Analysis of a dynamic renormalization-group technique

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A previously developed real-space renormalization-group approach to critical dynamics is examined. A correction to the formalism has to be introduced. The need for this correction is demonstrated through an exactly soluble one-dimensional Ising model. The new version imposes additional serious limitations on the usefulness of the method within standard approximation schemes. The second-order cumulant approximation of the two-dimensionall kinetic Ising model on a triangular lattice is used to demonstrate these limitations.

I. INTRODUCTION

The dynamics of systems near second-order phase transitions have been the subject of many studies in the last few decades. One of the phenomena encountered in these studies is the divergence of the typical relaxation time, τ, near criticality, referred to as *critical slowing down*. The conventional theory describes this divergence with a dynamical critical exponent z obeying the scaling relation

$$\tau \sim \xi^z ,$$

where ξ is the correlation length of the system (which diverges at the critical point).

The success of renormalization-group (RG) methods[1,2] in obtaining the critical exponents and universality classes of static problems led to several attempts to use RG ideas in critical dynamics. One example[3] is a generalization of the ϵ expansion that enables the calculation of time-dependent correlations. This method describes the dynamics in terms of a Langevin equation, and is useful near the upper critical dimension. Other methods[4–6] that generalize real-space RG techniques are preferable in low dimensions, or when a description of the dynamics by a master equation is used.

We focus on the dynamical real-space RG (DRSRG) technique proposed by Achiam and Kosterlitz.[6] This method has the advantage that it is very simple and transparent, yet it enables the determination of the dynamical critical behavior of certain systems very accurately. For this reason it has been used quite extensively in the last years (see, e.g., Refs. 7–14). Section II includes a detailed formal explanation of the DRSRG technique. In Sec. III we calculate the relaxation time and introduce a correction to the formalism. The main point is that in the generic case the relaxation time is given by the ratio of the *largest* eigenvalue of one matrix to the *smallest* one of another, and not by the ratio of the two largest eigenvalues.[6,9] A demonstration of the need for this correction is given in Secs. IV A and IV B through a one-dimensional Ising model. The dynamical critical behavior of the model is calculated exactly and with the DRSRG technique. In Sec. IV C, the limitations of standard approximation schemes in the framework of the DRSRG are demonstrated. For this purpose we examine the dynamics of the two-dimensional kinetic Ising model in the second-order cumulant approximation.[8]

II. DESCRIPTION OF THE METHOD

The DRSRG method is designed to give the long-time behavior of an Ising spin system, whose dynamics is described by a master equation of the form

$$\frac{dP(\{S\},t)}{dt} = -\mathcal{L}(\{S\})P(\{S\},t) , \tag{2.1}$$

where $\{S\}$ is a spin configuration, $P(\{S\},t)$ is the nonequilibrium probability distribution, and $\mathcal{L}(\{S\})$ is the Liouville operator. In the case of single-spin-flip dynamics,

$$\mathcal{L}(\{S\}) = \sum_i (1-p_i) W_i(\{S\}) , \tag{2.2}$$

where $W_i(\{S\})$ is the flipping rate of the spin S_i, p_i is a spin-flip operator defined by

$$p_i f(S_1, \ldots, S_i, \ldots, S_n) = f(S_1, \ldots, -S_i, \ldots, S_n) .$$

W_i satisfies the detailed-balance condition

$$\begin{aligned} &(1-p_i) W_i(\{S\}) P_{\rm eq}(\{S\}) = 0 , \\ &P_{\rm eq} = \frac{e^{-\mathcal{H}/k_B T}}{\mathrm{Tr}(e^{-\mathcal{H}/k_B T})} , \end{aligned} \tag{2.3}$$

where $\mathcal{H}$ is the Hamiltonian of the system.

A RG transformation (such as decimation or spin-block transformation) on the master equation (2.1) should yield a new equation of a similar form for a renormalized probability distribution. For example, in the case of a decimation transformation, the spins are divided into two groups denoted by $\{\sigma\}$ and $\{\mu\}$. Decimation of the spins $\{\sigma\}$ defines a new probability distribution,

$$P^{(1)}(\{\mu\},t) = \sum_{\{\sigma\}} P(\{\sigma\},\{\mu\},t) ,$$

and the new Liouville operator $\mathcal{L}^{(1)}(\{\mu\})$ is defined by

$$\sum_{\{\sigma\}} \mathcal{L}(\{\sigma\},\{\mu\}) P(\{\sigma\},\{\mu\},t) = \mathcal{L}^{(1)}(\{\mu\}) P^{(1)}(\{\mu\},t) .$$

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The assumption is that for an appropriate choice of the flipping rate W_i, $\mathcal{L}^{(1)}$ and $\mathcal{L}$ will have the same form.

In order to perform the RG transformation one writes the nonequilibrium probability distribution as

$$P(\{S\},t)=P_{\rm eq}(\{S\})\left[1+\sum_{i=1}^{m}\mathcal{O}_i(\{S\})h_i(t)\right]$$
$$=P_{\rm eq}(1+\mathcal{O}\cdot\mathbf{h})\ , \tag{2.4}$$

where $\mathcal{O}_i$ are spin operators[9] (examples will be given later), and $\mathcal{O}$ and $\mathbf{h}$ are vectors with components $\mathcal{O}_i$ and h_i, respectively. Equation (2.4) is exact if one includes all the $m=2^N$ independent spin operators in the expansion (N is the number of spins). However, usually one has to truncate this expansion in order to solve for the relaxation times, and (2.4) is used as an approximation for the nonequilibrium distribution function. The action of the RG transformation in the static parameter space (the parameters of the Hamiltonian) is known: It transforms $P_{\rm eq}[\mathcal{H}]$ to $P_{\rm eq}^{(1)}=P_{\rm eq}[\mathcal{H}^{(1)}]$. Forcing $P^{(1)}$ into the form (2.4) with the transformed equilibrium distribution defines the action of the RG transformation on the dynamic parameters h_i.

As in static RG, the strategy is to move away from the critical region by applying the transformation many times until one reaches a region in parameter space (the *high-temperature regime*) where the master equation may be solved. The dynamical critical behavior is then deduced from the solution.

With the definition (2.4), Eq. (2.1) takes the form

$$P_{\rm eq}\mathcal{O}\cdot\frac{d\mathbf{h}}{dt}=-\mathcal{L}P_{\rm eq}\mathcal{O}\cdot\mathbf{h}\ . \tag{2.5}$$

The general form of the equation after applying the RG transformation is

$$P_{\rm eq}^{(1)}\mathcal{O}\Lambda^{(1)}(k)\frac{d\mathbf{h}}{dt}=-\mathcal{L}^{(1)}P_{\rm eq}^{(1)}\mathcal{O}\Omega^{(1)}(k)\mathbf{h}\ ,$$

where $\Lambda^{(1)}(k)$ and $\Omega^{(1)}(k)$ are $m\times m$ matrices which depend on the parameters of the Hamiltonian k_i. The transformed operators $P_{\rm eq}^{(1)}$ and $\mathcal{L}^{(1)}$ are assumed to have the same form as $P_{\rm eq}$ and $\mathcal{L}$, respectively. After n steps of renormalization one gets

$$P_{\rm eq}^{(n)}\mathcal{O}\Lambda^{(n)}\frac{d\mathbf{h}}{dt}=-\mathcal{L}^{(n)}P_{\rm eq}^{(n)}\mathcal{O}\Omega^{(n)}\mathbf{h}\ ,$$

where

$$\Lambda^{(n)}=\Lambda^{(1)}(k)\Lambda^{(1)}(k^{(1)})\cdots\Lambda^{(1)}(k^{(n-1)})\ ,$$
$$\Omega^{(n)}=\Omega^{(1)}(k)\Omega^{(1)}(k^{(1)})\cdots\Omega^{(1)}(k^{(n-1)})\ .$$

If n renormalization steps take us to a *high-temperature regime* where $P_{\rm eq}^{(n)}\approx 1$ and $\mathcal{L}^{(n)}\approx\sum_i(1-p_i)$ (assuming single-spin-flip dynamics), we are left with the equation

$$\Lambda^{(n)}\frac{d\mathbf{h}}{dt}=-L\Omega^{(n)}\mathbf{h}\ , \tag{2.6}$$

where L is an $m\times m$ matrix defined by $\mathcal{L}^{(n)}\mathcal{O}=\mathcal{O}L$. The solution for the largest relaxation time τ is the largest eigenvalue of the matrix $(L\Omega^{(n)})^{-1}\Lambda^{(n)}$.

Two difficulties are encountered in carrying out the above program.

(i) The complicated form (2.4) of the nonequilibrium distribution does not permit exact calculations since it includes $m=2^N$ spin operators for an N-spin system. Therefore, one has to truncate the space of operators to a small subspace that meets the following requirements: First, it is closed under the RG transformation, and second, there is a solution of the master equation in the high-temperature regime that has the truncated form. This truncation procedure possibly eliminates some of the relaxation times from the solution. The hope is that the largest relaxation time remains in the truncated solution, which then reproduces the long-time behavior of the system correctly.

(ii) In many cases the RG transformation cannot be performed exactly even for the truncated problem. One then has to devise an approximation scheme that gives a reliable estimate of the largest relaxation time.

III. CALCULATION OF THE DYNAMIC EXPONENT

The first step in the calculation of the relaxation time is the evaluation of the matrices $\Lambda^{(1)}$ and $\Omega^{(1)}$. Since our starting point is very close to a fixed point of the static RG transformation, the dependence of the matrix elements on k_i can be approximated by linearizing around the fixed point. We will assume that the resulting matrices may be written as

$$\Lambda^{(n)}=\Lambda^n\ ,$$
$$\Omega^{(n)}=\Omega^n\ , \tag{3.1}$$

where the matrices Λ and Ω are evaluated at the fixed point and do not depend on k_i (as will be the case in the examples we give in the following sections). Combining (3.1) with (2.6) we see that the relaxation time is the largest eigenvalue of the matrix $(L\Omega^n)^{-1}\Lambda^n$. The claim of Refs. 6 and 9 is that when Λ and Ω do not commute, and in the limit of large n, this eigenvalue can be approximated by

$$\tau\sim\left[\frac{\lambda_{\max}}{\omega_{\max}}\right]^n\ , \tag{3.2}$$

where $\lambda_{\max}$ and $\omega_{\max}$ are the *largest* eigenvalues of the matrices Λ and Ω, respectively. In contradiction, we find that in fact τ is given by

$$\tau\sim\left[\frac{\lambda_{\max}}{\omega_{\min}}\right]^n\ , \tag{3.3}$$

where $\omega_{\min}$ is the *smallest* eigenvalue of Ω. In terms of the dynamical exponent z this is equivalent to

$$z=\frac{\ln(\lambda_{\max}/\omega_{\min})}{\ln(b)}\ , \tag{3.4}$$

where b is the length rescaling factor associated with the RG transformation.

Values of the exponent z calculated with the DRSRG method will change in a significant way due to this correction.[8,9,14] Moreover, there may be serious difficulties in getting a good approximation for $\omega_{\min}$ within standard approximation schemes when the RG transformation cannot be performed exactly. The reason for this is that commonly used approximation schemes in static RG are designed to give reliable results for $\lambda_{\max}$ and possibly also for $\omega_{\max}$. Using these methods, one usually truncates the space of operators that form the Hamiltonian, and the remaining operators are those with the largest eigenvalues with respect to the RG transformation. Thus, the smaller eigenvalues obtained by such a technique are unreliable. Note that these problems arise only when the matrices Λ and Ω do not commute. Our correction does not apply to cases where they do.

Recall that the largest relaxation time, τ, is the largest eigenvalue of the matrix $(L\Omega^n)^{-1}\Lambda^n$. In order to show that τ is given by (3.3) and not (3.2), we work in a basis of operators that diagonalizes Ω. In this basis there is a matrix A such that $A\Lambda A^{-1}$ is diagonal. Both matrices Ω and $A\Lambda A^{-1}$ can be ordered according to the size of their eigenvalues, such that ω_1,λ_1 are the largest eigenvalues and ω_m, λ_m are the smallest ones. For convenience, we consider the matrix $(\omega_m/\lambda_1)(L\Omega^n)^{-1}\Lambda^n$. The general form of its elements in this basis is

$$\left[\left(\frac{\omega_m}{\lambda_1}\right)^n (L\Omega^n)^{-1}\Lambda^n\right]_{ij} = \sum_{l=1}^{m} (L^{-1}A^{-1})_{il} A_{lj} \left(\frac{\omega_m}{\omega_i}\right)^n \left(\frac{\lambda_l}{\lambda_1}\right)^n . \quad (3.5)$$

Assuming ω_m is nondegenerate, all the elements of the preceding matrix are vanishingly small in the limit of large n, except for those of the last row. Each of the elements of this $i=m$ row is given by a sum that is dominated by a single term, the $l=1$ term. In the generic case (and in the following examples)

$$(L^{-1}A^{-1})_{m1}A_{1m}\neq 0 ,$$

and, therefore, this term does not vanish. Hence, the largest relaxation time is given by

$$\tau=(L^{-1}A^{-1})_{m1}A_{1m}\left(\frac{\lambda_1}{\omega_m}\right)^n , \quad (3.6)$$

which proves Eqs. (3.3) and (3.4).

To see why the original derivation[9] that led to (3.2) is wrong, consider Eq. (5.18) of Ref. 9. It correctly states that for the case $m=2$, the relaxation time is given by

$$\tau=\frac{(\lambda_{\max}/\omega_{\max})^n}{a+d(\omega_{\min}/\omega_{\max})^n} . \quad (3.7)$$

Equation (3.2) indeed results from this, provided $a\neq 0$. However, we found that $a=0$ identically, whenever Eq. (3.7) for τ holds [i.e., when the truncation of $P(\{S\},t)$ includes two spin operators ($m=2$)], and, therefore, Eq. (3.7) reduces to (3.3). We now demonstrate the validity of our result on an exactly soluble model.

IV. EXAMPLES

A. Linear chain—exact solution

Since our correction is irrelevant for $m=1$, we choose as an example the simplest model with $m\neq 1$. Consider a linear Ising chain with nearest-neighbor ferromagnetic couplings arranged in such a way that in a unit cell there are three bonds. Two of them are of strength J_1, while the third is of strength J_2 (see Fig. 1). The Hamiltonian is

$$\mathcal{H}=-\sum_i J_i S_i S_{i+1} , \quad (4.1)$$

where J_i takes the value J_1 or J_2 according to its location in the chain. First, we outline an analytic solution of the Glauber dynamics[15] of the model, and then solve the problem with the DRSRG method. We show that Eq. (3.3) [rather than Eq. (3.2)] leads to the exact result. The definition of the spin-flip rates in the single-spin-flip Glauber-type dynamics[16] is

$$W_i(S_i)=A_i(1-\alpha_i^{+}S_iS_{i+1}-\alpha_i^{-}S_{i-1}S_i) , \quad (4.2)$$

where

$$\alpha_i^{\pm}=\tfrac{1}{2}[\tanh(K_i+K_{i-1})\pm\tanh(K_i-K_{i-1})] ,$$

$$A_i=(1+\tanh|K_i-K_{i-1}|)^{-1} ,$$

with $K_i=J_i/k_BT$. Since the probability of flipping a spin depends only on its value relative to the values of its nearest neighbors, one has to consider the probability of occurrence of four possible events: creation of two domain walls (when the spin and its nearest neighbors are initially parallel), annihilation of two domain walls (when the spin is antiparallel to each of its nearest neighbors), diffusion of an existing domain wall (when the spin is parallel to one of its neighbors and antiparallel to the other) that either raises or lowers the energy of the system. The detailed balance condition (2.3) relates the probabilities of the first two possible events as well as the probabilities of the two latter ones. Therefore, there are only two independent spin-flip rates. We normalize these rates so that the rate of moving a domain wall and lower the energy is 1.

The critical exponent is independent of this normalization as long as the normalizing factor is temperature independent. One can, in principle, multiply the transition rates by a temperature-dependent coefficient, and by that change the critical exponent z.[17–19] For example, in Ref. 18 the dynamics of the q-state Potts model is examined, and the dynamical critical exponent is found to be $z=2$ for $q=2$ and $z=3$ for $q>2$ (in contradiction to $z=2$ for

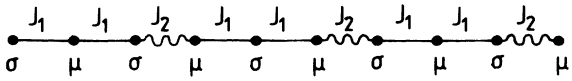

FIG. 1. Three unit cells of the linear Ising chain. The decimation of the spins denoted by σ creates a new linear chain with the same type of couplings.

any q in Glauber-like dynamics). An analysis of the spin-flip rates of Ref. 18 shows that the temperature dependence of the rates at low temperatures for $q=2$ is different from that of $q>2$. This is the origin of the difference in the dynamical exponents.

However, one has to be cautious in introducing temperature-dependent factors into the transition rates, since they can give rise to critical slowing down in finite small systems. The dynamics defined in Ref. 18 does not allow diffusion of domain walls at low temperatures, nor does it allow the creation of domain walls. Therefore, it leads to the divergence of the relaxation time of a finite small system as $T\to 0$. The large dynamical exponent for $q>2$ is the consequence of this artificial slowing down.

A unit cell of the lattice in Fig. 1 contains three spins, so there are three coupled Glauber equations. One has to diagonalize a 3×3 matrix to get three relaxation times associated with the magnetiztion of the model. One of them diverges when $T\to 0$, and it dominates the relaxation of the magnetization at long times. The result of this calculation is that for $K_1, K_2\to\infty$ the divergence of τ is exponential in K_1 and K_2:

$$\tau\sim e^{2(K_1+K_2)} . \tag{4.3}$$

A similar analytic solution of a one-dimensional kinetic Ising model with alternating couplings is given in Ref. 20. As will be explained later, the model studied there leads to an expansion of $P(\{S\},t)$ with a single operator [$m=1$ in Eq. (2.4)]. For that case our correction is, of course, irrelevant.

This result can be confirmed by using a heuristic argument of Henley.[21] According to this argument the relaxation time of a system of linear size L, at low temperature, is given by

$$\tau(L,T)\approx\tau_0 e^{\Delta E(L)/k_B T} , \tag{4.4}$$

where $\Delta E(L)$ is defined as follows. Consider all paths in phase space of which the initial and final states are fully magnetized, with $M=1$ and $M=-1$, respectively (M is the time-dependent magnetization of the system). With each path one can associate an energy barrier which is the maximal energy of the system along this path. Henley assumes that at low enough temperatures the path with the minimal energy barrier dominates the dynamics. $\Delta E(L)$ is that minimal barrier. Obviously, the minimal barrier for the system studied here (see Fig. 1) is

$$\Delta E(L)=2(J_1+J_2) .$$

Using this together with (4.4) we get Eq. (4.3).

B. Linear chain — RG solution

The RG transformation we choose is a decimation of the spins denoted by σ in Fig. 1. Any other decimation should give the same results, but this one is natural in the sense that it does not change the form of the Hamiltonian and the Liouville operator. A different decimation transformation is described in the Appendix. Our assumption is that we get the correct long-time behavior if the nonequilibrium probability distribution is truncated to

$$P(\{S\},t)=P_{\rm eq}(\{S\})\left[1+\sum_i h_i(t)S_i\right] .$$

Clearly, there are two types of spins in the chain: those which have two different bonds connecting them to their nearest neighbors (type 1), and those which are connected to their neighbors by two identical bonds (type 2). Therefore, there must be at least two fields, $h_i(t)$, for the two physically different types of spins. A uniform field leads to a contradiction as is illustrated in the Appendix. Let us consider a minimal choice where $h_i(t)=h_1(t)$ for spins of type 1, while $h_i(t)=h_2(t)$ for spins of type 2. Other choices may complicate the calcultions, but will eventually lead to the same result.

The decimation can be performed exactly if the spin-flip rates are chosen to be

$$W_i(S_i)=B_i\left[\frac{P_{\rm eq}(-S_i)}{P_{\rm eq}(S_i)}\right]^{1/2} , \tag{4.5}$$

where the coefficients B_i ensure that the rate of moving a domain wall and lower the energy is 1. These rates clearly satisfy detailed balance. Although they differ from the Glauber rates, the rate of moving a domain wall, and by that raising or lowering the energy, is the same in both types of dynamics. Since it is believed (in accordance with Henley's argument[21]) that low-temperature dynamics is controlled by diffusion of domain walls, the result for the relaxation time should be the same for both forms of transition rates.

The static recursion relations as obtained from the decimation of Fig. 1 are

$$v_1^{(1)}=v_1 v_2 ,$$
$$v_2^{(1)}=v_1^2 ,$$

where $v_i=\tanh(K_i)$. Linearizing around the fixed point $v_1^*=v_2^*=1$ we get

$$\delta v_1^{(1)}=\delta v_1+\delta v_2 ,$$
$$\delta v_2^{(1)}=2\delta v_1 ,$$

where $\delta v_i=1-v_i$. One can easily find the eigendirections in parameter space,

$$\begin{aligned}\delta x^{(1)}&=2\delta v_1^{(1)}+\delta v_2^{(1)}=2(2\delta v_1+\delta v_2)=2\delta x ,\\ \delta y^{(1)}&=\delta v_1^{(1)}-\delta v_2^{(1)}=-(\delta v_1-\delta v_2)=-\delta y .\end{aligned} \tag{4.6}$$

Decimating the left-hand side of the master equation we find that the matrix Λ depends on K_1 and K_2. After linearizing around the fixed point $v_1^*=v_2^*=1$, Λ takes the form

$$\Lambda=\begin{bmatrix} 1+\dfrac{1+(\delta y/\delta x)}{2-(\delta y/\delta x)} & \frac{1}{2} \\ 2\dfrac{1-2(\delta y/\delta x)}{2-(\delta y/\delta x)} & 1 \end{bmatrix} .$$

In a region where the linearized recursion relations (4.6) hold, δx is multiplied by 2 in each RG step while δy is multiplied by -1. After a few RG steps $\delta y/\delta x$ becomes very small, and it is negligible compared to 1 in almost all the RG steps. Hence, we can approximate Λ by

$$\Lambda \approx \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 1 \end{bmatrix},$$

with eigenvalues

$$\lambda_{\max}=2 ,$$

$$\lambda_{\min}=\tfrac{1}{2} .$$

The matrix A [see Eqs. (3.5) and (3.6)] is given by

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} .$$

Let us define n so that, after n RG steps, we get to a point in parameter space where $\delta x^{(n)} \approx 1$. This is equivalent to choosing n such that, after n RG steps, the temperature is high and the correlation length, $\xi^{(n)}$, is of order unity ($\delta x \sim 1/\xi$). Using the recursion relations (4.6) we get

$$\lambda_{\max}^{n}=2^{n}=\frac{\delta x^{(n)}}{\delta x} \approx \frac{1}{\delta x} . \tag{4.7}$$

The definition of δx gives (for $K_1, K_2 \to \infty$)

$$\delta x \approx 4e^{-2K_1}+2e^{-2K_2} \sim e^{-2\min\{K_1,K_2\}} . \tag{4.8}$$

Both Ω and L are diagonal in this representation. The decimation of the right-hand side of the master equation leads to

$$\Omega^{(1)}(K_1,K_2)\Omega^{(1)}(K_1^{(1)},K_2^{(1)}) \cdots \Omega^{(1)}(K_1^{(n-1)},K_2^{(n-1)}) \approx \begin{bmatrix} 4e^{-2K_2} & 0 \\ 0 & 4e^{-2K_1} \end{bmatrix}$$

and

$$L = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} .$$

Since all the elements of the matrix A are nonzero, we can use Eq. (3.3) for the relaxation time to get

$$\tau \sim \frac{e^{2\max\{K_1,K_2\}}}{\delta x} ,$$

which reduces to the analytical result [Eq. (4.3)] on substituting Eq. (4.8) for δx. Had we taken the largest eigenvalue of Ω, we would not have obtained the correct result.

The above example shows that the DRSRG method with the new correction does give the correct results when the RG transformation can be performed *exactly*. Next, we consider another example, where the RG transformation is *approximate*.

C. Two-dimensional Ising model

We now discuss the two-dimensional kinetic Ising model on a triangular lattice in the second-order cumulant expansion with nearest neighbors (NN), next-nearest neighbors (NNN), and third-nearest neighbors (TNN) interactions.[9] These operators form an invariant subspace of the RG transformation (a spin-block transformation with a majority rule projection operator). The truncation of the expansion of the nonequilibrium probability distribution contains four operators—the total magnetization and three linear combinations of triplets of spins. In the first combination all the spins in a triplet are NN, in the second, two spins in a triplet are NNN, while in the third, two spins in a triplet are TNN. The matrices Λ and Ω are, therefore, 4×4 matrices. The spin-flip rates used are those of Eq. (4.5) with $B_i=1$. For details about the calculation and the matrices see Ref. 9 and references therein.

The changes in the results due to the correction discussed earlier are easily seen from the eigenvalues

$$\lambda_{\max}=2.76;\ \omega_{\max}=0.83;\ \omega_{\min}=-0.04 .$$

Using Eq. (3.2) one gets $z=2.18$, which is in agreement with results of other calculations (Monte-Carlo, high-temperature expansion, etc., suggest[22–29] $1.8<z<2.2$). However, when (the correct) Eq. (3.3) is used, this agreement is destroyed. The value of $\omega_{\min}$ obviously does not make any sense because of its sign. Even if one has a reason to believe that the sign here is not important (i.e., one has to take the absolute value), the value of z obtained using Eq. (3.4) with $b=\sqrt{3}$ is $z=7.71$. It seems that, in the approximation scheme used, it is difficult to get a reliable value for $\omega_{\min}$, and this is the reason for the discrepancy in the results.

V. SUMMARY AND DISCUSSION

We have performed a detailed analysis of a dynamical real-space RG introduced originally by Achiam and Kosterlitz, and added a correction to the formalism. We showed that the longest relaxation time of the system is given by the ratio of the largest eigenvalue of one matrix to the smallest one of another, and not by the ratio of the two largest eigenvalues. The validity of this result was demonstrated by solving (both exactly and by the RG method) a one-dimensional kinetic Ising model with nearest-neighbor ferromagnetic couplings (Fig. 1). In the RG technique, the nonequilibrium probability distribution is represented as an expansion whose terms are spin operators with time-dependent coefficients. Practically, one has to truncate this expansion and include in it a small number of operators. Calculations that use a truncated expansion with a single operator are not influenced by our correction. However, in a large class of models more than one operator must be used. An expansion that includes a single operator leads to a solution with a single

relaxation time, and as we saw (in Sec. IV B and in the Appendix), if not all the sites of the lattice are equivalent, such a solution is unphysical and leads to contradictions. They arise from the fact that it is not possible in such systems to isolate the largest relaxation time, and the calculation involves other relaxation times as well. The calculation gets much more complicated as more relaxation times contribute, simply because one has to diagonalize very large matrices. In particular in hierarchical systems, where there is a hierarchy of time scales, this leads to formal difficulties.

Applying the correction to the two-dimensional kintic Ising model on a triangular lattice in the second-order cumulant approximation, we found that the result obtained with the RG technique completely disagrees with results from other studies. This suggests that conventional approximation schemes (that usually give a good estimation of large eigenvalues) are not reliable when combining them with the dynamical RG technique.

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APPENDIX

The linear chain in the model described in Sec. IV can be decimated in a different way than the one suggested in Sec. IV B. For example, one can decimate the spins denoted by σ in Fig. 2. The result is a linear chain with the same coupling between any pair of nearest neighbors. One of the fields disappears in the decimation and the resulting model has a uniform field $h(t)$, which is a linear combination of $h_1(t)$ and $h_2(t)$. Λ and Ω are 2×1 matrices, and one cannot diagonalize them. However, one

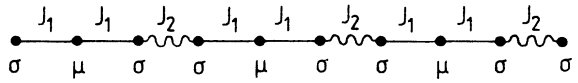

FIG. 2. Three unit cells of the linear Ising chain. Decimation of the spins denoted by σ leads to a linear chain with a single coupling coefficient, while decimation of the spins denoted by μ creates a chain with alternating couplings.

can still decimate the resulting chain to get an equation [the analog of (2.6)] for $h(t)$. Since this is a single equation with two unknowns $[h_1(t),h_2(t)]$, one needs another equation in order to find the solution. The second equation is obtained by starting with a different decimation. If one decimates the spins denoted by μ in Fig. 2, one gets a linear chain with alternating couplings. As in the previous case, one of the fields disappears in the decimation, and the resulting uniform field is a different linear combination of $h_1(t)$ and $h_2(t)$. A differential equation for this linear combination is obtained by decimating the alternating chain. The solution of these two equations gives two relaxation times of which the largest is $\tau\sim e^{2(K_1+K_2)}$. If one starts with a uniform field in the original problem, and performs the two decimations, one gets two contradicting equations for the field.

In Ref. 10 the DRSRG is used to solve a one-dimensional kinetic Ising model with hierarchical couplings. The magnetization is found to relax algebraically. The origin of the algebraic relaxation is the existence of a wide range of characteristic time scales. Although this result may be correct, there is an inconsistency in the calculations of Ref. 10. It is assumed there that a uniform field is sufficient to solve a problem with many time scales. As we have seen, this assumption leads to a contradiction when there is more than one relevant relaxation time. The attempt to avoid this contradiction in Ref. 10 has no formal justification, and a deeper understanding of this problem is still required.

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