} \left[-\frac{32(1+\lambda)^2[\eta - \lambda\Delta_1 - \lambda^2(\Delta_1+\eta)]N|B|^2\eta^2}{[(2\lambda+\eta)^2 - |B_0|^4]\det T_1} + \frac{(2-\lambda)(\lambda+10)[\lambda^2(\eta+\Delta_1)+\lambda\Delta_1-\eta]B_0^2}{\det T_1} - (1+\lambda)^2 \right] ,$$

$$C_1^0 = \frac{B_0}{3\det T_1} \left[\frac{8\eta N|B|^2(\lambda+1)^2[\lambda(\eta+\Delta_1)-\eta]}{(2\lambda+\eta)^2 - |B_0|^4} - \frac{(2-\lambda)(\lambda+10)[\lambda(\eta+\Delta_1)+\eta]}{2} \right] ,$$

$$\det T_1 = (2\lambda+1)^2(\eta+\Delta_1)^2 - (3\eta+\Delta_1)^2 ,$$

$$\Delta_1 = \frac{4N|B|^2\eta^2}{(2\lambda+\eta)^2 - |B_0|^4} ,$$

and

$$h_2 = \begin{bmatrix} h_2^1 \\ \vdots \\ h_2^1 \\ h_2^0 \end{bmatrix} ,$$

$$h_2^1 = C_2^1 \begin{bmatrix} 1 \\ i \end{bmatrix} ,$$

$$h_2^0 = C_2^0 \begin{bmatrix} 1 \\ -1 \end{bmatrix} ,$$

where

$$C_2^1 = \frac{4\eta B}{\eta^2 - |B_0|^4} \left[\frac{-16\eta^4}{(\eta^2 - |B_0|^4)\det T} + \frac{4\eta^2}{\det T} - \frac{1}{3} \right] ,$$

$$C_2^0 = \frac{4\eta B_0}{\det T} \left[1 - \frac{4\eta^2}{\eta^2 - |B_0|^4} \right] ,$$

$$\det T = (\eta+\Delta)^2 - (3\eta+\Delta)^2 ,$$

and

$$\Delta = \frac{4N|B|^2\eta^2}{\eta^2 - |B_0|^4} .$$

We are now in the position to evaluate the coefficient of the cubic term in the normal form (19). The coefficient of $|\alpha|^2\alpha$ in N is,

$$N_k = i2B_0 \left[h_{2,k} + h_{1,k} + \frac{iBB_0}{6}(2-\lambda)h_{20} + \frac{iBB_0}{6}(2-\lambda^*)h_{10} \right] + 2B^* i(h_{20}+h_{10}) - \frac{BB_0}{3}(2-\lambda) - \frac{BB_0}{6}(2-\lambda^*) ,$$

$$N_0 = 2B_0^* i \sum_k (h_{2,k}\psi_k + h_{1,k}\tilde{\psi}_k) + 2i \sum_k B(\psi_k h_{20} + \psi_{20} + \tilde{\psi}_k h_{10} + \psi_0 h_{2,k} + \tilde{\psi}_0 h_{1,k}) + i \sum_k (2\psi_k\tilde{\psi}_k\psi_0 + \psi_k^2\psi_0) ,$$

where $h_{2,k}$ and $h_{1,k}$ are the kth component of h_1 and h_2. After entering the numbers (see Appendix B), we find

$$\langle \xi, N \rangle = -\frac{4}{\Gamma}[(1+\sqrt{2})i + \sqrt{2}]|\alpha|^2\alpha$$

so

$$\beta = -\frac{4}{\Gamma}[(1+\sqrt{2}i)+\sqrt{2}]/\langle \xi, \psi \rangle ,$$

$$= -\frac{1}{\Gamma}\left[1 + \frac{3}{2}\sqrt{2} + \sqrt{2}i \right] ,$$

where

$$\Gamma = \frac{\eta^2 - |B_0|^4}{\eta^2} .$$

Since $\mathrm{Re}\beta < 0$, we thus confirmed the existence of the limit cycle and the type of instability being a Hopf bifurcation. The size of that limit cycle a little above the bifurcation point is

$$|\alpha|^2 = -\frac{\mathrm{Re}\lambda}{\mathrm{Re}\beta} .$$

The application of the center-manifold theory is somewhat shadowed by the presence of $N-1$ zero λ's, as discussed in Appendix B. This is a result of the assumption that ρ_k equals the constant and all B_k's are the same and the thermal and quantum fluctuations are absent. As soon as the assumption is removed, those λ's become negative. The question then raised is how far away those λ's are from the imaginary axis so that the center-manifold calculation is still meaningful. We do not have a complete answer to this question. Our confidence in using the center-manifold theory comes from two sources. The first is from numerical calculation of three spin waves, i.e., six modes. We can clearly see the unstable pair crossing the imaginary axis without dragging any other modes with it. The second is from the experimental observations that exhibit a low dimensionality as the auto-oscillation occurs.

V. DISCUSSION

Throughout this paper we have used mode expansions as our starting point. This leads to several difficulties especially in the case of parallel pumping where truncation of interaction becomes more questionable. An alternative method is to use Newell-Whitehead-Segel envelope equations[15] and it is planned to present such a treatment in a later publication.

The damping rate η we used in this paper corresponds to ordinary ferromagnetic resonance linewidths at a small signal level. Usually for YiG, η is of the order of hundred KHz or a MHz. This is also the range of auto-oscillation frequency observed in experiments. The observation that the auto-oscillation frequency decreases when the system is cooled down[16] makes our result more convincing to the extent that η is a monotonically decreasing function of temperature. It would be interesting to determine if doping or roughening the surface of the sample sufficiently to increase the η causes the auto-oscillation frequency to increase.

The use of center-manifold theory has helped to clarify the effective dimensionality of the system, i.e., the effective number of spin-wave modes involved. Various experimentally observed time series indicate a low dimensionality[9,11,12] and it appears to be the case that two spin-wave mode computations presented in the past captures qualitatively much of the observed behavior. In this paper we have furnished an explanation for this empirical result: In fact, the entire degenerate spin-wave manifold is excited, but organizes itself into new eigenmodes of which only a few are active. Therefore, the dynamics of the system is low dimensional even though the number of degrees of freedom is infinite. On the other hand, under conditions of total chaos, progress has also been made in estimating the dimensionality.[17] That will provide us clues on whether further instabilities will occur on the low-dimensional center manifold we have constructed.

Finally, when the dc field direction is turned away from the easy magnetization axis, the coupling constants $\rho_{kk'}$ will change. This can be seen easily if an anisotropy field $h = AM_z$ is added in the equation of motion. Some of the couplings that are neglected in our treatment may then become important and the linear spectrum would change accordingly. The pair of complex conjugates crossing the imaginary axis may not occur at all or there are more than two eigenvalues may cross simultaneously. In the former case, the absorption rate will remain steady and in the latter, irregular oscillations, even chaotic behavior may set in directly.[4(c),7] Unfortunately, our theory does not yield these results automatically. *Ad hoc* assumptions, on keeping some special modes, can simulate several abnormal oscillations,[18] but have not been corroborated by any theory to date.

ACKNOWLEDGMENTS

We are grateful to John Greene for bringing our initial attention to the center-manifold theory. We want to thank J. D. Crawford for several tutorial discussions on the theory. This research is partly funded by the Center for Magnetic Recording Research at The University of California at San Diego and by a grant from the Naval Research Laboratory, Grant No. N00014-87-K-0059.

APPENDIX A: DEGENERACY OF THE COLLAPSED SPECTRUM

The characteristic equation for the eigenvalue of linear operator L, $\det(L-\lambda I)=0$, has the form,

$$\det\begin{bmatrix} A' & & & & B \\ & A' & & & B \\ & & \ddots & & \vdots \\ & & & A' & B \\ C & C & \cdots & C & D \end{bmatrix} = 0 , \qquad (\mathrm{A1})$$

where B, C, and D are defined in Sec. IV and

$$A' = \begin{bmatrix} -\eta-\lambda & iB_0^2 \\ -iB_0^{*2} & -\eta-\lambda \end{bmatrix} .$$

If we subtract the second column from the first, and then add the first row to the second, (A1) becomes

$$\det\begin{bmatrix} A' & & & & B \\ & A' & & & 2B \\ & & \ddots & & \vdots \\ & & & A' & B \\ 0 & C & \cdots & C & D \end{bmatrix} = 0 .$$

Follow the same procedure to the rest of the columns and rows, the determinant becomes,

$$\det\begin{bmatrix} A' & & & & & B \\ & \ddots & & & & 2B \\ & & A' & & & \vdots \\ & & & \ddots & & (N-1)B \\ & & & & A' & NB \\ & & & & C & D \end{bmatrix}=0$$

or

$$(\det A')^{N-1}\det\begin{bmatrix} A' & NB \\ C & D \end{bmatrix}=0\ ,$$

so, $\det A'=0$ has $N-1$ degree of degeneracy.

$$\det\begin{bmatrix} A' & NB \\ C & D \end{bmatrix}=0$$

is identical to the two-mode characteristic equation with B replaced by NB. N is the total number of nonuniform modes.

APPENDIX B: CALCULATION OF THE CENTER MANIFOLD

Since the final goal in the entire calculation is to determine the sign of β, actual numbers have to be substituted for various expressions. The steady-state calculation is Sec. II can certainly be used. But since the linear operator in the center-manifold calculation is inverted by assuming all the nonuniform modes are the same, we might as well, for simplicity, use that assumption for the steady-state expression. Also, since thermal noise in general has a random phase, which makes it hard to determine the phase of each B_k, we neglect it for the present purpose. The negative consequence of this will be discussed. The steady-state equations are simplified to

$$\begin{aligned} \eta B &= iB_0^2 B^*\ , \\ \eta B_0 &= -i\omega_s + iNB^2 B_0^*\ , \end{aligned} \tag{B1}$$

with solutions

$$N\,|B\,|^2=\frac{\omega_s}{|B_0|}-\eta,\quad |B_0|^2=\eta\ ,$$

$$\varphi_0=-\pi/2,\quad \varphi=-\pi/4\ ,$$

where φ_0 and φ are the phases of B_0 and B.

We next illustrate in detail the inversion of L. Putting a unit matrix beside it, we have

$$\begin{bmatrix} A & & & & B \\ & A & & & B \\ & & \ddots & & \vdots \\ & & & A & B \\ -B^* & -B^* & \cdots & -B^* & D \end{bmatrix}\begin{bmatrix} E & & & & \\ & E & & & \\ & & \ddots & & \\ & & & E & \\ & & & & E \end{bmatrix}. \tag{B2}$$

A, B, and D are defined in Sec. IV. Multiply every row except the last one by B^*A^{-1} and add to the last row. Then (B2) becomes

$$\begin{bmatrix} A & & & & B \\ & A & & & B \\ & & \ddots & & \vdots \\ & & & A & B \\ & & & & T_0 \end{bmatrix}\begin{bmatrix} E & & & & \\ & E & & & \\ & & \ddots & & \\ & & & E & \\ B^*A^{-1} & B^*A^{-1} & \cdots & B^*A^{-1} & E \end{bmatrix},$$

where $T_0=D+NB^*A^{*-1}B$. Then multiply the last row by BT^{-1} and subtract it from every other row; this gives

$$\begin{bmatrix} A & & & & \\ & A & & & \\ & & \ddots & & \\ & & & A & \\ & & & & T_0 \end{bmatrix}\begin{bmatrix} E-W & -W & \cdots & -W & -BT_0^{-1} \\ -W & E-W & & -W & -BT_0^{-1} \\ -W & -W & \ddots & -W & \\ \vdots & \vdots & & \vdots & \vdots \\ -W & -W & & E-W & -BT_0^{-1} \\ B^*A^{-1} & B^*A^{-1} & \cdots & B^*A^{-1} & E \end{bmatrix},$$

where

$$W=BT_0^{-1}B^*A^{-1}\ .$$

Finally, multiply A^{-1} into every row except the last one, and multiply T_0^{-1} into the last row. The inverted matrix is

$$L^{-1}=\begin{bmatrix} A^{-1}(E-W) & -A^{-1}W & \cdots & -A^{-1}W & -A^{-1}BT \\ -A^{-1}W & A^{-1}(E-W) & -A^{-1}W & \cdots & -A^{-1}BT \\ \vdots & -A^{-1}W & \ddots & -A^{-1}W & \vdots \\ -A^{-1}W & -A^{-1}W & \cdots & A^{-1}(E-W) & -A^{-1}BT \\ TB^{*}A^{-1} & TB^{*}A^{-1} & \cdots & TB^{*}A^{-1} & T \end{bmatrix}, \tag{B3}$$

where $T=T_0^{-1}$.

All the entries of L^{-1} are evaluated, with

$$A^{-1}W=\frac{16\eta^2|B|^2}{(\eta-|B_0|^4)\det T_0}\begin{bmatrix}1 & i\\ i & 1\end{bmatrix},$$

$$TB^{*}A^{-1}=\frac{4iB_0^{*}B^{*}\eta^2}{\det T_0(\eta^2-|B_0|^4)}\begin{bmatrix}-i & -1\\ i & 1\end{bmatrix},$$

$$A^{-1}BT=\frac{4iB_0B\eta^2}{(\eta^2-|B_0|^4)\det T_0}\begin{bmatrix}i & -i\\ -1 & 1\end{bmatrix}.$$

The term $1/\eta^2-|B_0|^4$ as it stands is divergent. This is due to several reasons. First, it is a direct result of steady state without fluctuation. With fluctuation, as stated in the main text of Sec. IV, $\eta^2-|B_0|^4\sim\eta|\tilde{\xi}|$ with $|\tilde{\xi}|$ being the fluctuation $(\xi\xi^{*})^{1/2}$. Second, it is a result of assuming all the nonuniform modes are identical. That assumption leads to $\lambda=-\eta+|B_0|^2$ with $N-1$ degeneracy. Together with the above oversimplified steady-state calculation, there are $N-1$ zero value λ's. Physically, it simply means that while the amplitude of each nonuniform mode is undetermined, only the sum of B_kB_{-k} is determined. Since we have developed our mathematical scheme on the basis that only two modes cross the imaginary axis simultaneously, any other zero eigenvalue will certainly cause divergency through matrix inversions. Fortunately, the true single-particle modes are decay modes, $\lambda_k<0$. This is because when SI occurs, the system reacts to it to restore stability in such a way as to hold those eigenvalues below zero. More rigorous demonstrations can be found from a two-mode calculation.

From now on, we use the steady-state equation (B1) to substitute in the numbers, but treat $\Gamma=\eta^2-|B_0|^4$ as finite. Both $\langle\zeta,N\rangle$ and $\langle\tilde{\zeta},N\rangle$ are evaluated to be zero from the following calculations. The coefficient of α^2 in $\langle\zeta,N\rangle$ and $\langle\tilde{\zeta},N\rangle$ is

$$\begin{aligned}\langle\zeta,N\rangle_{\alpha^2}&=\frac{B^{*}B_0^{*}}{6}(2-\lambda)(\tfrac{4}{3}B+2iB^{*})N\\&\quad+\frac{iB^{*}B_0^{*}}{6}(2-\lambda)(\tfrac{4}{3}B^{*}-2iB)N\ ,\\&=\tfrac{4}{3}(B_0+B_0^{*})=0\ ,\end{aligned}$$

$$\langle\zeta^{*},N\rangle_{\alpha^2}=\tfrac{4}{3}(B_0^{*}+B_0)=0\ .$$

The coefficients of $|\alpha|^2$ in $\langle\zeta,N\rangle$ and $\langle\tilde{\zeta},N\rangle$ vanish for the same reason. We now show the detailed calculation of h_1 and h_2. The coefficients of α^2 and $|\alpha|^2$ in vector N can be expressed in a more convenient way,

$$N|_{\alpha^2}=\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

with

$$u_1=\left[\frac{\eta(2-\lambda)}{3}-1\right]\begin{bmatrix}B\\ B^{*}\end{bmatrix},$$

$$u_2=\frac{\eta}{2}(2-\lambda)\left[\eta\frac{(2-\lambda)}{6}-2\right]\begin{bmatrix}B_0\\ B_0^{*}\end{bmatrix},$$

and

$$N|_{|\alpha|^2}=\begin{bmatrix}v_1\\ v_2\end{bmatrix},$$

with

$$v_1=\left[\frac{4}{3}\eta-2\right]\begin{bmatrix}B\\ B^{*}\end{bmatrix},$$

$$v_2=2\eta(\eta-2)\begin{bmatrix}B_0\\ B_0^{*}\end{bmatrix},$$

$$h_1=-(L-2\lambda)^{-1}N|_{\alpha^2}=-L_{\eta+2\lambda}^{-1}N|_{\alpha^2}$$

($L_{\eta+2\lambda}$ means to replace η in L by $\eta+2\lambda$),

$$h_2=-L^{-1}N|_{|\alpha|^2}=-L^{-1}N|_{|\alpha|^2}\ .$$

Applying (B3) we have,

$$h_1=\begin{bmatrix}h_1^1\\ h_1^0\end{bmatrix},$$

where

$$h_1^1=NA_{\eta+2\lambda}^{-1}W_{\eta+2\lambda}U_1+A_{\eta+2\lambda}^{-1}BT_{\eta+2\lambda}U_0-A_{\eta+2\lambda}^{-1}U_1\ ,$$

$$h_1^0=-(NT_{\eta+2\lambda}B^{*}A_{\eta+2\lambda}^{-1}U_1+T_{\eta+2\lambda}U_0)\ ,$$

and

$$h_2=\begin{bmatrix}h_2^1\\ h_2^0\end{bmatrix},$$

where

$$h_2^1=NA_\eta^{-1}W_\eta V_1+A_\eta^{-1}BT_\eta V_0-A_\eta^{-1}V_1\ ,$$

$$h_2^0=-(NT_\eta^{-1}B^{*}A_\eta^{-1}V_1+T_\eta^{-1}V_0)\ .$$

The above ten terms in h_1 and h_2 are evaluated. For notation simplicity, we denote $\det T_{0_\eta}$ as $\det T_0$ and

$\det T_{0_{\eta+2\lambda}}$ as $\det T_1$. Therefore,

$$NA^{-1}_{\eta+2\lambda}W_{\eta+2\lambda}U_1=K_1\begin{bmatrix}1\\ i\end{bmatrix},$$

$$K_1=-\frac{32}{3}\frac{(1+\lambda)^2N\,|B\,|^2B\eta^3[\eta-\lambda\Delta_1-\lambda^2(\Delta_1+\eta)]}{[(2\lambda+\eta)^2-|B_0|^4]^2\det T_1},$$

$$A^{-1}_{\eta+2\lambda}BT^{-1}_{\eta+2\lambda}U_0=K_2\begin{bmatrix}1\\ i\end{bmatrix},$$

$$K_2=\frac{2B_0^2B\eta(2-\lambda)(\lambda+10)[\lambda^2(\eta+\Delta_1)+\lambda\Delta_1-\eta]}{3\det T_1[(\eta+2\lambda)^2-|B_0|^4]},$$

$$A^{-1}_{\eta+2\lambda}U_1=K_3\begin{bmatrix}1\\ i\end{bmatrix},$$

$$K_3=\frac{2}{3}\frac{\eta(\lambda+1)^2B}{[(2\lambda+\eta)^2-|B_0|^4]},$$

$$NT^{-1}_{\eta+2\lambda}B^*A^{-1}_{\eta+2\lambda}U_1=K_4\begin{bmatrix}1\\ -1\end{bmatrix},$$

$$K_4=\frac{8\eta}{3}\frac{B_0^*N\,|B\,|^2(\lambda+1)[\lambda(\lambda+1)(\eta+\Delta_1)-\eta-\lambda]}{\det T_1[(2\lambda+\eta)^2-|B_0|^4]},$$

$$T^{-1}_{\eta+2\lambda}U_0=K_5\begin{bmatrix}1\\ -1\end{bmatrix},$$

$$K_5=\frac{(2-\lambda)(\lambda+10)[\lambda(\eta+\Delta_1)+\eta]B_0}{6\det T_1},$$

$$NA^{-1}_{\eta}W_\eta V_1=K_6\begin{bmatrix}1\\ i\end{bmatrix},$$

$$K_6=-\frac{64}{3}\frac{\eta^4N\,|B\,|^2B}{(\eta^2-|B_0|^4)^2\det T_0},$$

$$A^{-1}_{\eta}BT^{-1}_{\eta}V_0=K_7\begin{bmatrix}1\\ i\end{bmatrix},$$

$$K_7=\frac{16\eta^3B}{(\eta^2-|B_0|^4)\det T_0},$$

$$A^{-1}_{\lambda}V_1=K_8\begin{bmatrix}1\\ i\end{bmatrix},$$

$$K_8=\frac{4\eta B}{3(\eta^2-|B_0|^4)},$$

$$NT^{-1}_{\eta}B^*A^{-1}_{\eta}V_1=K_9\begin{bmatrix}1\\ -1\end{bmatrix},$$

$$K_9=-\frac{16}{3}\frac{N\,|B\,|^2B_0^*\eta^2}{\det T_0(\eta^2-|B_0|^4)},$$

$$T^{-1}_{\eta}V_0=K_{10}\begin{bmatrix}1\\ -1\end{bmatrix},$$

$$K_{10}=-\frac{4\eta B_0}{\det T_0}.$$

---

*Present address: Control Data Corp., 7801 Computer Ave., S., Minneapolis, MN 55437.

[1]S. Wang, G. Thomas, and Ta-Lin Hsu, J. Appl. Phys. **39**, 2719 (1968).

[2]H. Suhl, J. Phys. Chem. Solids **1**, 209 (1957).

[3]V. E. Zakharov, V. S. L'vov, and S. S. Starobinets, Usp. Fiz. Nauk **112-114**, 609 (1974) [Sov. Phys.—Usp. **17**, 896 (1975)].

[4](a) K. Nakamura, S. Ohta, and K. Kawasaki, J. Phys. C **15**, L143 (1982); (b) X. Y. Zhang and H. Suhl, Phys. Rev. A **32**, 2530 (1985); (c) F. M. deAguiar and S. M. Rezende, Phys. Rev. Lett. **56**, 1070 (1986).

[5]H. Suhl and X. Y. Zhang, Phys. Rev. Lett. **57**, 1480 (1986).

[6]In the presence of thermal agitation, the fixed point, even at low signal powers, is already slightly away from the uniform-mode axis, but moves away from it rapidly as the pump power approaches the SI threshold to within a thermal power.

[7]J. D. Bierlein and P. M. Richards, Phys. Rev. B **1**, 4342 (1970).

[8]When the thermal drive tends to zero and the SI threshold is approached so that some spin waves start to attain nonthermal values, $\eta-\rho_{k0}|B_0|^2$ scales with $|\xi_k|$, as can be seen from Ref. 4(a). The limit of zero thermal drive restricts the excited spinwaves to those that satisfy $\eta=\rho_{k0}|B_0|^2$. Because $|B_0|^2/\eta$ is k independent, the excited spin waves all have the same coupling constant ρ and therefore their individual amplitudes equal an undertermined phase governed by the random drive, $B_k\propto -i\eta(\xi_k/|\xi_k|)-\rho B_0^2(\xi^*_{-k}/|\xi_k|)$. The constant of proportionality can be evaluated from Ref. 5.

[9]J. Smyth and S. Schultz (private communication).

[10]The value of λ that corresponds to the plus sign is close to zero, which reflects the fact that the distribution of amplitudes among nonuniform modes remains undetermined. Once we restore the k dependence of $\bar{\rho}_k$, the distribution of spin-wave amplitudes is fixed and $\lambda=-\eta+|\bar{\rho}_{k0}|\,|B_0|^2$ will never reach zero as argued in Sec. II.

[11]Paul Bryant, Carson Jeffries, and Katsuhiro Nakamura, Phys. Rev. A (to be published).

[12]Hitoshi Yamazaki and Mike Warden, J. Phys. Soc. Jpn. **55**, 4477 (1986).

[13]J. Carr, *Applications of Center Manifold Theory* (Springer-Verlag, New York, 1984), Vol. 28. See also, John Guckenheimer and Philip Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer-Verlag, Nrew York, 1983), Vol. 42.

[14]John David Crawford, Contemp. Math. **28**, 377 (1984).

[15]A. C. Newell and J. A. Whitehead, J. Fluid Mech. **38**, 279 (1969); L. A. Segel, *ibid.* **38**, 203 (1969).

[16]G. Thomas and Goh Komoriya, J. Appl. Phys. **46**, 883 (1975).

[17]Tepper L. Gill and W. W. Zachary, J. Appl. Phys. **61**, 4130 (1987).

[18]X. Y. Zhang, Ph. D. thesis, University of California, San Diego, 1987 (unpublished).