

Equilibrium behavior of the spin-glass ordered phase

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A phenomenological theory of the ordered phase of short-range Ising spin glasses is developed in terms of droplet excitations and presented in detail. These excitations have free energies with a broad distribution whose characteristic magnitude grows with length scale L as L^θ . A small fraction of droplets of all scales are thermally active; these dominate much of the physics. The mean-square correlation functions are found to decay with distance as $1/r^\theta$ for all $T < T_c$ and the auto-correlations decay logarithmically with time because of large activation barriers for creation and annihilation of the droplet excitations. A renormalization procedure is sketched in order to define excitations at positive temperature. It is found that the long-distance equilibrium correlation functions are extremely sensitive to small temperature changes, yielding breakdown of certain relations between fluctuations and thermodynamic derivatives. The behavior near to the critical temperature is discussed and some of the ideas are extended to systems with power-law interactions and to spin glasses with X - Y or Heisenberg symmetry. The inequality $\theta \leq (d-1)/2$ is also derived.

I. INTRODUCTION AND SUMMARY

In spite of a large volume of both theoretical and experimental work¹ on spin glasses there has been little contact made between theory and experiment at temperatures below the point at which ordering or freezing occurs. The main reasons for this state of affairs are twofold: On the theoretical side, much of the effort has been focused on questions of the *existence* of an equilibrium phase transition at positive temperatures and on the possibility of many pure states at temperatures below such a transition. However, experimental implications of the latter are by no means clear. On the experimental side, the intrinsic equilibrium or near-equilibrium phenomena have only recently started to be disentangled from nonequilibrium effects such as metastability, hysteresis, remanence, etc.²

Unfortunately, the only analytical calculations of dynamic quantities in mean-field theory below T_c are restricted to equilibrium³; for nonequilibrium effects Monte Carlo simulations and hand-waving arguments have had to be relied on. Thus, at this stage, it is useful to have a phenomenological picture of the spin-glass ordered phase in terms of which the interplay between near-equilibrium and far-from-equilibrium phenomena can be discussed.

In this paper we elaborate on a phenomenological scaling approach to the static and dynamic properties of finite-range spin glasses which was briefly introduced in Ref. 4. Although a considerable amount of earlier work on scaling in the spin-glass ordered phase exists,⁵⁻⁷ only the seminal work of McMillan⁸ had previously attempted investigation of the dynamics. Here we will concentrate on equilibrium and near-equilibrium phenomena and attempt to further justify the scaling picture. In a com-

panion paper,⁹ extensions of some of the ideas to nonequilibrium phenomena are discussed.

For most of this paper we consider the Edwards-Anderson¹⁰-model spin glass with Ising spins and nearest-neighbor interactions on a d -dimensional lattice. The Hamiltonian is

$$\underline{H} = - \sum_{\langle i,j \rangle} J_{ij} S_i S_j - H \sum_i S_i, \quad (1.1)$$

where the first sum runs over all pairs of interacting spins $S_i = \pm 1$, the $\{J_{ij}\}$ are quenched random variables, independently and identically distributed, and H is a uniform magnetic field. Some aspects of the general picture we develop also apply, with some changes, to long-range interactions with $J_{ij}^2 \sim |i-j|^{-2\sigma}$, $\sigma > d/2$, and to systems with vector X - Y or Heisenberg, rather than Ising, spins, as is discussed at the end of this paper (Secs. IX and X). The overbar denotes an average over configurations of the disorder.

We have developed an internally consistent theory of the spin-glass ordered phase based on a scaling ansatz for the nature, statistics, energetics, and dynamics of the low-lying large-scale excitations about an ordered state. The approach is, in a certain sense, very traditional, being in the same spirit as a spin-wave theory for a ferromagnetic ordered phase or a phonon theory for a crystal. However, unlike the latter two cases the properties of the low-lying excitations in spin glasses have not been directly calculated, so at this point an ansatz is required to develop a theory based on them.

We have attempted to produce what we believe to be the simplest ansatz that is consistent with known results. The only limit in which the properties of the low-lying

excitations above a ground state can be calculated in detail is the one-dimensional Ising spin glass.¹¹ Our ansatz is motivated by the scaling properties found there and in numerical work by McMillan¹² and Bray and Moore⁷ on finite-size two- and three-dimensional systems.

The scaling ansatz is discussed in detail in Sec. II of this paper. We argue that for an Ising spin glass the lowest energy excitation of length scale L around a particular point will be a compact droplet of, of order L^d spins coherently flipped with an energy cost of order L^θ . The nontrivial exponent θ increases with increasing dimensionality d . For $d=1$, $\theta=-1$, (Ref. 7) while the latest estimates (for Ising systems) are $\theta \simeq -0.25$ for $d=2$ (Ref. 13) and $\theta \simeq 0.2$ for $d=3$.^{7,12} An upper bound on θ of $\theta \leq (d-1)/2$ is derived in Appendix A. The distribution of droplet energies at length scale L is broad, with weight down to zero energy. The resulting *thermally active droplets*, with excitation energies less than or of order the temperature T , play a very important role in the equilibrium properties of the spin-glass ordered phase. The dynamics of the droplet excitations is governed by thermal activation over barriers whose height scales as L^ψ , with $\theta \leq \psi \leq (d-1)$.

In a pioneering paper, McMillan⁸ proposed a phenomenological renormalization-group theory for the spin glass. Our ansatz incorporates much of the scaling behavior he assumed, although we allow $\psi > \theta$, while he appears to have assumed $\psi = \theta$. McMillan⁸ did not explore many of the consequences of his theory for the ordered phase; for a more recent treatment that does this within a phenomenological renormalization-group framework, see Ref. 14. We have instead chosen to work more directly in terms of the low-lying droplet excitations in the spin-glass ordered phase, using renormalization-group ideas only to justify our scaling assumptions. Of course, our various results can be rederived by suitable generalizations of McMillan's approach (and some were first obtained through this approach),^{8,14} but we feel they are generally more readily obtained and understood within our droplet picture. Many of our results are in striking contrast to the behavior found in the infinite-range Sherrington-Kirkpatrick (SK) model.¹ In what sense (if any) this infinite-range model represents the high-dimensional limit of finite-range systems is still unclear.

A summary of this paper is as follows. In Sec. II we introduce and discuss our basic ansatz for the nature and energetics of the low-lying large length scale excitations available in the spin-glass ordered phase. Through most of the paper we focus on Ising spin glasses with nearest-neighbor interactions where these excitations are compact droplets of coherently flipped spins. We choose a precise definition for these droplets. The assumption is that the droplets of size L have excitation energies that scale as L^θ and we argue that the exponent $\theta \leq (d-1)/2$. The ordered phase exists at positive temperature when $\theta > 0$. The distribution of excitation energies is broad with a nonzero density of states at arbitrarily low energies. The surfaces of the droplets are very rough, having a nontrivial fractal dimension $d > d_s > (d-1)$.

In Sec. III we discuss the consequences of these assumptions for the equilibrium static behavior in the or-

dered phase. Much of the behavior is dominated by the thermally active droplets, namely those with free energies less than or of order T . These give average spatial correlations that fall off with distance r as $r^{-\theta}$ and a divergent nonlinear susceptibility throughout the ordered phase. Because the exponent θ is so small, the ordered phase is unstable to a uniform (or random) magnetic field and no long-range spin-glass ordering occurs in the presence of a magnetic field.

In Sec. IV we examine the dynamic behavior in the ordered phase. Here we introduce another exponent by assuming that the free-energy activation barriers that must be crossed to flip a droplet scale as $B \sim L^\psi$, with $\theta \leq \psi \leq (d-1)$. These divergent barriers give rise to very slow, thermally activated dynamics. We find that the spin autocorrelation function decays with time t as $\sim (\ln t)^{-\theta/\psi}$ and the ac nonlinear susceptibility diverges with frequency $\omega \rightarrow 0$ as a power of $\ln \omega$.

Section V addresses the spin-glass critical point, focusing in particular on the approach to the critical point from within the ordered phase. Although we expect this critical point to obey conventional static and dynamic scaling, we do examine other possibilities and their consequences.

In Sec. VI we discuss a schematic coarse-graining procedure that is necessary to define the droplet excitations and their free energies at positive temperature. For droplet excitations of length scale L , coarse graining to a length scale $\lambda_L \sim L^{d_s/(d_s-\theta)}$ is necessary to determine their free energies to an accuracy of order T .

This coarse graining to a large length scale is necessary because the droplet excitations and the spin-glass state itself are very sensitive to the entropy of thermal fluctuations. The resulting sensitivity to temperature changes, which has important experimental consequences,^{2,9} is discussed in Sec. VII. We argue that for $T > 0$ the free energy of a large droplet excitation is a near cancellation of much larger energy and entropy contributions (via $F = E - TS$). Thus a small temperature change will cause this balance to change and many large droplets will flip, completely rearranging the ordered state on sufficiently large length scales.

Section VIII examines the behavior near the lower critical dimension, based on phenomenological renormalization-group equations supported by droplet arguments. An apparently new result is that the order parameter exponent β behaves as $\beta \sim \theta^{-1/2}$ for $\theta \rightarrow 0^+$ in the simplest scenario.

In Sec. IX we discuss systems with long-range random interactions decaying with distance as $r^{-\sigma}$, $\sigma > d/2$. For small enough σ these interactions are relevant and cause the free energy of suitably defined droplet excitations to scale with exponent $\theta_\sigma = d - \sigma$.

In Sec. X we discuss extensions of some of the results to spin glasses with a continuous X-Y or Heisenberg spin symmetry. New subtleties occur which we do not go into in detail. Finally, a brief discussion of the observability of the equilibrium phenomena discussed herein is included in Sec. XI. Arguments for the exponent inequality $\theta \leq (d-1)/2$ and counterarguments to criticisms of Villain's¹⁵ concerning the destruction of the ordered

spin-glass phase by a magnetic field are relegated to Appendices A and B, respectively.

II. SCALING ANSATZ FOR DROPLET EXCITATIONS

For simplicity we primarily consider Ising systems with nearest-neighbor interactions $\{J_{ij}\}$. In order to avoid exact degeneracies, we restrict consideration to a continuous distribution $w(J)$ of independent exchanges.¹⁶ For most purposes, we further restrict the distribution to be symmetric under $J_{ij} \leftrightarrow -J_{ij}$, i.e., $w(J) = w(-J)$, thus eliminating the possibility of ferromagnetic or other such behavior; however, we do give some results for asymmetric distributions.

There are two basic possibilities for the low-temperature equilibrium behavior of Ising spin glasses. The simplest possibility, which will certainly occur in low enough dimensions, is that the system is paramagnetic at all positive temperatures. More interesting behavior occurs if there is a phase transition as the temperature T is decreased to a phase in which the global spin reversal symmetry is broken. In this case, which is an equilibrium spin-glass phase, each spin S_i will attain a nonzero expectation value $\langle S_i \rangle$ whose sign will depend on the site i , and thus the Edwards-Anderson¹⁰ order parameter,

$$q_{\text{EA}} = \frac{1}{V} \sum_i \langle S_i \rangle^2 \quad (2.1)$$

(with V the volume) will be nonzero. The angular brackets here denote a thermal (or temporal) average within a given pure state¹⁷ in the thermodynamic limit $V \rightarrow \infty$. Throughout this paper the thermodynamic limit $V \rightarrow \infty$ is implied in expressions like Eq. (2.1) normalized by V .

More complicated possibilities might occur in principle, however, the most likely scenario is that there is no transition for dimensions d less than a lower critical dimension d_l while for $d > d_l$ there exists a positive transition temperature T_c below which q_{EA} is nonzero.

The philosophy of this paper (as discussed in the Introduction) is to, as far as possible, try to describe the low-temperature behavior in terms of ground states and their excitations. If the system has no spin-glass phase, then this description must fail at long distances. If $T_c > 0$, on the other hand, one might hope (and we will argue) that such a description will work qualitatively at sufficiently long length scales and low frequencies for all $T < T_c$.

We thus start by considering the properties of ground states and their excitations. We will generally focus on a particular realization (sample) of the disorder in an infinite system, only averaging over realizations of the disorder (samples) when necessary. Some quantities of interest are not self-averaging, meaning their values are either not well defined or differ from sample to sample in the thermodynamic limit; these quantities must therefore be averaged over samples. For most purposes we can replace volume averages such as in Eq. (2.1) by configuration averages denoted by an overbar.

Ground states have the defining property that their energy cannot be lowered by flipping any *finite* collection of spins. (Note: The total energy of a configuration of an

infinite system is not well defined; some prescription is needed for taking the thermodynamic limit and the result will depend on this procedure.)

There will certainly be at least two configurations of the infinite system which are ground states, since if some configuration Γ is a ground state, then its global spin reversal $\bar{\Gamma}$ will also be. An important question concerns the existence of more than one pair of ground states. We have argued elsewhere¹⁸ that scaling arguments strongly suggest that there exists *exactly one pair* of ground states, and we will generally assume this to be the case. However, for much of the discussion of this paper it will not matter whether this is correct since most of the statistical properties of a single pair of states, which we discuss here, will hold for any other pair as well.

For definiteness, we focus on a particular ground state Γ . Each configuration of the Ising system can be considered as a collection of walls separating regions in which the spins are in the same direction as the reference ground state Γ from regions in which they are in the opposite direction, i.e., just the spin flip of the reference ground state $\bar{\Gamma}$. We will particularly be interested in connected excitations which consist of a single wall enclosing a connected cluster C_N of N spins which are reversed from the ground state, as in Fig. 1. There will be many such clusters which include any particular spin, say the j th, however, because of the random exchange and the frustration, the energies of the flipped clusters will vary considerably. We expect that those with the lowest energies will dominate the physics.

Most of the low-energy excitations will involve only one or a few spins and will not directly contribute to the low frequency or long length scale phenomena in which we are interested. However, there can also be large-scale coherent fluctuations which involve a large number of connected spins and thus will contribute to long-distance correlations. Since such excitations intrinsically involve some degree of coherent flipping of spins, the characteristic time scales for such excitations are likely to be long and will hence influence the low-frequency, long-time dynamics of interest. We must thus ask: What are the

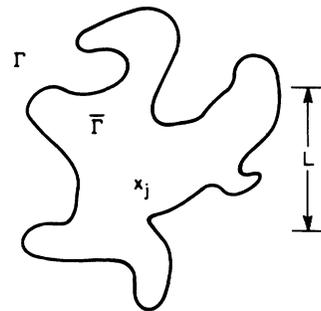


FIG. 1. Schematic picture of the droplet of length scale L containing site j . Outside the droplet the spins are aligned as in the ground state Γ , while inside the droplet the spins are reversed, as in the ground state $\bar{\Gamma}$, which is just the global spin flip of Γ . The surface of the droplet is actually fractal, as vaguely suggested by this drawing.

properties of the dominant large-scale coherent excitations in spin glasses? In order to distinguish them from generic clusters, we will call connected clusters whose energy (or more generally free energy) is minimal *droplets*. Specifically we define the droplet $D_L(j)$ of length scale L which encloses site j to be the minimum energy connected cluster which includes site j and contains between L^d and $(2L)^d$ spins. The energy of $D_L(j)$ is

$$F_L(j) = \min_{\substack{C_N \supset j \\ L^d \leq N < (2L)^d}} (E_{C_N}), \quad (2.2)$$

where the minimum is over all connected clusters C_N of N spins with N between L^d and $(2L)^d$ which contain site j , and E_C is the energy cost of flipping cluster C . At positive temperature, the free energy will replace the energy of the droplet, and we thus use F generally. The manner in which a typical droplet free energy scales with its size will determine many properties of the system. For example, in an Ising ferromagnet, F_L scales as L^{d-1} for large L with the coefficient simply the interfacial tension which is the *stiffness modulus* of a ferromagnet. In a spin glass, a *generic* cluster on scale L with a smooth boundary will also have an energy which scales as L^{d-1} ; however, we expect the minimal clusters, i.e., the droplet excitations, to have much smaller energy.

The natural scaling ansatz is that the typical (e.g., median) droplet free energy will scale as

$$F_L(j) \sim L^\theta, \quad (2.3)$$

with some exponent θ . This exponent will play a very important role in what follows. A more general conjecture which we make is that *all* appropriate measures of the *stiffness* of the ordered phase on length scale L will typically scale as L^θ , see, for example, the discussion in Ref. 18.

In contrast to ferromagnets, we expect the exponent θ to be *less* than $d-1$ in spin glasses due to the frustration and the competition between various configurations as potential ground states. Indeed, we argue that

$$\theta \leq \frac{d-1}{2} \quad (2.4)$$

is an upper bound on θ . Arguments supporting (2.4) are given in Appendix A. Since the total free energy of the wall of a droplet of scale L grows less rapidly than the surface area of the wall, small sections of the wall will have free energies which are almost random in sign. This suggests that the distribution of droplet free energies will have a width which is comparable to its median, since the sums of the free energies of these small sections will vary considerably, subject to the constraint that their sum be positive. In particular, due to the presence of configurations which are almost degenerate with Γ in the region of a droplet, the distribution of F_L will have weight down to zero energy. The natural scaling ansatz is that the distribution $\rho_L(F_L)dF_L$ of droplet free energies at scale L will, for large L , assume a scaling form

$$\rho_L(F_L) \approx \frac{1}{\Upsilon L^\theta \tilde{\rho}} \left[\frac{F_L}{\Upsilon L^\theta} \right], \quad (2.5)$$

where Υ is a generalized stiffness modulus which will be of order the characteristic exchange $J \equiv (\bar{J}_{ij}^2)^{1/2}$ at zero temperatures. (Here and throughout this paper we measure lengths in units of the lattice spacing.) From the argument above we expect $\tilde{\rho}(0)$ to be positive.¹⁹ It is convenient to choose the normalization of Υ so that $\tilde{\rho}(0)=1$. The distribution is normalized so that $\int dF_L \rho_L(F_L) = 1$. The expected form of the distribution for large L is shown schematically in Fig. 2.

A. Shape of droplets

Because of the randomness, the large-droplet excitations can (as suggested by Fig. 1) have complicated shapes and even nontrivial topologies with overhangs handles, etc. An important question concerns the typical area A_L of the surface of droplets of size L . In a previous paper¹⁸ we have argued that $A_L/L^d \rightarrow 0$ for large L so that the droplets are compact in the sense that reasonable measures of their diameters scale as L . The broad distribution of free energies of the droplets and sections of the droplets, will, however, make their surface fractal with

$$A_L \sim L^{d_s}. \quad (2.6)$$

This can readily be seen as follows: First note that, for a given j , the probability that $F_{2L}(j) < F_L(j)$ is positive. For some positive fraction of such cases the walls of $D_L(j)$ and $D_{2L}(j)$ intersect. In these cases their walls will coincide over some regions, and over other parts where they do not coincide, the wall of the larger droplet $D_{2L}(j)$ will have lower free energy. Thus the scale- L droplet could (if permitted) lower its free energy by expanding some section of its wall by a distance of order L . It should be apparent that this will also occur for smaller sections of large droplets, i.e., given a $(d-2)$ -dimensional curve of scale l lying in the wall of a scale- L

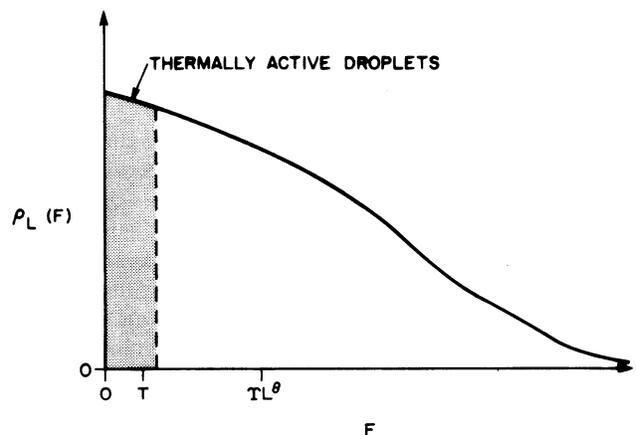


FIG. 2. Schematic distribution $\rho_L(F)$ of droplet free energies F at length scale L . The mean droplet free energy is of order ΥL^θ . The thermally active droplets are those with free energy less than or of order T . For $T \ll \Upsilon L^\theta$ these represent only a small fraction of all droplets. Here we show $\rho_L(F)$ having a finite slope for $F \rightarrow 0$; this corresponds (3.20) to the case $\phi=1$, which seems likely.

droplet, the section of the wall spanning the curve will, with positive probability for large l , minimize its free energy by deviating by amounts of order l from the minimum area position. This implies that on all scales sections of droplet wall will deviate from minimal area by relative amounts of order unity. Thus the droplet walls will have nontrivial fractal dimension

$$d_s > d - 1. \quad (2.7)$$

Bray and Moore²⁰ have measured the areas of domain walls in small two-dimensional Ising spin-glass samples where they find $d_s \cong 1.3$.

B. Correlations between droplet free energies

The scaled distribution $\tilde{\rho}$ clearly depends on the specific choice for the definition of F_L which we made in Eq. (2.2), in particular, on the choice of the allowed range $L^d \leq N < (2L)^d$. We have chosen a definition of this general form, with a coarse graining at each length scale, in order that the correlations between the $F_L(j)$ also exhibit simple scaling forms. If we consider droplets of different sizes which include the same site, $D_{L_1}(j)$ and $D_{L_2}(j)$, then there will be correlations between $F_{L_1}(j)$ and $F_{L_2}(j)$ which are large for $L_1 \sim L_2$, but small if $L_1 \gg L_2$ since in that case the free energy of the larger droplet will only depend weakly on the region within distance of order L_2 of the site j . Similarly, if we take two different sites i and j a distance r_{ij} apart, then $F_L(i)$ and $F_L(j)$ will be almost perfectly correlated for $|r_{ij}| \ll L$, since the same droplet will usually contain both sites. For $|r_{ij}| \gg L$, on the other hand, we expect approximate independence. These expectations are all consistent with the simple scaling form for the joint distribution of $F_{L_1}(j_1)$ and $F_{L_2}(j_2)$:

$$\begin{aligned} & \rho_{L_1, L_2, r_{ij}}^{(2)}[F_{L_1}(j_1), F_{L_2}(j_2)] \\ & \cong \frac{1}{\Upsilon^2 L_1^\theta L_2^\theta} \tilde{\rho}^{(2)} \left(\frac{F_{L_1}}{\Upsilon L_1^\theta}, \frac{F_{L_2}}{\Upsilon L_2^\theta}, \frac{|r_{ij}|}{L_1}, \frac{|r_{ij}|}{L_2} \right) \end{aligned} \quad (2.8)$$

for L_1 and L_2 both large. The scaled distribution approaches a δ function in the difference between its first two arguments for $L_1 \rightarrow L_2$ and $(|r_{ij}|/L_1) \rightarrow 0$, and factorizes as $\tilde{\rho}^{(2)}(a, b; c, d) \approx \tilde{\rho}(a)\tilde{\rho}(b)$ for $L_1/L_2 \rightarrow 0$ or ∞ or $|r_{ij}|/\min(L_1, L_2) \rightarrow \infty$. Homogeneous forms of probability distributions as a function of length scale, such as Eqs. (2.5) and (2.8), are consequences of a zero-temperature fixed point which controls the spin-glass ordered phase. This will be discussed further in Sec. VI below.

In order to assure that the excitations at a given scale will be dominated by the droplets, we need to be sure that there are not too many clusters which cost almost the same free energy to flip as the minimal cluster of the same scale in the same region, i.e., the droplet. From the general scaling properties of the excitations and their correlations, we expect that the only such clusters which differ significantly (differ in a volume of order L^d) from

the droplets will also typically differ in energy by of order L^θ . Thus for $\theta > 0$ and large L , they are much higher in energy than the droplets, and restricting our attention to the droplets is appropriate.

The complicated correlations between the droplet free energies make calculations rather cumbersome. It is thus useful to consider a toy model of independent droplets which will reproduce the essential features of the real system. We consider only length scales $L_k \equiv 2^k L_0$, with k an integer and L_0 a microscopic length. At each of these length scales, we consider a cubic array of points with lattice spacing L_k and coordinates $\mathbf{n}L_k$ with \mathbf{n} an integer vector. We then define a droplet of scale L_k enclosing each of the points on this lattice as in Eq. (2.2). We do not define droplets about other points, although of course other points will be included in the selected droplets. With all these selected droplets assumed independent, the only additional information we need is the distribution of their shapes which determines the probability $\Pr[D_{L_k}(j) \supset i]$ that a droplet around site j contains site i . This will approach unity for $|i - j| \ll L_k$ and fall off for $|i - j| > L_k$. The normalization is such that

$$L_k^d \leq \sum_i \Pr[D_{L_k}(j) \supset i] < (2L_k)^d. \quad (2.9)$$

In this toy model, correlations between the droplet free energies do not have the full translational invariance; however, they are simple and have the correct scaling form since the joint distribution trivially satisfies Eq. (2.8). Thus this toy model should produce the correct scaling forms and exponents for the properties of the spin-glass phase; a more realistic model is necessary only to obtain details about the scaling functions and subtle effects such as temperature dependence (see Sec. VII).

It should be clear that the behavior of the system at small positive temperature and long length scales will radically depend on the sign of θ . If θ is *negative* then large low-energy excitations will be so common that at any low-temperature T , there will be a scale $\xi_+ \sim (\Upsilon/T)^{1/|\theta|}$ at which there is a high density of roughly independent excitations with energies $\lesssim T$. Above this length scale the entropy will clearly dominate and the droplet picture breaks down. The natural conclusion is that the system will be paramagnetic for all positive T since each spin can be flipped with arbitrarily small free-energy cost. The scale ξ_+ is simply the paramagnetic correlation length and $1/|\theta|$ is the correlation length exponent ν for the zero-temperature transition. Thus $\theta < 0$ implies $d < d_f$.^{7,8} (We note that an analogous mechanism disorders nonrandom Heisenberg ferromagnets for $d < 2$.)

If θ is *positive*, on the other hand, then very few of the large scale droplets will be thermally activated since their free energy is typically much larger than T . However because $\rho_L(F_L)$ has weight near zero, a certain fraction of the droplets will be thermally excited at *any* positive temperature. These droplets with F_L less than or of order T , which we call the *active droplets*, will dominate most of the equilibrium physics. We will henceforth consider only the case $\theta > 0$ unless otherwise specified.

III. STATIC BEHAVIOR FOR $T < T_c$

We are now in a position to deduce from our ansatz the behavior of equilibrium correlation functions at low temperatures for $\theta > 0$. The simplest correlation functions are just the single-site expectation values $\langle S_j \rangle$. If we ignore the effects of droplets within droplets, which naively appears valid for $T \ll J$, then for a given spin,

$$|\langle S_j \rangle_T| \approx \min_L \tanh[F_L(j)/2T], \quad (3.1)$$

since, to leading order in T , the possibility that two different sized droplets enclosing j both have free energy of order T (are active) is negligible. From the approximate independence of droplets of scales differing by factors of 2, the probability distribution of $F_m(j) \equiv \min_L F_L(j)$ is roughly

$$\begin{aligned} \Pr[F_m(j) < f] &\sim 1 - \prod_L \Pr[F_L(j) > f] \\ &\approx 1 - \prod_L \left[1 - \frac{f}{\Upsilon L^\theta} \bar{\rho}(0) \right] \\ &\approx \frac{f}{\Upsilon} \sum_L \frac{1}{L^\theta} \approx \frac{cf}{\Upsilon L_0^\theta}, \end{aligned} \quad (3.2)$$

where the second line holds for small f and L_0 is a short-distance cutoff on the sums and products. We have introduced the notation

$$\sum_L \equiv \sum_{k=0}^{\infty} \sim \int_{L_0}^{\infty} \frac{dL}{L}, \quad (3.3)$$

and likewise \prod , and c which will be used throughout to denote any positive finite dimensionless constant. Note that the minimum-free-energy droplet enclosing j is likely to be small since $\theta > 0$.

The Edwards-Anderson order parameter q_{EA} in Eq. (2.1) is the average over sites (or configurations) of $\langle S_i \rangle^2$. For small T the deviation of q_{EA} from unity is thus

$$1 - q_{EA} \approx \frac{cT}{\Upsilon L_0^\theta}, \quad (3.4)$$

which is dominated by the active droplets with $F_m(j)$ less than or of order T . Thus, with $\theta > 0$, it appears that the broken spin symmetry is stable at low T , and we hence expect a positive transition temperature T_c below which the global spin symmetry is broken. Thus $\theta > 0$ implies $d > d_l$ and the lower critical dimension d_l is where θ passes through zero. A similar argument yields a specific heat at low temperatures which is linear in T . This arises from the density of excitations with energies near zero, which again are primarily of small length scale. The linear behavior at low temperatures is in contrast to the T^2 dependence of the specific heat found in the infinite-range SK model.²¹

As for conventional Ising systems, the low-temperature behavior of the order parameter and energy density are dominated by small scale fluctuations. Long-distance correlation functions, on the other hand, are necessarily

dominated by large-scale excitations.

We consider the truncated correlations between two spins i and j a large distance r_{ij} apart:

$$C_{ij} \equiv \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle. \quad (3.5)$$

As for Ising ferromagnets in $d = 2$,^{22,23} this will be dominated at low T by large-droplet excitations which include both i and j , since these will cause both spins to fluctuate together. Ignoring other droplets, we have, for low T ,

$$|C_{ij}| \approx \{1 - \tanh^2[F_m(ij)/2T]\}, \quad (3.6)$$

where $F_m(ij)$ is the free energy of the minimum-free-energy cluster which contains *both* i and j . The typical value of $F_m(ij)$ will be of order Υr_{ij}^θ . Thus for a typical pair of spins, the logarithm of the truncated correlations will satisfy²⁴ $-\ln |C_{ij}| \sim r_{ij}^\theta$. Let us define the Edwards-Anderson correlation function $C^2(r)$ as the average of C_{ij}^2 over all pairs of spins with a given separation $r_{ij} = r$. This average will be dominated by those pairs with $F_m(ij) \lesssim T$, namely those pairs in active droplets. For fixed f we have

$$\Pr[F_m(ij) < f] \sim \sum_{L > |r_{ij}|} \frac{cf}{\Upsilon L^\theta} \sim \frac{cf}{\Upsilon r_{ij}^\theta}. \quad (3.7)$$

Thus

$$C^2(r) \sim \frac{T}{\Upsilon r^\theta}. \quad (3.8)$$

The Edwards-Anderson (EA) "susceptibility,"

$$\chi_{EA} \equiv \frac{1}{V} \sum_{i,j} C_{ij}^2 \sim \int d^d r C^2(r) \quad (3.9)$$

is a nonlinear susceptibilitylike quantity associated with the spin-glass order parameter q_{EA} . Because $\theta < d$, the sum in Eq. (3.9) diverges yielding

$$\chi_{EA} = \infty \quad (3.10)$$

in the entire ordered phase $T \leq T_c$. This is analogous to the divergence of the order parameter susceptibility in nonrandom Heisenberg ferromagnets below T_c due to spin-wave excitations.

The ferromagnetic susceptibility

$$\chi = \frac{1}{TV} \sum_{i,j} C_{ij} \quad (3.11)$$

is for symmetric disorder [$w(J) = w(-J)$] simply given by

$$\chi = (1 - q_{EA})/T, \quad (3.12)$$

since the sum of the terms in Eq. (3.11) with $i \neq j$ vanishes in the thermodynamic limit $V \rightarrow \infty$. For an asymmetric distribution of couplings, Eq. (3.12) is not valid. It is then convenient to introduce for later use a related order parameter:

$$q_M = \frac{1}{V} \sum_{i,j} \overline{\langle S_i \rangle \langle S_j \rangle} = \frac{1}{V} \overline{\langle M \rangle^2}, \quad (3.13)$$

with M the total magnetization.

We then have generally

$$\chi = \left[\frac{1}{V} \overline{\langle M^2 \rangle} - q_M \right] / T. \quad (3.14)$$

For a symmetric distribution of exchanges, $q_M = q_{EA}$; however, if the system has some ferromagnetic tendencies, then $q_M > q_{EA}$, while antiferromagnetic tendencies give $q_M < q_{EA}$. Droplets whose size is larger than the (anti-) ferromagnetic correlation length will have rms magnetic moments of size $(q_M L^d)^{1/2}$. In either case the susceptibility will go to a constant for $T \rightarrow 0$; its value is dominated by small-scale excitations. Note that we have defined χ in terms of the normalized second moment of the magnetization fluctuations, not as dm/dH , where $m = M/V$. Since it is dominated by small excitations, we expect χ is self-averaging and

$$\chi = \frac{dm}{dH}. \quad (3.15)$$

In general, the fluctuation-dissipation theorem implies that equations like (3.14) for χ will yield the zero-frequency limit of response functions; these may however not be simply related to thermodynamic derivatives.

The nonlinear susceptibility χ_3 we define as

$$\chi_3 \equiv \frac{1}{V} [\langle (M - \langle M \rangle)^4 \rangle - 3 \langle (M - \langle M \rangle)^2 \rangle^2]. \quad (3.16)$$

In the absence of symmetry breaking χ_3 is simply related to the EA susceptibility and $d^3 m/dH^3$; however, for $T < T_c$ this is no longer so. Note that in zero field χ_3 is generally negative. For a symmetric distribution of exchanges

$$\begin{aligned} \chi_3 = & \frac{1}{VT^3} \sum_{i,j} (3 \overline{\langle \delta S_i^2 \delta S_j^2 \rangle} - 3 \overline{\langle \delta S_i^2 \rangle \langle \delta S_j^2 \rangle} - 6 \overline{C_{ij}^2}) \\ & + \frac{1}{VT^3} \sum_i (\overline{\langle \delta S_i^4 \rangle} - 3 \overline{\langle \delta S_i^2 \rangle^2}), \end{aligned} \quad (3.17)$$

where $\delta S_i \equiv S_i - \langle S_i \rangle$. The last term in (3.17) is non-singular and the largest contributions of the first, second, and third terms, which behave (after averaging) as $r_{ij}^{-\theta}$, cancel. Thus the average contribution to χ_3 from large thermally active droplets is due to *corrections* to the leading behavior of these correlation functions.

To see where this cancellation comes from let us examine the contribution to χ_3 from a given active droplet D_L with excitation free energy F_L . The total magnetic moment of D_L is of order $(q_M L^d)^{1/2}$ so its contribution to χ_3 is of order

$$\frac{q_M^2 L^{2d}}{T^3} [2 \operatorname{sech}^2(F_L/2T) - 3 \operatorname{sech}^4(F_L/2T)]. \quad (3.18)$$

This contribution can be either positive or negative, and if the distribution of energies $\rho_L(F_L)$ were uniform then the average contribution would actually vanish. Thus χ_3 arises from the variation of $\rho_L(F_L)$ for F_L less than or of order T :

$$\begin{aligned} \chi_3 \sim & \frac{q_M^2}{T^3} \sum_L L^d \int dF [\rho_L(0) - \rho_L(F)] \\ & \times [3 \operatorname{sech}^4(F/2T) - 2 \operatorname{sech}^2(F/2T)]. \end{aligned} \quad (3.19)$$

If $\rho_L(F) - \rho_L(0)$ scales as in (2.5) and the scaling function is of the form

$$1 - \bar{\rho}(x) \sim x^\phi \quad (3.20)$$

for $x \rightarrow 0$, then we obtain

$$\chi_3 \sim - \frac{q_M^2}{T^3} \sum_L L^d \left[\frac{T}{\Upsilon L^\theta} \right]^{1+\phi}. \quad (3.21)$$

This results in a nonlinear susceptibility that is infinite when $d > (1 + \phi)\theta$. A perfectly smooth $\bar{\rho}$ has $\phi \leq 2$ (unless both its first and second derivatives happen to vanish) so it seems we can safely assume $\phi \leq 2$. (Probably $\phi \leq 1$.) This means that near the lower critical dimension where θ is small, and for $d=3$ in particular, χ_3 should be infinite. We note that there is a possibility that due to corrections to scaling $\rho_L(F) - \rho_L(0)$ does not scale as naively implied by Eq. (2.5) for large L and F of order T . For most reasonable forms of $\rho_L(F)$, however, one will still roughly have

$$\chi_3 \sim \frac{q_M^2}{T^2} \sum_L L^d [\rho_L(3T) - \rho_L(T)]. \quad (3.22)$$

This is because the droplets with F near $3T$ contribute positively to χ_3 , while those near T contribute negatively. In order to obtain a finite nonlinear susceptibility for $d=3$ one would need an extremely slow variation of $\rho_L(F)$. Thus we conclude that χ_3 is almost certainly infinite for $T \leq T_c$ in $d=3$, and indeed is likely to be infinite for all dimensionalities.

We now extend the results away from the temperature regime $T \ll J$ to any temperature less than the presumed T_c .

At positive temperatures the droplets must be defined far more carefully to take into account the entropic effects, and the ground state must be replaced in the discussion by the configuration $S_{i0} = \operatorname{sgn} \langle S_i \rangle$, which as we argue below will not be simply related to the zero-temperature ground state. These important and subtle features will be discussed in detail later in Secs. VI and VII; however, for the present purposes it is sufficient to consider that the large-scale droplets behave *statistically* the same as those at zero temperature but with the overall scale of the free energy Υ and the order parameters q_{EA} , q_M dependent on temperature. Each large droplet will now flip effective spin magnitudes whose values are reduced on average from unity to $\sqrt{q_{EA}}$ by the small-scale fluctuations. The mean-square correlation function will then have the asymptotic form

$$\overline{C^2(r)} \approx K_2 \frac{q_{EA}^2 T}{\Upsilon r^\theta}, \quad (3.23)$$

where K_2 is a *universal* constant which will depend on the distribution of the droplets which enclose two far away sites. The EA susceptibility and χ_3 will be infinite for all $T < T_c$ by the same arguments given earlier.

A. Effects of a magnetic field

We now consider the singular effects of applying a small uniform magnetic field, which are suggested by the divergent nonlinear susceptibility. Since the spins are (spatially) randomly oriented in equilibrium, a uniform field breaks the global spin symmetry like a random field. Following McMillan⁸ we may analyze its effects by use of a variant of the argument Imry and Ma²⁵ used to determine the stability of Ising ferromagnets to random fields.

Flipping a droplet of scale L will change the magnetization by a random amount of order $L^{d/2}$, whose mean-square value for large droplets is proportional to $q_M L^d$. For half the droplets this will result in a decrease in magnetic free energy of order $H(q_M L^d)^{1/2}$ when they are flipped in the presence of a small uniform field H . The exchange energy increase is of order ΥL^θ , which for large L is smaller than the magnetic energy decrease since $\theta \leq (d-1)/2 < d/2$. Thus at a scale

$$\xi_H \approx K_H \left(\frac{\Upsilon}{H \sqrt{q_M}} \right)^{2/(d-2\theta)} \quad (3.24)$$

the field will flip half the spins and the zero-field states will be destroyed. We have argued elsewhere¹⁸ that there are only two pure states in $H=0$, and thus that the spin-glass transition will be destroyed by the magnetic field, as is the case for random-field Ising ferromagnets in $d < 2$. The phase diagram of an Ising spin glass is illustrated in Fig. 3. For $H > 0$, the system is hence a paramagnet and ξ_H is the correlation length. With this definition of ξ_H , the coefficient K_H should be universal. Villain¹⁵ has recently criticized the above conclusion; we rebut his arguments in Appendix B.

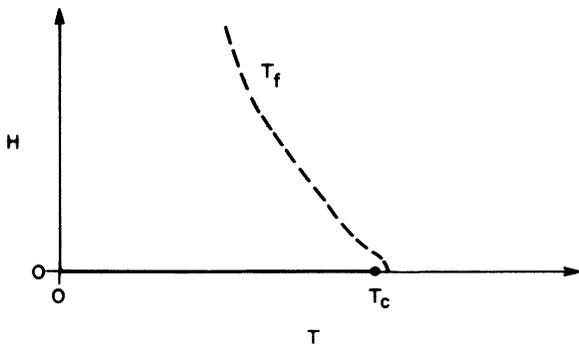


FIG. 3. Schematic phase diagram for the Ising spin glass with dimensionality $d > d_l$. Long-range order at equilibrium only occurs for zero magnetic field $H=0$ and temperature T less than the critical temperature T_c . The system falls out of equilibrium on accessible laboratory time scales for temperatures less than $T_f(H)$, dashed line. The position of this freezing line is weakly dependent on the time allowed for equilibration (5.25).

The magnetization has a singularity as a function of magnetic field because most droplets of size of order ξ_H are fully aligned with the field. This results in

$$\chi H - m \approx K_M \left(\frac{q_M}{\xi_H^d} \right)^{1/2} \sim H^{d/(d-2\theta)}, \quad (3.25)$$

where K_M is universal and χ is the zero-field linear susceptibility. Note that χH exceeds m by this amount because the low-field linear response of the droplets with $L \gtrsim \xi_H$ has saturated. The third derivative of m therefore diverges as

$$-\frac{d^3 m}{dH^3} \sim H^{-2(d-3\theta)/(d-2\theta)} \sim \xi_H^{d-3\theta} \quad (3.26)$$

for $H \rightarrow 0$. This diverges only for $\theta < d/3$, while χ_3 (3.21) is, for $\phi < 2$, divergent for larger θ . The difference arises from noncommutability of the $H \rightarrow 0$ and thermodynamic limits due to the phase transition at $H=0$.

B. Zero-temperature fixed point

The scaling behavior and universality in the spin-glass phase are due to a zero-temperature fixed-point ensemble of random Hamiltonians which control the whole phase, as discussed by various authors. In contrast to conventional ordered systems, this fixed point is highly nontrivial, yielding the novel features discussed in this paper. The exponent θ is just minus the renormalization-group eigenvalue of the temperature, and

$$y_H = \frac{d}{2} - \theta \quad (3.27)$$

is the eigenvalue of the magnetic field. Unfortunately, due to the dangerous irrelevancy of temperature, most of the interesting properties of the spin-glass phase are difficult to obtain from simple scaling arguments without the input of additional physics such as the droplet picture (see Ref. 26 for modified scaling arguments). For example, truncated correlation functions such as C_{ij} are zero at the fixed point (since there are no thermal fluctuations) and their leading behavior is determined by corrections to scaling. In order to obtain physical results, it is thus easier to work directly in the droplet picture. However, justification of the results, the universality of various quantities, and considerations of the definitions of droplets at positive temperatures require rather careful renormalization group arguments which will be given in Sec. VI. Before doing this, we continue with the physical picture and discuss the dynamics of the spin-glass phase.

IV. DYNAMIC BEHAVIOR FOR $T < T_c$

In the previous section, we have shown how the long-distance correlations of the Ising spin-glass phase are determined by droplet excitations, especially the active droplets. In this section we argue that these will similarly determine the equilibrium low-frequency dynamics.

Small droplets with scale of order one will, if active, appear and disappear with a characteristic microscopic time scale τ_0 which will depend on the detailed dynamics

of the system. We assume, for simplicity, that there are no conservation laws which constrain the dynamics, although local conservation laws would not in fact affect most of what follows. Large-scale coherent excitations have relaxation times much longer than τ_0 due to their collective nature. This is already true for droplet excitations in pure Ising ferromagnets for which droplets of scale L last (in the absence of conservation laws) for a time²⁷ $\tau_L \sim L^2$. For spin glasses and other random systems, the time scales can be much longer due to the presence of large free-energy barriers.²⁷ These barriers exist because the configurations of the droplet excitations are locally optimal so that small changes in them will generally increase their free energy. This is especially true for large active droplets, which by definition have anomalously low free energy $F_L \sim T \ll \Upsilon L^\theta$.

In order to form (flip) a large active droplet with volume $\gtrsim L^d$, the system will have to pass through configurations in which there is a droplet (or higher-energy cluster) of scale $L/2$ in the same region. The free energy of these intermediate configurations will typically be at least of order $\Upsilon(L/2)^\theta$ which is thus a minimum size for the typical free-energy barrier for creation of the larger active droplet. In fact, the barrier is likely to be considerably larger since configurations which are far from locally optimal droplets must be passed through. In addition, more generic droplets with $F_L \sim L^\theta$ (rather than $\sim T$) will also typically have barriers for their formation and annihilation. Whether the complicated optimization procedure needed to obtain the optimum path in configuration space for flipping a droplet will yield a barrier which grows as L^θ or one which grows more rapidly is not at all clear. In the absence of any solid arguments one way or the other, we make the natural scaling ansatz that the typical free-energy barriers $B_L(j)$ for *annihilation* of a droplet grow as a power of L ,

$$B_L \sim L^\psi, \quad (4.1)$$

with an independent exponent ψ satisfying

$$\theta \leq \psi \leq d - 1. \quad (4.2)$$

The barrier for the *creation* of the droplet $D_L(j)$ is just $B_L(j) + F_L(j)$.

The upper bound on the exponent ψ [which will hold at least provided the large J_{ij} tail of the exchange distribution $w(J)$ is not too long] arises from the observation that any compact connected cluster of scale L whose free energy is not larger than L^{d-1} can be created with a maximum barrier of size L^{d-1} by moving the wall without worrying about any local optimization along the way. Droplet barriers will, like the droplet free energies, have a broad distribution, $\Phi_L(B_L)$ with a width the same order as the median. For large L we likewise expect Φ to have a scaling form

$$\Phi_L(B_L) \approx \frac{1}{\Delta L^\psi} \bar{\Phi} \left[\frac{B_L}{\Delta L^\psi} \right], \quad (4.3)$$

where Δ sets the overall free-energy scale of the barriers; for $T \ll T_c$, Δ will be of order J . The characteristic time

$\tau_L(j)$ that a droplet of scale L will last for is exponentially activated:

$$\tau_L(j) \sim \tau_0 e^{B_L(j)/T}. \quad (4.4)$$

Because the B_L are broadly distributed, the distribution of τ_L has an extremely long tail and thus does not have a scaling limit for large L . The quantity which has a well-defined scaling distribution should rather be $\ln(\tau_L/\tau_0)$ since²⁶

$$\ln(\tau_L/\tau_0) \sim L^\psi/T. \quad (4.5)$$

We will often be interested in times t which are extremely long compared to microscopic time scales. Because of the extremely broad distribution of relaxation times, it is important to distinguish between excitations or other processes whose time scale τ is really the same order as t , i.e., $t \sim \tau$, from those whose *logarithmic* time scale $\ln \tau$ is of the same order as $\ln t$. To avoid confusion, we will use *time scale* to mean $t \sim \tau$ and *epoch* to mean $\ln(t/\tau_0) \sim \ln(\tau/\tau_0)$ which, for very large t , includes times for which $t \ll \tau$ or $t \gg \tau$. Epochs correspond to length scales, since a factor of 2 in length scale corresponds, by (4.5), to a factor of 2^ψ in $\ln \tau_L$. We will often measure times in units of τ_0 and leave τ_0 out of expressions.

The spins which are in large active droplets with large barriers have very long-time autocorrelations. Indeed, the spatial average $\bar{C}(t)$ of the autocorrelation function

$$C_j(t) = \langle S_j(0)S_j(t) \rangle - \langle S_j \rangle^2 \quad (4.6)$$

is dominated at long times by those spins S_j in such large active droplets. For a given spin at a given time, the autocorrelation function will be dominated by a droplet of some length scale L which depends on the time t . Including the effects of all droplets containing spin j , we have

$$C_j(t) \approx \langle S_j \rangle^2 \left[\prod_L \left[1 + \frac{e^{-t/\tau_L(j)}}{\sinh^2[\frac{1}{2}F_L(j)/T]} \right] - 1 \right], \quad (4.7)$$

where

$$|\langle S_j \rangle| \approx \prod_L \tanh[\frac{1}{2}F_L(j)/T]. \quad (4.8)$$

Since most spins are in no large thermally active droplets, for a *typical* spin at long times the product in Eq. (4.7) can be expanded and the reduction of $|\langle S_j \rangle|$ from one will be primarily due to small-scale excitations. This yields

$$C_j(t) \sim \langle S_j \rangle^2 \sum_L (e^{-t/\tau_L(j)} e^{-F_L(j)/T}), \quad (4.9)$$

whereupon with, for a typical spin, $\ln(\tau_L/\tau_0) \sim \Delta L^\psi/T$ and $F_L \sim \Upsilon L^\theta$, the sum on L is dominated by the scale L_t for which $\tau_{L_t} \sim t$, yielding

$$-\ln C_j(t) \sim \frac{\Upsilon}{T} \left[\frac{T \ln(t/\tau_0)}{\Delta} \right]^{\theta/\psi}. \quad (4.10)$$

For a given spin the fluctuations around this result will be large due to the discrete set of droplets with variable F_L

and B_L which contribute. Note that, since $\theta/\psi \leq 1$, the autocorrelation of a typical spin decays as a power law of time (if $\psi = \theta$) or even more slowly (if $\theta < \psi$).

Although the autocorrelations of a typical spin are dominated by typical droplets which are inactive, the spatially averaged autocorrelation function $\bar{C}(t)$ will be dominated by those spins which lie in active droplets. This can easily be seen by examining Eq. (4.7): If a spin is in an active droplet with lifetime $\tau_L(j)$, then at times $t \lesssim \tau_L(j)$ the autocorrelations of that spin will be of order

$$C_j(t) \approx \prod_{\substack{L \\ \tau_L(j) < t}} \tanh^2[\frac{1}{2}F_L(j)/T] \left[1 - \prod_{\substack{L \\ \tau_L(j) > t}} \tanh^2[\frac{1}{2}F_L(j)/T] \right]. \quad (4.11)$$

In the approximation that droplets with scales differing by factors of 2 are independent, we can factorize the averages over j . The first factor in Eq. (4.11) averages to q_{EA} for large L , and the average of all the terms in the second product will be close to one and can thus be expanded, yielding

$$\bar{C}(t) \sim q_{EA} \sum_L \frac{T}{\Upsilon L^\theta} \int_{T \ln(t/\tau_0)}^\infty \Phi_L(B_L) dB_L. \quad (4.12)$$

Using the scaling form Eq. (4.3) for the distribution Φ of barriers of large scale droplets, we obtain

$$\bar{C}(t) \sim q_{EA} \frac{T}{\Upsilon} \left[\frac{\Delta}{T \ln(t/\tau_0)} \right]^{\theta/\psi}, \quad (4.13)$$

an extremely slow logarithmic decay of the autocorrelations. [Note: If the scaled barrier distribution $\tilde{\Phi}(x)$ has a long tail which decays more slowly than $1/x^{1+\theta/\psi}$ for large x , then the averaged autocorrelation function at time t will not be dominated by droplets with size $L_t \sim (\ln t)^{1/\psi}$ but rather by those rare smaller droplets which have anomalously large barriers. In this case, which seems unlikely, except perhaps for distributions of J with long tails for which the barrier scaling could break down, Eq. (4.13) represents a lower bound for $\bar{C}(t)$.]

The autocorrelations of the magnetization density,

$$C_M(t) = \frac{1}{V} \sum_{i,j} [\langle S_i(t) S_j(0) \rangle - \langle S_i \rangle \langle S_j \rangle], \quad (4.14)$$

will be identical to Eq. (4.13) if the distribution of the J_{ij} 's is symmetric, and generally differ by a factor

$$C_M(t) \approx \frac{q_M}{q_{EA}} \bar{C}(t) \quad (4.15)$$

for long times if $\omega(J_{ij})$ is asymmetric. The magnetic susceptibility can be obtained from $C_M(t)$ by Fourier transforming and use of the fluctuation-dissipation theorem: $\chi''(\omega) = (\omega/2T) C_M(\omega)$. The real part of $\chi(\omega)$ can then be obtained from the Kramers-Kronig relations. As discussed in Refs. 26 and 28, these transforms of slowly varying functions of $\ln t$ can be simply obtained for small ω via

unity, i.e., much larger than the $\exp[-c(\ln t)^{\theta/\psi}]$ of a typical spin. In a given factor of 2 range of length scale, corresponding to a given epoch, a fraction of order $L^{-\theta}$ of the droplets will be active. These active droplets dominate the spatial average at long times.

At a fixed long time, almost all droplets will either have $\tau_L \ll t$ or $\tau_L \gg t$. Thus to a good approximation either e^{-t/τ_L} is very near one, or else it is extremely small. We can therefore approximate the relaxation of a given droplet in Eq. (4.7) by a step function, yielding

$$C(\omega) \approx \frac{-\pi}{|\omega|} \frac{\partial}{\partial \ln t} C(t) \Big|_{t=1/\omega} \quad (4.16)$$

and

$$\chi''(\omega) = \frac{-\pi}{2} \frac{\partial}{\partial \ln \omega} \chi'(\omega), \quad (4.17)$$

yielding

$$\begin{aligned} \chi'(\omega) &\approx \chi(\omega=0) - \frac{1}{T} C_M(t=1/\omega) \\ &\approx \chi_0 - \frac{K_\omega q_M}{\Upsilon} \left[\frac{\Delta}{T |\ln \omega|} \right]^{\theta/\psi}, \end{aligned} \quad (4.18a)$$

$$\chi''(\omega) \approx \frac{\pi}{2} \frac{\theta}{\psi} K_\omega \frac{q_M}{\Upsilon} \left[\frac{\Delta}{T |\ln \omega|} \right]^{1+\theta/\psi}, \quad (4.18b)$$

and

$$C_M(\omega) \approx \frac{T}{\omega} \frac{\pi \theta}{\psi} K_\omega \frac{q_M}{\Upsilon} \left[\frac{\Delta}{T |\ln \omega|} \right]^{1+\theta/\psi}, \quad (4.19)$$

with K_ω a constant which is universal, given some convention to define Δ , such that ΔL^ψ is the median barrier height at scale L .

Note that the real part of the susceptibility, Eq. (4.18a) could have been obtained directly by observing that any processes with $\tau > 1/\omega$ do not contribute to $\chi(\omega)$, i.e., exactly the opposite of $C(t)$ which is determined by processes with $\tau > t$. From Eq. (4.19), we see that the spin-glass ordered phase should exhibit *equilibrium* $1/f$ noise with weak logarithmic corrections, which is consistent with recent experiments.²⁹

In addition to the linear susceptibility, we are interested in the nonlinear ac susceptibility since it should more closely couple to the spin-glass order. From the previous section we expect that the static cubic nonlinear susceptibility will be infinite, but at any finite frequency we expect only a finite result. We define the real part of cubic nonlinear susceptibility as the in-phase 3ω magnetization response $M(3\omega)$, to a small time-dependent applied field $h(t) = h_\omega \cos \omega t$,

$$\chi_3(3\omega; \omega) \equiv \lim_{h\omega \rightarrow 0} \frac{24M(3\omega)}{Vh\omega^3}, \quad (4.20)$$

which reduces to χ_3 as $\omega \rightarrow 0$.

From similar arguments to those for χ , it is clear that $\chi_3(\omega)$ will be dominated by the active droplets with lifetimes $\tau_L \lesssim 1/\omega$. For each of these droplets, the contribution to χ_3 is virtually the same as to the static χ_3 discussed in Sec. III. Thus we have the same cancellation of the naive divergence as in Eq. (3.19) and obtain, assuming scaling of $\rho_L(0) - \rho_L(F)$ for F of order T

$$\chi_3(3\omega; \omega) \sim -\frac{q_M^2}{T^3} \left[\frac{T}{\Upsilon} \right]^{1+\phi} \left[\frac{T |\ln \omega|}{\Delta} \right]^{[d-(1+\phi)\theta]/\psi}, \quad (4.21)$$

which diverges as a power of $\ln \omega$. If, on the other hand, the temperature dependence of the distribution of active droplets does not scale in this precise way, then the divergence is modified as discussed in Sec. III.

The scaling of the various out-of-phase parts of the cubic nonlinear magnetic response will be subtle and there are no simple general laws (such as Kramers-Kronig and the fluctuation-dissipation theorem) which determine them. One expects, however, that, as for the linear susceptibility, the in-phase response will be larger than the out-of-phase response by powers of $\ln \omega$.

A. Dynamics in a field

We now turn to consideration of the dynamics in a small field which, as discussed above, we expect destroys the equilibrium spin-glass phase. On a length scale of order the correlation length ξ_H (3.24), the characteristic relaxation time, τ_{ξ_H} , will (4.5) be given by

$$\ln(\tau_{\xi_H}/\tau_0) \sim \frac{\Delta}{T} \xi_H^\psi, \quad (4.22)$$

and thus will diverge extremely rapidly for $H \rightarrow 0$. As for the droplet relaxation times τ_L, τ_{ξ_H} will have an extremely broad distribution due to the distribution of barriers. Therefore, an overall characteristic time scale, τ_H , cannot be uniquely defined, but only the characteristic epoch, i.e.,

$$\ln(\tau_H/\tau_0) \sim \frac{\Delta}{T} \xi_H^\psi. \quad (4.23)$$

Dynamic correlation functions will assume activated dynamic scaling forms²⁶ for small H with $\ln t / \ln \tau_H$ the scaling variable rather than the usual t/τ . For example, the singular part of the magnetization noise in a small field should scale as

$$C_M(\omega, H) \sim \frac{1}{\omega |\ln \omega|^{1+\theta/\psi}} \Gamma_M \left[\frac{|\ln \omega \tau_0|}{\ln(\tau_H/\tau_0)} \right], \quad (4.24)$$

where $\Gamma_M(u)$ is a scaling function which approaches unity for small u . For very large times, $\ln t \gg \ln \tau_H$, the correlations will be those of a strongly disordered paramagnet. Randeria *et al.*³⁰ and the present authors²⁷

have argued that the very long-time correlations in strongly disordered paramagnets decay as $\exp[-(\ln t)^y]$ with $y = d/(d-1)$ due to exponentially rare (in $\ln t$) low free-energy excitations which have large barriers. This should apply in the present case of a small field and temperatures $T < T_c$. However since $y > 1$, these long-time correlations decay faster than any power of t so that at asymptotically low frequencies outside the scaling regime $|\ln \omega \tau_0| \gg \ln(\tau_H/\tau_0)$, we expect $C_M(\omega, H)$ will saturate to a finite constant.

The activated scaling behavior in Eq. (4.24) might be more readily seen in $\chi''(\omega) \propto \omega C_M(\omega)$ since the rapid $1/\omega$ dependence on frequency is then absent.

V. CRITICAL BEHAVIOR

So far, we have concentrated on behavior relatively far below T_c , where the Edwards-Anderson order parameter q_{EA} is near unity and the free-energy scales Υ and Δ are both of order J . The dominant temperature dependences in this regime are due to the fraction of thermally active droplets, which is proportional to T , and to the Arrhenius form of the dynamics. Near to a second-order spin-glass transition, however, there will be strong temperature dependence due to the critical phenomena.

A. Statics

We first consider the static critical behavior. We define the reduced temperature

$$\varepsilon \equiv \frac{T - T_c}{T_c} \quad (5.1)$$

(to avoid confusion with times). As $T \rightarrow T_c^-$, the order parameter goes to zero as

$$q_{EA} \sim |\varepsilon|^\beta. \quad (5.2)$$

The spin-glass stiffness modulus will also go to zero as

$$\Upsilon \sim |\varepsilon|^\mu, \quad (5.3)$$

defining μ . A correlation length below T_c , ξ_- , can be defined as the length scale at which the mean-square correlation function $\overline{C^2(r)}$ crosses over from its critical power-law form at T_c ,

$$\overline{C^2(r)} \sim \frac{1}{r^{d-2+\eta}}, \quad (5.4)$$

to the noncritical power-law form below T_c , Eq. (3.8). This correlation length diverges as

$$\xi_- \sim |\varepsilon|^{-\nu'}. \quad (5.5)$$

We expect $\nu' = \nu$, where ν is the correlation length exponent defined from *above* T_c .

We could also define a correlation length from the typical decay of the correlation function

$$-\ln |C_{ij}| \sim (|i-j|/\xi_-)^\theta, \quad (5.6)$$

which should be valid for $\theta < 1$. However, if $\theta > 1$, then $\ln |C|$ will decrease linearly with distance²⁴ and not be

dominated by isotropic droplets. In this case it is possible that a different correlation length (larger than ξ_-) could enter in (5.6) due to dangerous irrelevant operators (see below).

The singular part of the specific heat will behave as

$$C_s \sim |\varepsilon|^{-\alpha}, \quad (5.7)$$

which should be nondivergent (see below). Simple scaling laws relate the various exponents. Hyperscaling yields

$$d\nu = 2 - \alpha, \quad (5.8)$$

which is a consequence of the singular part of the free energy in a correlation volume being of order T_c . Similarly, the stiffness free energy in a correlation volume $\Upsilon \xi_-^\theta$ should also be of order T_c , yielding

$$\mu = \theta\nu, \quad (5.9)$$

analogous to the Josephson^{31,32} and Widom³³ scaling laws for X - Y and Ising ferromagnets, respectively. A general inequality for disordered systems has been proven³⁴ which yields

$$d\nu \geq 2, \quad (5.10)$$

implying, from Eq. (5.8) that $\alpha \leq 0$.

Scaling with magnetic field yields the usual

$$\beta = \frac{1}{2}(d - 2 + \eta)\nu \quad (5.11)$$

and

$$2 - \alpha = 2\beta + \gamma, \quad (5.12)$$

where γ is the exponent of the diverging nonlinear susceptibility χ_3 for $T \rightarrow T_c^+$. The characteristic crossover field scales as

$$H^* \sim |\varepsilon|^{(\beta+\gamma)/2}, \quad (5.13)$$

consistent with the scaling relations. At $H^*(T)$ for $T < T_c$, the magnetic correlation length ξ_H from Eq. (3.24) is of the same magnitude as ξ_- . From Eq. (3.23) and the scaling laws above, we see that the mean-square correlation function for $r \gg \xi_-$ scales as $\xi_-^{\theta - (d-2+\eta)r - \theta}$ which is of the same order as the critical correlation function Eq. (5.4) for $r \sim \xi_-$, as expected.

B. Breakdown of scaling

For conventional systems, some of the scaling laws break down in high dimensions d due to the presence of a dangerous irrelevant operator. In particular, for Ising ferromagnets in $d > 4$,

$$d\nu \neq 2 - \alpha, \quad \mu \neq (d-1)\nu,$$

and

$$\beta \neq \nu(d-2+\eta)/2.$$

For spin glasses, the critical behavior *above* T_c will certainly be modified for $d > 6$ due to the presence of a dangerous irrelevant operator. For example, in this regime the behavior should be mean field so that $\alpha = -1$

and $\nu = \frac{1}{2}$ and hence $d\nu \neq 2 - \alpha$. The critical behavior below T_c is more problematical. Fisher and Sompolinsky³⁵ have shown that if the Parisi picture for the ordered phase is assumed to naively apply in high dimensions, then the presence of *two* dangerous irrelevant operators at the critical Gaussian fixed point will modify the mean-field exponents (such as the scaling of the de Almeida–Thouless line) for $8 > d > 6$ and the simple scaling will work only for $d < 6$. Unfortunately, since we have argued that the Parisi picture of the ordered-phase picture does not apply in any finite dimension,¹⁸ these results cannot be used. However, it is very likely that the dangerous irrelevant operators will in any case affect the critical behavior below T_c for $d > 6$ so that some of the scaling laws break down. In particular it should be possible to define several correlation lengths which diverge differently: The smallest of these is the true correlation length ξ_- whose exponent should still be equal to that of ξ_+ above T_c .

A better understanding of the high-dimensional limit is needed to find the regime of the validity of the scaling laws in spin glasses. The most likely result is that they all apply for $d_l < d < 6$.

C. Dynamic scaling

Conventional dynamic scaling predicts that the characteristic critical relaxation time τ_c will diverge near the critical point as

$$\tau_c \sim \xi_-^z \sim |\varepsilon|^{-z\nu}, \quad (5.14)$$

defining the dynamic exponent z . Frequency-dependent correlation functions then scale as functions of $\omega\tau_c$. Because the critical point in spin glasses appears to be relatively conventional, it is natural to expect that conventional dynamic scaling will apply. We thus first assume this is the case and then afterwards examine the possible failure of conventional dynamic scaling.

The ac nonlinear susceptibility in spin glasses is expected to scale for $T \rightarrow T_c$ as

$$\chi_3(3\omega; \omega) \sim |\varepsilon|^{-\gamma} X_3(\omega\xi_-^z). \quad (5.15)$$

Experimental support for this scaling form as $T \rightarrow T_c^+$ has been found recently by Levy and Ogielski.³⁶ For $T > T_c$, $X_3^+(y \rightarrow 0)$ is a constant while at T_c , χ_3 diverges with frequency as

$$\chi_3(3\omega; \omega) \sim \omega^{-\gamma/z\nu}. \quad (5.16)$$

As found in Sec. IV the contribution of large droplets makes the $\omega \rightarrow 0$ nonlinear susceptibility remain infinite below T_c . We thus expect that for $T < T_c$

$$X_3^-(y) \sim (\ln y)^{[d - (1+\phi)\theta]/\psi} \quad \text{for } \omega\tau_c \ll 1,$$

i.e., $y \rightarrow 0$. In this limit, we thus have

$$\chi_3(3\omega; \omega) \sim |\varepsilon|^{-\gamma} [\ln(\omega\tau_c)]^{[d - (1+\phi)\theta]/\psi}. \quad (5.17)$$

The prefactor of Eq. (5.17) has the same form as Eq. (4.21) provided that

$$\Delta \sim |\varepsilon|^{\psi\nu}. \quad (5.18)$$

This is, however, just what we expect: At a length scale of order ξ_- , the barriers should start developing so that $B_{\xi_-} \sim \Delta \xi_-^\psi \sim T_c$ yielding Eq. (5.18).

To obtain the critical behavior of the singular part of other dynamic correlation functions in the limit $\omega\tau_c \ll 1$ we can similarly use the critical behavior of the coefficients Υ , Δ , and q_{EA} and then replace the microscopic time τ_0 with the diverging correlation time τ_c in the arguments of the logarithms. Thus, for example, the average spin autocorrelation function should scale as

$$\bar{C}(t) \approx t^{-\beta/z\nu} \Gamma(t/\tau_c), \quad (5.19)$$

with

$$\Gamma^-(y) \sim y^{\beta/z\nu} / (\ln y)^{\theta/\psi} \quad (5.20)$$

for $T < T_c$ and $y \gg 1$ in order to match (4.13). The asymptotic behavior of $\Gamma^+(y)$ for $T > T_c$ is not known. Ogielski,³⁷ based on Monte Carlo simulations, has suggested it is of a stretched exponential or Kohlrausch form $\Gamma^+(y) \sim e^{-y^n}$ with $n \approx 1/3$. The asymptotic form of $\bar{C}(t)$ for $T > T_c$ is

$$\bar{C}(t) \sim \exp[-A (\ln t/t_0)^{d/(d-1)}], \quad (5.21)$$

due to rare unfrustrated regions,^{30,27} but it appears that the coefficient A and microscopic time t_0 remain finite and nonsingular for $T \rightarrow T_c^+$ so that this part of $\bar{C}(t)$ does not contribute to the scaling functions

$$\Gamma^\pm(y) \equiv \lim_{T \rightarrow T_c^\pm} [(y\tau_c)^{\beta/z\nu} \bar{C}(y\tau_c)]. \quad (5.22)$$

In the presence of a magnetic field, the relaxation time will not diverge except below T_c as $H \rightarrow 0$. However, for H small the relaxation time will diverge very rapidly as T is decreased. At $T = T_c$, the relaxation time τ_H will diverge as

$$\tau_H(T_c) \sim H^{-2z\nu/(\beta+\gamma)}, \quad (5.23)$$

and below T_c when $\xi_H(T) \sim \xi_-(T)$, τ_H will be of order τ_c . For H less than the crossover field $H^*(T)$, Eq. (5.13), $\xi_H \gg \xi_-$ and the dynamics will be activated. In this regime

$$\ln[\tau_H/(\tau_0 |\varepsilon|^{-z\nu})] \sim |\varepsilon|^{(\gamma+\beta)\psi/(d-2\theta)} H^{2\psi/(d-2\theta)}. \quad (5.24)$$

For a given experimental measuring time t_m , the system will not be able to equilibrate when $T < T_f(t_m, H)$, where the freezing temperature T_f is given by

$$\frac{T_c - T_f(t_m, H)}{T_c} \sim H^{2/(\gamma+\beta)} [\ln(t_m/\tau_0)]^{(d-2\theta)/(\gamma+\beta)\psi}, \quad (5.25)$$

where we have neglected $\ln \ln$ corrections. This freezing temperature is shown schematically in the phase diagram in Fig. 3. Thus, even though there is no true de Almeida–Thouless line, as discussed in Sec. III, the system will fall out of equilibrium at a weakly time-

dependent temperature whose shift from T_c scales with H in the same way as the de Almeida–Thouless line would have been expected to.

D. Nonconventional scaling

The scenario outlined above is the simplest and most conventional kind of dynamic critical scaling. The basic justification is that the critical point, in contrast to the ordered phase, results from a competition between exchange energy and entropy. Since the entropic contribution to the free energy is of order T_c , it is natural to expect that the singular parts of the free energy in a correlation volume—including the random parts—will be of order T_c , i.e., the transition is controlled by a finite-temperature fixed point. This yields the hyperscaling relation Eq. (5.8). In addition, it suggests that the scale of the free-energy barriers, to the extent that they exist, will also be of order T_c so that no appreciable activation is needed to surmount them. However, it is by no means clear that this simple scenario is correct.

For random-field Ising ferromagnetic critical points Villain³⁸ and Fisher³⁹ have argued that conventional dynamic scaling fails since in that system the characteristic scales of the free energy in a correlation volume and the barriers both grow with length scale leading to activated dynamic scaling in the critical region. This is because the dominant competition which causes the transition is between the exchange and random-field energies. In this case, the critical point itself, like the spin-glass ordered phase, is controlled by a zero-temperature fixed point.

In spin glasses, there is no compelling reason to expect this sort of behavior. However it is possible that, although the static behavior is controlled by a finite-temperature fixed point at which hyperscaling is satisfied, there might still be divergent barriers at the critical point whose magnitude grows with length scale as

$$B_L \sim T_c L^{\psi_c}. \quad (5.26)$$

In this case the critical dynamics would be activated and, assuming a broad distribution of barriers, the ac correlation functions would scale as functions of $(\ln \omega \tau_0)/\xi^{\psi_c}$. The cubic nonlinear susceptibility would then diverge at T_c as

$$\chi_3(3\omega; \omega) \sim (\ln \omega \tau_0)^{\gamma/\psi_c \nu}. \quad (5.27)$$

Unfortunately, this behavior is rather hard to distinguish from the power-law behavior in Eq. (5.16) with a large exponent z . Similarly, a particular measure of the relaxation time τ_c would diverge in the activated scenario as

$$\tau_c \sim \exp \left[\frac{b}{|\varepsilon|^{\psi_c \nu}} \right], \quad (5.28)$$

which again is hard to distinguish from the conventional Eq. (5.14) for large z . It is important to note, however, that if a different measure of the characteristic time is chosen, then activated scaling implies that a different value of b in (5.28) will generally be needed.

Several authors⁴⁰ have shown that data for $T \rightarrow T_c^+$

can be fitted to an activated form for the dynamics. However, a crucial test which, to our knowledge, has not been performed is to check, over as wide as possible a range of frequencies, whether the scaling is as a function of $(\ln\tau/\ln\tau_c)$ as in activated scaling or as a function of (τ/τ_c) as in the conventional scaling form. This test is rather more sensitive than attempts to fit a characteristic time with either Eq. (5.14) or (5.28); see, for example, the behavior found in Ref. 41. It is important to note that in addition to these two critical scaling scenarios, there are other possibilities.

(i) *Conventional with $\nu = \infty$.* In this case, which occurs at the 2D X - Y Kosterlitz-Thouless transition, the susceptibility $\chi \sim \exp(k/|\varepsilon|^{\bar{\nu}})$ and $\tau_c \sim e^{b/|\varepsilon|^{\bar{\nu}}}$. The scaling variable is $\omega\tau_c$. All measures of characteristic times τ_c will yield the *same* b in contrast to the activated dynamic scaling. This possibility appears unlikely in spin glasses except perhaps if T_c remains positive at the lower critical dimension.

(ii) *Conventional with $z = \infty$.* If there are divergent barriers *without* a broad distribution and ξ diverges as $|\varepsilon|^{-\nu}$ then

$$\tau_c \sim \exp(\xi^{\psi_c}) \sim \exp(|\varepsilon|^{-\psi_c\nu}).$$

This can probably occur for certain pure systems with zero-temperature critical points. As for (i), however, the scaling variable is still $\omega\tau_c$.

Both the special cases will look rather similar to the large- z limit of conventional scaling. In particular, the functional form of dynamic scaling functions will not be like those for activated scaling. A very large range of ω , preferably going up to close to microscopic frequencies, is needed to distinguish the various possible forms of critical dynamic scaling.

VI. POSITIVE TEMPERATURE AND COARSE GRAINING

In the preceding sections we have taken into account the effect of thermal fluctuations on large-scale droplet excitations by simply discussing droplet free energies instead of energies and renormalizing the various coefficients. Although this procedure works in conventional systems, it is by no means clear that it is a consistent procedure in spin glasses. In particular, the presence of low-energy excitations on all length scales will certainly complicate matters considerably. In this section we sketch a renormalization-group procedure to take into account thermal fluctuations in a consistent manner. In the course of the analysis, several subtle and novel features of the spin-glass phase will emerge.

In order to establish the general procedure and to draw the crucial contrasts, we first go back to an ordered Ising ferromagnet. The aim is to produce an effective Hamiltonian to describe the large-scale properties which includes the effects of small-scale thermal fluctuations and entropy. Since we are especially interested in droplet fluctuations, we focus on an interface between up and down regions which will be flat at zero temperature.

Thermal fluctuations in the local position of the interface and the effect of the interface on nearby droplet excitations will change the free energy of the system *with* the interface by a different amount than the free energy in the *absence* of the interface. Thus the free energy of the interface will be modified at positive temperatures by some amount per unit area, i.e., a renormalization of the interfacial tension. If a coarse-grained free energy is defined on a scale somewhat larger than the correlation length ξ_- , then the longer length fluctuations and droplets of size much larger than ξ_- will be determined by this renormalized interfacial tension and the intermediate-scale fluctuations will not significantly alter the large-scale properties. This will be true even for rough interfaces, such as in two dimensions, although in this case the long-length-scale fluctuations cause divergent fluctuations in the position of the interface. This is because there is very little free energy in the large-scale fluctuations. We conclude that on scales much larger than ξ_- , the properties of ordered ferromagnets are described in terms of the fluctuations about a renormalized zero-temperature state described by an effective Hamiltonian which is similar to the original H but with a renormalized magnetization and interfacial tension. The work of Abraham and others^{42,43,22,23} demonstrates this conclusively for the two-dimensional (2D) Ising model. We have argued elsewhere²⁷ that the *dynamics* of ordered ferromagnets can also be analyzed in terms of renormalized droplet excitations.

The natural guess for spin glasses, made in Sec. II, is that the large-scale properties of a spin glass at temperatures $0 < T < T_c$ can be described as droplet fluctuations about a "ground state" of an effective renormalized Hamiltonian which is similar to the zero-temperature Hamiltonian but with renormalized q_{EA} , Υ , etc. Since the fixed-point Hamiltonians are intrinsically random there is, however, no reason to believe that, for a specific realization of the microscopic Hamiltonian, the renormalized Hamiltonian at one temperature is simply related to that at another temperature. Indeed, as we will see, it will in general be quite different in some important respects.

We start by estimating the contribution of smaller-scale fluctuations to the free energy of a putative droplet of scale L . Since $\theta < 0$ implies that the large-scale droplets will all be active and destroy the ordered phase and the droplet picture, we restrict consideration to $\theta > 0$.

We need to consider the difference S_L between the entropy in the vicinity of the droplet wall with and without a particular droplet excited. Note that since $\theta < d - 1$, locally there is no average energy associated with the wall of a large droplet. The entropy of the droplet arises from deformations of the wall and small-scale droplets near the wall whose properties depend on the presence or absence of the large-scale droplet. Indeed, deformations of the large droplet are just such small-scale droplets. Due to the absence of a wall energy per unit area, small sections of the wall of size $l \ll L$ are almost statistically similar to the ground state. We thus expect that the local entropy differences of the small sections will be random in sign with variance much larger than their mean. The local-entropy differences of small sections of the wall which are

far apart are only weakly correlated so that the total-entropy difference S_L will have contributions from small scales l which is a sum of $(L/l)^{d_s}$ random terms. Therefore the deformations at scales of order one will contribute to S_L an amount of order $L^{d_s/2}$ with a random sign (as well as an amount $\sim L^\theta$ which is more likely to be positive). Since, for large L , $L^{d_s/2}$ is larger than the free energy L^θ , it appears that there is a problem with defining large droplets for $T > 0$. However, as we shall see, this can be resolved by appropriate renormalization of the effective exchanges.

For simplicity, we consider $T \ll J$, where all the subtleties already occur and then argue by continuation that the results can be extended to all $T < T_c$. We first consider droplets of scale $L=2$ about a ground state Γ . A fraction of order T of these droplets will have contributions to their entropy of random sign from deformations of scale 1 or droplets of scale 1 whose walls overlap the walls of the scale-2 droplets. These will alter the scale-2 droplet free energies $F_2(j)$ by an amount of order T (by both entropic and energetic contributions). In some small fraction of droplets this will change the sign of the droplet free energy causing a change in the renormalized "ground state," to Γ_2 and hence in the sign of the effective couplings on scale 2 which determine the renormalized ground state and the droplet free energies. We now go to scale 4 and consider the effects of the renormalized scale-2 deformations and droplets on scale-4 droplets about the renormalized ground state Γ_2 . These will be active with probability $T/\Upsilon_2 2^\theta$, where $\Upsilon_2(T)$ is the renormalized stiffness at scale 2. The active scale-2 droplets will contribute to the $F_4(j)$ of droplets they affect an amount of order T . This again changes the effective ground state to Γ_4 and changes some (but a smaller fraction than before) of the effective couplings.

We continue this procedure to larger and larger scales defining a renormalized "ground state" Γ_L , a renormalized stiffness Υ_L , and renormalized droplet free energies at each scale. (Note that it is by no means clear how this can be done in practice; the arguments given here are purely schematic.) All the features at scale L will be given by a renormalized Hamiltonian which will in general be more complicated than the original bare Hamiltonian. In going from scale L to scale $2L$, the fraction of scale $2L$ droplets which have active sections of scale L is $T/\Upsilon_L L^\theta$ which is very small for large L . Thus we see that with probability which approaches unity, we do not have to consider the effects of entropy on scales close to the size of a particular large droplet of interest. This implies that on some scale $\lambda_L \ll L$, the large-scale L droplets will usually consist of a well-defined flipped cluster which optimizes the free energy coarse grained to scale λ_L .

The scale λ_L can be determined by considering the contribution due to scales $\geq \lambda_L$ to the entropy $S_L(j)$ of a large droplet. This will be dominated by the scale λ_L itself. The entropy can easily be estimated as the sum of the entropy of the active pieces which will be a fraction $T/\Upsilon_L L^\theta$ of $(L/\lambda_L)^{d_s}$ total sections of size λ_L . For scales greater than

$$\lambda_L \sim \left(\frac{T}{\Upsilon} \right)^{1/(d_s+\theta)} L^{d_s/(d_s+\theta)} \quad (6.1)$$

there will typically be no active sections of the large droplet. The droplet free energy $F_L(j)$ will then have no contributions of order T from these scales. An accuracy of this order is what is needed in order to determine which droplets are active; for greater accuracy a scale greater than λ_L is needed.

The free energies of typical droplets of scale L are of order ΥL^θ . In order to get a rough estimate of a given droplet's free energy, the total free energy of the neglected smaller-scale deformations must be less than this. If we coarse grain as discussed to scale λ , the free energies of the remaining active deformations of larger scales will be each of order T and add randomly to the total free energy of the droplet. In order to have the total free energy of the remaining active modes typically less than ΥL^θ we need

$$T \left[\frac{T}{\Upsilon \lambda^\theta} \left(\frac{L}{\lambda} \right)^{d_s} \right]^{1/2} \ll \Upsilon L^\theta, \quad (6.2)$$

or

$$\lambda \gg \tilde{\lambda}_L \sim \left(\frac{T}{\Upsilon} \right)^{3/(d_s+\theta)} L^{(d_s-2\theta)/(d_s+\theta)}. \quad (6.3)$$

Thus even for a rough estimate of the droplet's free energy the required length scale $\tilde{\lambda}_L$ still diverges as a power of L since $2\theta < d_s$.

The behavior we find is in striking contrast to conventional pure systems. In order to get even a rough estimate of the droplet free energies at scale L , we need to renormalize to a scale $\tilde{\lambda}_L$ which grows as a power of L , while in the pure systems just renormalizing to the fixed scale ξ_- suffices.

VII. SENSITIVITY TO TEMPERATURE CHANGES

If we go back and examine the renormalization procedure carefully, it becomes evident that the large-scale droplets are local minima of an effective Hamiltonian which is very different from the original Hamiltonian. Indeed, the effective ground state Γ_L about which the droplets are excitations is itself quite different from the ground state of the original Hamiltonian. (Note that the infinite-scale effective ground state Γ_∞ will have each spin S_j pointing in the direction of $\langle S_j \rangle_T$.) The large-scale active droplets will thus have different positions at temperature T from zero temperature. It follows that they will also differ from any one temperature to another. This is due to the mechanism which we noted earlier. Although the free energy of large-scale droplets is obtained by optimizing an effective Hamiltonian, neither the energy nor the entropy are optimized separately and both will hence be of order $L^{d_s/2}$. If we take equilibrium states of the *same* microscopic Hamiltonian at different nearby temperatures T_1 and T_2 , then these states will differ (by other than a global spin flip) at length scales larger than the scale $L_{\Delta T}$ at which droplet orientations at T_1

significantly differ from those at T_2 . At this length scale, a finite fraction of regions will flip in going at equilibrium from T_1 to T_2 , causing large changes in the state. For this discussion we assume the result argued for in Ref. 18, i.e., that only two pure states exist at a *fixed* temperature.

The scale $L_{\Delta T}$ can be easily estimated by noting that the derivative of $F_L(j)$ with respect to T is just $S_L(j)$ so that, for small ΔT

$$F_L(j, T_2) \sim F_L(j, T_1) - \Delta T S_L(j, T_1), \quad (7.1)$$

where

$$\Delta T = T_2 - T_1. \quad (7.2)$$

Outside the critical region, i.e., for $T \lesssim T_c/2$, the entropy is dominated by deformations at scale unity. A fraction $T/J \sim T/\Upsilon$ of these will be active so that the sum over the random terms gives

$$S_L(j, T) \sim \left[\frac{T}{\Upsilon} L^{d_s} \right]^{1/2}. \quad (7.3)$$

Note that as mentioned above there will also be a nonrandom component which scales as L^θ and causes the temperature dependence of Υ . We see that the second term on the right-hand side of Eq. (7.1) is the same magnitude as the first when $L \sim L_{\Delta T}$ with

$$L_{\Delta T} \sim \left[\frac{\Upsilon^{3/2}}{T^{1/2} |\Delta T|} \right]^{2/(d_s - 2\theta)}. \quad (7.4)$$

This length scale will determine the decay of the average *overlap correlation function* which we define by

$$\Xi(i, j, \Delta T) \equiv \overline{\langle S_i S_j \rangle_T \langle S_i S_j \rangle_{T+\Delta T}}. \quad (7.5)$$

This overlap correlation function is invariant under a global spin flip at each temperature, so it does not distinguish between the two states at a fixed temperature. For conventional ferromagnets,

$$\Xi \rightarrow m^2(T) m^2(T + \Delta T)$$

as $|r_{ij}| \rightarrow \infty$, where $m(T)$ is the spontaneous magnetization density.

For "toy" spin glasses (such as the Mattis⁴⁴ model) in which the spin correlations below T_c have the same sign as in the ground state, Ξ would approach $q_{\text{EA}}(T) q_{\text{EA}}(T + \Delta T)$ for large $|r_{ij}|$. However, because of the sensitivity to temperature discussed above, in real spin glasses it will in fact decay to zero for any nonzero ΔT . A natural guess is that for large distances

$$\Xi(i, j, \Delta T) \sim \exp[-(|r_{ij}|/L_{\Delta T})^\sigma], \quad (7.6)$$

with some exponent σ . In the companion paper,⁹ we show that this dramatic sensitivity to temperature changes has important experimental consequences.

In the critical region near T_c , the estimate Eq. (7.3) for the entropy must be modified due to critical fluctuations which dominate at scales up to the correlation length ξ_- . On the scale ξ_- , the singular part of the entropy will be of order $1/|\varepsilon|$ and, because the spin-glass critical point is intrinsically random, the spatial variations and the

dependence on proximity to a droplet wall will be of the same order. Thus we expect

$$S_L \sim \frac{1}{|\varepsilon|} \left[\frac{L}{\xi_-} \right]^{d_s/2}, \quad (7.7)$$

so that for $T \leq T_c$

$$L_{\Delta T} \sim \left[\frac{\Delta T}{\Upsilon |\varepsilon| \xi_-^{d_s/2}} \right]^{-2/(d_s - 2\theta)} \\ \sim \xi_- \left[\frac{\Delta T}{|\varepsilon| T_c} \right]^{-2/(d_s - 2\theta)} \quad (7.8)$$

for $(\Delta T/T_c) < |\varepsilon|$. As might be expected on general scaling grounds, for $(\Delta T/T_c) \sim |\varepsilon|$, $L_{\Delta T} \sim \xi_-$.

We can invert Eqs. (7.4) and (7.8) and, for a given pair of spins a distance r apart, ask how often will the sign of their correlations change as the temperature is lowered? The flips will start when $\xi_- \sim r$ and initially occur with a spacing of order $\Delta T_r \sim T_c r^{-1/\nu}$. At low temperatures, $T \lesssim \frac{1}{2} T_c$,

$$\Delta T_r \sim \frac{\Upsilon^{3/2}}{T^{1/2} r^{(d_s/2) - \theta}}. \quad (7.9)$$

The total number of flips will be of order $r^{(d_s/2) - \theta}$.

We have found that the sensitivity of the state of the system to temperature, which is caused by the inequality $\theta < d_s/2$, implies that it is *not* possible to simply construct a low-temperature state out of a ground state and its excitations. Thus this procedure, which Bovier and Frohlich¹⁶ (BF) suggest, cannot work at long length scales. However, as we have seen, this does not imply that low-temperature states which are statistically similar to ground states and excitations cannot be constructed. Indeed we have argued that they can, provided $\theta > 0$.

BF (Ref. 16) defined an exponent which should be equal to θ provided it is positive. They argue that entropic contributions to the wall free energy will scale as L^{d-1} (which would be valid if $\theta = d - 1$), and therefore that $\theta = d - 1$ is necessary for stability of the ordered state. We believe that this argument, which is the root of the disagreements between BF (Ref. 16) and ourselves¹⁸ about the existence of many states in high dimensions, is not valid. What is essential is that since $\theta < d - 1$, the entropy no longer scales as L^{d-1} . Furthermore, even if the entropy *always* dominates as we have argued, an ordered state should still exist provided $\theta > 0$.

A. Energy fluctuations

One of the consequences of the extreme sensitivity to temperature is that the usual relationships between fluctuations and thermodynamic derivatives can break down, as we have seen already for the nonlinear susceptibility χ_3 . This is because, in a certain sense, the spin-glass ordered phase as a function of temperature is the limit of an infinite sequence of infinitesimal first-order transitions. As such, it is hardly surprising that thermodynamic derivatives do not commute straightforwardly with the infinite volume limit.

Let us consider moments of the energy fluctuations in a volume V . These will be dominated by the active droplets. Thus for considering the effects of the large-scale droplets we can assume they are independent. From the above discussion, we expect the energy $E_L(j)$ of a droplet of scale L to have a random component of order $L^{d_s/2}$. Thus the second moment of the energy fluctuations will

$$\frac{1}{V} \overline{\langle (E - \langle E \rangle)^4 \rangle} - 3 \overline{\langle (E - \langle E \rangle)^2 \rangle^2} \sim \sum_L \frac{L^{2d_s}}{L^d} \int [4 \operatorname{sech}^2(\frac{1}{2}\beta F_L) - 6 \operatorname{sech}^4(\frac{1}{2}\beta F_L)] \rho(F_L) dF_L. \quad (7.11)$$

Because of cancellations similar to those in the nonlinear susceptibility (see Sec. III), for large L the leading term in Eq. (7.11) cancels but, assuming scaling, there is a correction of the form $\sum_L L^{2d_s - d - (1+\phi)\theta}$ which does not vanish. Thus the fourth cumulant of the energy will be infinite provided

$$2d_s - d - (1 + \phi)\theta > 0, \quad (7.12)$$

which is likely to be the case, at least in 3D. By contrast ($\partial^4 F / \partial \beta^4$), which is usually related to the fourth energy cumulant, will be finite.

VIII. AN EXPANSION ABOUT THE LOWER CRITICAL DIMENSION

There should be some lower critical dimension d_l where $\theta(d)$ goes through zero; here we briefly analyze the limit $d \rightarrow d_l^+$, where θ is small and positive. The behavior for $d < d_l$ and the crossover from $d > d_l$ behavior to $d < d_l$ behavior in finite-size samples (films) is discussed in a separate publication.⁴⁵

In the spirit of spin-wave expansions about the lower critical dimension of Heisenberg ferromagnets,⁴⁶ we attempt an expansion of the critical behavior about d_l . We can choose to renormalize so that as a function of length scale the characteristic free energy remains fixed: The renormalized temperature must then change. From the definition of θ , it is clear that the leading term for low T is just $dT/d(\ln L) \approx -\theta T$; however, we need the form of the first correction.

We are thus interested in the renormalization of, for example, the average droplet free energy \bar{F}_L on scale L due to smaller-scale thermal fluctuations. The fluctuations with scale $l \ll L$ are weakly correlated with F_L so do not renormalize it directly. On the other hand, fluctuations on scales of order L , say $L/2$, will be correlated with F_L . With probability of order T/L^θ , there will be active fluctuations of scale- L droplets at scale $\sim L/2$. These will contribute to F_L amounts of order T . Thus the mean \bar{F}_L can be renormalized by amounts of order $T^2/\bar{F}_{L/2}$. The sign of this systematic effect is not obvious, however it is natural to guess, as discussed below, that it will decrease F_L on average. We thus can write

$$\bar{F}_L = \bar{F}_{L/2} 2^\theta - \frac{cT^2}{\bar{F}_{L/2}} + \dots \quad (8.1)$$

be given by the sum over scales of L^{d_s} times the number of active droplets of scale L , so that

$$\frac{1}{V} \langle (E - \langle E \rangle)^2 \rangle \sim \sum_L TL^{d_s - d - \theta}, \quad (7.10)$$

which is finite and equal to $T^2 C_V$. The fourth moment of E is

(Note that the effect of the fluctuations on scale $l \ll L$ will be to renormalize F_{2l} hence F_{4l} , etc.) Defining the renormalized temperature by

$$T(L) = \frac{T}{\bar{F}_L}, \quad (8.2)$$

we have

$$\frac{dT}{d(\ln L)} = -\theta T + cT^3 + \dots \quad (8.3)$$

(Note that this is the same as found by McMillan⁸ in the Migdal-Kadanoff approximation.)

From Eq. (8.3) we see that for θ small and positive, i.e., $d \gtrsim d_l$, there is an unstable fixed point at $T^* \sim \sqrt{\theta}$ which is natural to identify as the critical fixed point separating the spin glass from the paramagnetic phase. The eigenvalue at the critical fixed point is $1/\nu = 2\theta$, so that we have for $d \rightarrow d_l^+$,

$$\nu\theta \rightarrow \frac{1}{2}. \quad (8.4)$$

If we make the natural assumption that $\theta(d)$ goes through zero linearly, then $T_c(d) \sim \sqrt{d - d_l}$. If the coefficient of the T^3 term in Eq. (8.3) were negative, this would imply a stable fixed point at a low positive temperature for $d < d_l$ which would imply a quasicrystalline spin-glass phase in this regime; this seems rather unlikely and suggests that the coefficient of T^3 is indeed positive. Indeed, one expects that due to the excess free-energy density near the droplet, active excitations which affect the droplet free energy are more likely to occur when the droplet is present thus yielding a decrease in \bar{F}_L as in Eq. (8.1). Of course, there remains the possibility that the coefficient of T^3 vanishes and something less generic happens, but since we are unaware of any reason to expect this, we will not examine this possibility here.

In order to obtain the renormalization of a uniform magnetic field, let us consider the effect of smaller-scale fluctuations on the total magnetic moment \bar{M}_L of a droplet of size L . At zero temperature, the typical magnetic moment of a droplet is simply proportional to the square root of its volume. For $T > 0$ the probability of a droplet of size $L/2$ being thermally active and thus having a reduced moment when its fluctuations are integrated over is of order $T/L^\theta \sim T(L)$. Droplets at size L consist of 2^d droplets of size $L/2$, with moments added together with

random signs and reduced by thermal fluctuations. We can thus write

$$\tilde{M}_L = \tilde{M}_{L/2} [2^{d/2} - cT(L) + \dots], \quad (8.5)$$

to lowest order in temperature. Defining the renormalized field as

$$h(L) = H\tilde{M}_L / \bar{F}_L, \quad (8.6)$$

we have

$$\frac{dh}{d(\ln L)} = h \left[\frac{d}{2} - \theta - cT + \dots \right]. \quad (8.7)$$

Note that here the leading correction is at relative order T , unlike (8.3) where the leading correction is at relative order T^2 .

Now the critical fixed point is at $h^* = 0$, $T^* \sim \sqrt{\theta}$, at which the renormalization-group eigenvalue of field is

$$y_h = \frac{d}{2} - c\sqrt{\theta} + O(\theta). \quad (8.8)$$

This eigenvalue is related to the critical exponent η via

$$4y_h = d + 2 - \eta, \quad (8.9)$$

so we have

$$\beta = \frac{1}{2}(d - 2 + \eta)\nu \sim \theta^{-1/2}, \quad (8.10)$$

a divergence of the order parameter exponent for $d \rightarrow d_1^+$. This is consistent with the low-temperature behavior of the order parameter, which from the discussion preceding (3.4), varies as

$$1 - q_{EA} \sim T \int_{L_0}^{\infty} \frac{dL}{L^{1+\theta}} \sim \frac{T}{\theta} \quad (8.11)$$

for $\theta \rightarrow 0^+$. Thus q_{EA} drops rapidly at lowest order in T , a naive linear extrapolation suggesting it vanishes at a temperature of order θ . However, the order parameter exponent β is very large and q_{EA} actually does not vanish until the critical temperature, which is of order $\sqrt{\theta}$. Qualitatively, we expect

$$q_{EA} \sim \left[\frac{T_c - T}{T_c} \right]^\beta, \quad (8.12)$$

with $T_c \sim \sqrt{\theta}$ and $\beta \sim \theta^{-1/2}$, which is perfectly consistent with (8.11).

IX. LONG-RANGED INTERACTIONS

So far, we have discussed only Ising spin glasses with nearest-neighbor interactions. However, many experimental systems have further neighbor interactions or in some cases long-range power-law interactions with

$$\bar{J}_{ij}^2 \sim \frac{1}{r_{ij}^{2\sigma}}. \quad (9.1)$$

In this section we discuss the extension of some of our results to systems with such power-law interactions. In order for the total energy per spin to be finite, we require

$$\sigma > \frac{d}{2}. \quad (9.2)$$

For long-range interactions it is not appropriate to use the definition of droplets in terms of connected clusters as was done for the short-range case in Sec. II. Various alternative definitions of droplets at scale L are possible. Here we choose to modify the definition preceding Eq. (2.2) to allow disconnected clusters of flipped spins, but to forbid spins at distances larger than, say, $2L$ from the central site j . The restriction on the total number of flipped spins $L^d \leq N < (2L)^d$ is kept. Droplets which are far apart compared to their scales will be approximately independent, but now interact weakly via the long-ranged interactions. These interactions induce certain correlations and various subtle effects analogous to those studied in the "Coulomb glass,"^{47,48} which we will not delve into here. For the purposes of this paper, we will confine ourselves to the question as to when long-range interactions are relevant and, if so, what is the resulting θ_σ which describes the growth of the characteristic free energy with length scale. This question has been addressed from a somewhat different perspective by Bray, Moore, and Young.⁴⁹

We first consider the relevance of added long-range interactions to a system with short-range interactions and stiffness exponent θ_s . To do this, we simply consider the contribution of an additional long-range part of the interactions to the energy of a droplet D of scale L . The change in energy due to the added long-range interactions J'_{ij} is

$$\Delta F_D = 2 \sum_{\substack{i,j \\ i \in D \\ j \notin D}} J'_{ij} \langle S_i S_j \rangle, \quad (9.3)$$

so that

$$(\Delta F_D)^2 \sim \sum_{\substack{i,j \\ i \in D \\ j \notin D}} \frac{1}{r_{ij}^{2\sigma}} \sim L^{2(d-\sigma)}, \quad (9.4)$$

for $\sigma < d$. The absence of cross terms in Eq. (9.4) is because the additional random exchanges are uncorrelated with the original ground-state configuration. We expect the added long-range interactions to be irrelevant when $\Delta F_D \ll L^{\theta_s}$, i.e., for

$$\sigma > \sigma_c(d) = d - \theta_s(d). \quad (9.5)$$

(This is analogous to the result for critical points of pure systems⁵⁰ for which added long-range interactions are irrelevant for $\sigma > d + 2 - \eta$.) It might be expected that the result Eq. (9.5) is too naive, for the effect of the short-range parts of J'_{ij} will modify the ground state and droplet configuration and thus invalidate the simplest estimate based on assuming independence. However, since the long-range parts of J'_{ij} are independent of the short-range parts, this effect can be taken into account iteratively by including first the new interactions of scale l , finding a new ground state and droplet excitations, then repeating successively at scales $l_m = 2^m$ until scale L is reached.

The contribution from the scale L will still be of order $L^{d-\sigma}$ which, if $\sigma > \sigma_c$, will be small compared to L^θ , implying that, indeed, the long-range part of J'_{ij} is irrelevant. We note that this irrelevance at the zero-temperature ordered fixed point does not necessarily imply that all the finite-temperature properties are unaffected. This is because, as noted in Sec. III, the finite-temperature behavior for $T < T_c$ is controlled by the dangerous irrelevant temperature, so that interplay between irrelevant long-range interactions and temperature can affect the positive temperature behavior. We will not investigate this possibility further here.

A. Relevant long-range interactions

We now consider the case $\sigma < \sigma_c$. In this case, θ will increase due to the long-range interactions. To find θ_σ we use a similar argument to that of Weinrib and Halperin⁵¹ for long-range correlated disorder in ferromagnets. Specifically, we consider a small interaction $J'_{ij} \sim r_{ij}^{-\sigma'}$ added to the already present long-range interaction $J_{ij} \sim r_{ij}^{-\sigma}$, where J_{ij} and J'_{ij} are independent. The change in energy of a droplet at zero temperature can again be estimated as above, yielding

$$\Delta F_L \sim L^{d-\sigma'} . \quad (9.6)$$

If $\sigma' > \sigma$, we expect that the addition of the more rapidly decaying long-range interaction J'_{ij} should be irrelevant, implying $d - \sigma' < \theta_\sigma$. Conversely, a longer-ranged interaction should be relevant, so that for $\sigma' < \sigma$, $d - \sigma' > \theta_\sigma$. These inequalities imply that

$$\theta_\sigma = d - \sigma \quad (9.7)$$

for all

$$\sigma_c > \sigma > \frac{d}{2} . \quad (9.8)$$

Some care must be taken in deriving the estimate Eq. (9.6), since the droplet excitations will not be connected in this case. However, as discussed above we can consider just the part of J'_{ij} with say, $r_{ij} > L$. Each spin S_i in the droplet interacts with of order L^d spins $\{S_j\}$ outside of the droplet which are at distances between L and $2L$ from site i , this is true even if the spin S_i is well separated from the rest of the droplet. Therefore, since the long-distance part of the sum over j in Eq. (9.4) is convergent for $\sigma > d/2$, the estimate Eq. (9.6) is correct.

We note that in one dimension, the critical value of σ below which a phase transition is possible is given by $\theta_\sigma = 1 - \sigma = 0$, which agrees with rigorous results⁵² on the absence of a transition for $\sigma > 1$.

Since $\sigma > d/2$, we have for long-range interactions a slightly weaker inequality for θ_σ than Eq. (2.4) for θ_s :

$$\theta_\sigma < \frac{d}{2} . \quad (9.9)$$

Indeed, the result Eq. (9.5) for θ_c , provides additional support for such an inequality for θ_s , since, if $\theta_s > d/2$ then all long-range interactions would be irrelevant up to the limit $\sigma = d/2$ at which the interaction becomes

effectively infinite ranged. This would be greatly peculiar.

Qualitatively, we expect much of the behavior for long-range interactions with $\theta_\sigma > 0$ to be similar to the short-range case: low-lying droplet excitations, barriers, etc. However, as mentioned above, the interactions between the droplets can give rise to quantitative differences. For example, because of the restrictive definition of droplets given above, the lowest-energy excitations in a given region may involve collections of droplets rather than individual droplets. In addition, it is by no means clear what quantity plays the role of d_s , or, for example, how the energy and entropy of a droplet for $T > 0$ scale with L . We leave these questions for future investigations.

X. X-Y AND HEISENBERG SPIN GLASSES

In this section we discuss possible extensions of some of the results to spin glasses with continuous spin symmetry, in particular to X-Y and Heisenberg systems.

For Ising systems, our basic ansatz is that all appropriate measures of the stiffness free energy of the ordered phase at scale L scale as L^θ . For example, one can define the stiffness as the typical free energy of droplet excitations as in Sec. II. Alternatively, following McMillan^{8,12} and others,⁵⁻⁷ we may define the stiffness $\Sigma(L)$ as the difference in free energy between a system of size L^d with periodic boundary conditions and the same system with antiperiodic boundary conditions in one of the directions. This forces in a single domain wall across the system. However, since antiperiodic boundary conditions are just another realization of the same ensemble (change the sign of the J_{ij} 's on the boundary) $\Sigma(L)$ can be either positive or negative with equal probability and has average zero. The stiffness can be defined as the typical magnitude of $\Sigma(L)$ or (provided long tails are not important, which appears to be the case), as the root-mean-square value of $\Sigma(L)$ for different realizations of the $\{J_{ij}\}$:

$$\{[\Sigma_-(L)]^2\}^{1/2} \sim \Upsilon L^\theta . \quad (10.1)$$

For X-Y and the Heisenberg systems we would like to define the stiffness in terms of altered boundary conditions, as in (10.1). The simplest choice is just the same as for the Ising case. In a cube with periodic boundary conditions, change the sign of the J_{ij} 's along one ($d-1$)-dimensional surface across the cube, and measure the free-energy change $\Sigma_-(L)$. For an X-Y system, this forces in a proper rotation of π across the cube, while for Heisenberg systems, it forces an improper rotation. In general, it might be possible that proper and improper rotations will have different stiffnesses associated with them, and one would like to be able to probe each of them, defining proper and improper stiffness exponents θ_p and θ_i . For the X-Y system

$$\Sigma_-(L) \sim L^{\theta_p} , \quad (10.2)$$

while for the Heisenberg system

$$\Sigma_-(L) \sim L^{\theta_i} . \quad (10.3)$$

It is not so obvious how to define the other exponents. One possibility is to couple the spins across the boundary by a coupling $J_{ij} \mathbf{S}_{ij} \cdot \underline{\mathbf{R}} \cdot \mathbf{S}_j$ with $\underline{\mathbf{R}}$ a rotation matrix. When $\underline{\mathbf{R}}$ is the inversion matrix, this just yields Σ_- as above. For

$$\underline{\mathbf{R}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the X - Y case or

$$\underline{\mathbf{R}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

in the Heisenberg case, different kinds of stiffness will be probed. Unfortunately, the new Hamiltonians with altered boundary conditions are not realizations of the same random ensemble (although this cannot be detected locally since the rotation can equivalently be performed on any surface) so that the average of the free-energy difference $\Sigma_{\underline{\mathbf{R}}}(L)$ will not in general be zero. (This can be readily seen in the one-dimensional case, when $\underline{\mathbf{R}}$ is a rotation of angle $< \pi$.) However, we still expect that $\Sigma_{\underline{\mathbf{R}}}(L)$ will typically scale with L in the same way as Σ_- if $\underline{\mathbf{R}}$ has the same signature as the inversion matrix. (It is plausible that $\overline{\Sigma_{\underline{\mathbf{R}}}} \ll [(\overline{\Sigma_{\underline{\mathbf{R}}}}^2)]^{1/2}$ for large L so that the system with the rotated sheet is essentially equivalent to a member of the same random ensemble for large L .)

We now make the natural conjecture that if $\underline{\mathbf{R}}$ is a proper rotation,

$$\Sigma_{\underline{\mathbf{R}}}(L) \sim g_p(\underline{\mathbf{R}}) L^{\theta_p}, \quad (10.4)$$

and if $\underline{\mathbf{R}}$ is improper

$$\Sigma_{\underline{\mathbf{R}}}(L) \sim g_i(\underline{\mathbf{R}}) L^{\theta_i}, \quad (10.5)$$

where g_p and g_i depend on the magnitude of the rotation. We expect that $\theta_i \geq \theta_p$, since improper rotations of the boundary conditions can also have proper components and will generally couple to proper rotations as well. The simplest possibility is that $\theta_i = \theta_p$.

Certain frustrated magnets with no randomness have ground states in which spins are noncollinear and exhibit different stiffnesses for proper and improper rotations⁵³; the proper rotations cause a slowly varying spin-wave-like excitation while improper rotations of the boundary conditions cause a domain wall with much higher free-energy cost. In general, we expect stiffness exponents to be no larger than the corresponding Ising exponent, so that

$$\theta_i, \theta_p \leq \frac{d-1}{2}, \quad (10.6)$$

although a stronger bound may obtain.

We note that if $\theta_i > \theta_p$, then $g_i(\underline{\mathbf{R}})$ in Eq. (10.5) should be independent of $\underline{\mathbf{R}}$ for all improper rotations, since different $\underline{\mathbf{R}}$ will only differ by the smaller proper stiffness.

A. Phase diagram

We now consider the phase diagram in various cases.

(i) $\theta_p \leq \theta_i < 0$. In this case, as for the Ising system, there will be no long-range order at positive temperature.

(ii) $\theta_i \geq \theta_p > 0$. In this case the rotational symmetries will be broken in a positive-temperature ordered phase with

$$\langle \mathbf{S}_i \rangle \neq 0. \quad (10.7)$$

The simplest possibility is that both the proper and improper rotational symmetries break at the same transition temperature, although it might be possible that the improper rotational symmetry breaks first so that there is an intermediate phase like that described below. The phase with both broken symmetries is unstable to the addition of a magnetic field in the x direction since $\theta_i, \theta_p < d/2$. The magnetic field will hence destroy the broken improper rotation symmetry of $S_x \rightarrow -S_x$. However, the rotational symmetry about the field can still be broken so that a phase transition breaking the transverse symmetries will still occur. This occurs also in the infinite-range models; the transition line $T_c(H)$ is called a Gabay-Toulouse line.

By analogy with the existence of only two pure states in the Ising spin glass, we expect that the only pure states in X - Y or Heisenberg systems will be simply related by global proper or improper spin rotations. As for the Ising case, addition of a random field with components in all directions will destroy the transition entirely.

(iii) $\theta_i > 0 > \theta_p$. This rather interesting possibility implies that proper rotational symmetry will not be broken at any positive temperature but that improper rotational symmetry will. The spin expectations

$$\langle \mathbf{S}_i \rangle = 0, \quad (10.8)$$

so that $q_{EA} = 0$. However, if we form scalar products of nearby spins these can attain expectation values. In the X - Y case we would have

$$\langle \mathbf{S}_i \times \mathbf{S}_j \rangle \neq 0 \quad (10.9)$$

for nearby spins so that the global reflection symmetries, e.g., $S^y \rightarrow -S^y$, are broken. (We note that such an X - Y state occurs in some 2D frustrated but nonrandom models.⁵³) In the Heisenberg case,

$$\langle \mathbf{S}_i \cdot (\mathbf{S}_j \times \mathbf{S}_k) \rangle \neq 0 \quad (10.10)$$

for i, j, k near each other.

As a transition to such phases is approached from above, the normal Edwards-Anderson susceptibility will not diverge. However it is straightforward to show that the high-order nonlinear susceptibilities ($\partial^7 M / \partial h^7$), and ($\partial^{11} M / \partial h^{11}$) will diverge for the X - Y and Heisenberg cases, respectively. A phase with stiffness only to improper rotations also could occur as an intermediate phase between a paramagnetic phase and a true spin-glass phase which has both θ_i and θ_p positive.

In the presence of a uniform magnetic field, it should still be possible to break the improper rotation symmetry about the field and an intermediate type-(iii) phase can

still occur. Investigation of the interesting properties of such a pseudo-spin-glass phase are left for future work.

B. Properties of the ordered phase

We now assume that both θ_i and θ_p are positive, and discuss some of the properties of the resulting ordered phase. One would like to be able to describe the ordered phase in terms of large-scale excitations analogous to the picture developed in earlier sections for Ising systems. Because of the continuous degrees of freedom, however, things are rather more complicated, and we will restrict ourselves here to a few general remarks.

In Heisenberg and X - Y spin glasses, there will be two kinds of excitations: *smooth* excitations analogous to spin waves in which the spins are rotated continuously from the ground state without crossing any free-energy barriers, and *activated* excitations analogous to droplets in the Ising case which will involve overcoming barriers to their formation. Henley⁵⁴ has studied various types of excitations in the Heisenberg case and finds activated excitations of various kinds, including (1) improperly rotated droplets which consist of a large region which is improperly rotated from the ground state with a relatively sharp wall separating it from the exterior and with some relaxation by proper rotations; and (2) dislocations or other textures involving purely proper rotations. One expects, by analogy with the Ising case, that excitations of these types will occur down to arbitrarily low energies even on long length scales. As for the Ising case, these will give rise to logarithmic decay of the temporal correlations due to the barriers. By contrast, the smooth excitations will yield power-law decay of temporal correlations and thus not affect the very long time dynamics. It is tempting to believe that the stiffness to proper rotations $\Sigma_R(L)$ will probe the energy of smooth excitations. It is quite likely, however, that this is not the case. It may be that even a small rotation of the boundary condition yields nonsmooth changes in the ground state of a large system associated with the interchange of stability of two configurations differing by large angle rotations in some region. We leave this and other subtle questions concerning the different types of stiffnesses in Heisenberg spin glasses for future investigation.

C. Anisotropy

In the presence of random-axis dipolar or Dzyaloshinskii-Moriya anisotropy, there will no longer be arbitrarily low-energy excitations involving small rotations. The system will crossover on long length scales to an anisotropic spin-glass fixed point. However, because of the statistical rotational symmetry, this fixed point may not be the same as the Ising fixed point, although many of its features should be qualitatively similar.

XI. CONCLUSIONS AND OBSERVABILITY OF EQUILIBRIUM PHENOMENA

In this paper we have analyzed the equilibrium behavior of the ordered phase of short-range spin glasses. Based upon what we believe to be the simplest possible

ansatz, consistent with what is known, we predict power-law decay of spatial correlations, and logarithmic decay of temporal correlations, both caused by the presence of large, low free-energy, active droplet excitations. Although, as argued in a previous paper,¹⁸ this picture naturally leads to a spin-glass phase with only two pure states for the Ising system which are related by the global spin-flip symmetry, these states are extremely sensitive to temperature changes. We find that the presence of a magnetic field destroys the Ising spin-glass phase and gives rise to a divergent nonlinear susceptibility for all $T \leq T_c$. We have argued that the characteristic equilibration times for the system will diverge as the ordered phase is approached either by decreasing temperature or decreasing the magnetic field to zero below T_c . Thus, in the ordered phase true equilibrium will never be reached, so we must question the observability of the predicted equilibrium phenomena. In the companion paper, we discuss in detail the approach to equilibrium and various nonequilibrium effects. Here we briefly note under what conditions the response and fluctuations of the system will mimic equilibrium. An important feature of spin glasses is that the local order cannot be readily measured experimentally—we are generally restricted to measuring quantities averaged over relatively large regions of space. Thus if the correlation functions are statistically similar to those in equilibrium, it will be exceedingly difficult to distinguish the system from one in true equilibrium.

In order to attempt to reach equilibrium, the natural procedure is to wait at the desired temperature $T < T_c$ for as long as possible, say a time t_w , and then make measurements at frequency ω . After waiting time t_w , the system will be in equilibrium on a scale

$$R_\omega \sim \left[\frac{T \ln(t_w/\tau_0)}{\Delta} \right]^{1/\psi}, \quad (11.1)$$

i.e., the characteristic scale of the frozen-in domain walls will be R_ω . Smaller-scale fluctuations will then be similar to equilibrium since, statistically, the presence or absence of large-scale domain walls will have only a small effect on the distribution of small-scale droplets. If

$$|\ln \omega \tau_0| \ll \ln(t_w/\tau_0), \quad (11.2)$$

then the scale of droplets probed at frequency ω will be much less than the equilibrium scale and the response should mimic equilibrium. In the companion paper we discuss in detail the corrections to equilibrium behavior and various far-from-equilibrium phenomena. Unfortunately, the condition Eq. (11.2) is almost impossible to satisfy, we therefore must consider in detail how the response functions approach equilibrium.

If we were able to know the equilibrium state and use microprobes to measure the local spin orientations (averaged over some time) then the presence of the domain walls could be observed. However, in the absence of such a possibility, direct measurements of the length scale of the deviations from equilibrium cannot be done.

Microprobes could also be used, in principle, to address (with some accuracy) questions of the number of pure states by cycling the system to above T_c and back

down and then carefully trying to extract long time averages of correlation functions. Probably the subtlest open questions concern whether or not a more complicated consistent picture of short-range spin glasses can be constructed which exhibits many pure states such as exist in the standard interpretation of the Parisi solution of the SK model.¹ At this stage, we feel it is incumbent upon those who believe that realistic spin glasses exhibit such features to make arguments which go beyond the pathologies of the SK model, since, as we have argued elsewhere,¹⁷ the approaches used for the SK model are not useful for short-range systems.

A related question concerns the high-dimensional limit of our picture of a spin glass: Does it, in some sense, approach the SK model as $d \rightarrow \infty$? A plausible scenario is that the de Almeida–Thouless transition, which we have argued does not exist for finite d , becomes a sharper and sharper crossover as $d \rightarrow \infty$. Work on this question would certainly be instructive.

APPENDIX A: ARGUMENTS FOR $\theta \leq (d-1)/2$

In this appendix we present an argument to support the inequality

$$\theta \leq \frac{d-1}{2}, \quad (\text{A1})$$

which was introduced in Sec. II and plays an important role in many aspects of this work. We first give an intuitive, although overly naive and thus not entirely correct, argument and then proceed with a careful derivation of the result by assuming the converse and arriving at a contradiction.

We define θ as in Ref. 12 and Sec. X via the difference in free energy Σ_L between periodic and antiperiodic boundary conditions in one of the directions of a cube of size L^d :

$$(\overline{\Sigma_L^2})^{1/2} \sim L^\theta. \quad (\text{A2})$$

In order to go from periodic to antiperiodic boundary conditions the signs of a plane of bonds J_{ij} have been changed. Since the distribution of J_{ij} 's is invariant under $J_{ij} \rightarrow -J_{ij}$, the average free energy (of many realizations) after changing the sign of any fraction of these bonds will be the same. Clearly the rms change in the free energy due to changing the sign of just one bond is of order unity. If the free-energy changes due to changing different bonds were independent, then we would immediately find that Σ_L was the sum of L^{d-1} terms of random sign and hence $\theta = (d-1)/2$. This assumption of independence is clearly not valid, especially for nearby bonds. However, the correlations in the free energy change due to changing bonds a distance r apart will decay with r . By the general scaling assumption, we expect these correlations to decay as $1/r^\theta$. Thus the larger the exponent θ , the more rapidly the correlations will decay and the more plausible the argument using approximate independence.

To proceed, we will assume that $\theta > (d-1)/2$ and then, using the approximate independence, find a contradiction. For simplicity, we work at zero temperature; similar results at positive temperature can, in principle,

be derived using the coarse-graining procedure outlined in Sec. VI. We use energy units of $J \sim \Upsilon = 1$. We also require the assumption that the distribution of droplet free energies, $\rho_L(F_L) \leq c/L^\theta$ for $F_L \ll L^\theta$, where $\rho_L(F_L)$ is defined prior to (2.5). Note that we do not need to assume that the full distribution scales, only that $\rho_L(F_L)$ is sufficiently small for low energies. We take periodic boundary conditions in all d directions on the box of size L^d , and then change the signs of a $(d-1)$ -dimensional sheet of bonds perpendicular to the sheet, yielding antiperiodic boundary conditions in the perpendicular direction. We label the $(d-1)$ -dimensional hypercubic array of perpendicular bonds in the sheet by the coordinate \mathbf{x}_\parallel with \mathbf{x}_\parallel a $(d-1)$ -dimensional integer vector. We will make the sign change of bonds $J_{\mathbf{x}_\parallel}$ not all at once, but in a series of steps. At the first step, we change the signs of a subarray of the bonds in the sheet with integer lattice constant $a \gg 1$, i.e., those bonds with coordinate

$$\mathbf{x}_\parallel = \mathbf{n}a \quad (\text{A3})$$

with \mathbf{n} an integer vector.

We are interested in the probability $\tilde{P}_a(B)$ that changes in the ground-state configuration occur a distance B from the sheet with

$$1 \ll a \ll B \ll L. \quad (\text{A4})$$

We will argue that for $\theta > (d-1)/2$, $\tilde{P}_a(B)$ is sufficiently small that even after all the bonds have been changed, the probability $\tilde{P}_1(B)$ that the ground state has changed a distance B from the sheet is still small for a and $B \ll L$ chosen appropriately. This is inconsistent, because changing all the bonds will almost always change the ground state at distances of order L .

We first allow relaxations of the ground state to occur by flipping clusters contained in boxes of size a^d centered on the changed bonds. These relaxations will occur in a positive fraction of the boxes but will usually only involve spins within distances much smaller than a of the changed bonds at the centers of the boxes. After this process, there will be an anomalous amount of energy left in the slab of thickness a about the sheet because we have not allowed relaxation outside the small boxes. The average of this excess energy per box will be less than or of order $1/a^\theta$ since it will primarily arise from those changed bonds which would have caused a cluster of size $\gtrsim a$ to flip if we had allowed all possible relaxations. This occurs only with probability $\lesssim 1/a^\theta$ since it implies that a droplet of scale $\gtrsim a$ with energy of order J which passed through the changed bond must have existed in the original ground state.

We now consider whether a low-energy droplet D of size B in the original ground state Γ_0 can flip due to the changes in the slab. Because of the correlations between droplets discussed in Sec. II we need only consider one such putative droplet in each cube of size B^d which intersects the slab. The energy E_B of the droplet is its original energy E_B^0 in Γ_0 plus a change ΔE from the changes in the slab of thickness a . The change in energy will consist of two parts: a random piece of magnitude

$$\Delta E_R \lesssim \left(\frac{B}{a} \right)^{(d-1)/2} \quad (\text{A5})$$

which arises from the random changes in each box of size a through which the droplet passes, and an error part ΔE_E due to the extra energy in the slab discussed above, which will be of order the number of boxes through which the droplet passes in which further relaxation outside the boxes could have occurred. This error term is of order

$$\Delta E_E(B) \lesssim \left(\frac{B}{a} \right)^{d-1} \frac{1}{a^\theta}. \quad (\text{A6})$$

In deriving Eqs. (A5) and (A6) we have assumed that the droplet D passes through each box of size a and has its energy affected by an amount of order one in each box. If $d_s < d$, as we expect to be the case, then the droplet will only pass near some fraction of the altered bonds and thus (A5) and (A6) are overestimates. We would like to make the error term small compared to the estimate (A5) for the random part of ΔE so that we should choose a and B so that

$$\Delta E_E \ll (B/a)^{(d-1)/2}. \quad (\text{A7})$$

This requires

$$B^{(d-1)/2} \ll a^{(d-1)/2} a^\theta. \quad (\text{A8})$$

In this case, the probability that $E_B < 0$, which is just the probability that the droplet will flip in the new ground state, is

$$p_a(B) \sim \frac{\Delta E_R}{B^\theta} \lesssim \left(\frac{B}{a} \right)^{(d-1)/2} \frac{1}{B^\theta}. \quad (\text{A9})$$

Since for $\theta > (d-1)/2$, $p_a(B)$ decreases with B , the total probability of any droplet flipping on scales $\geq B$ is dominated by scale B and hence is bounded above by $(L/B)^{d-1} p_a(B)$, which is just the sum of the $(L/B)^{d-1}$ different regions of size B^d which intersect the slab. Thus

$$\tilde{P}_a(B) \lesssim \left(\frac{L}{B} \right)^{d-1} \left(\frac{B}{a} \right)^{(d-1)/2} \frac{1}{B^\theta}. \quad (\text{A10})$$

We now allow all further relaxations to achieve a new ground state Γ which is statistically similar to the original ground state. We next repeat the process with another array of bonds with spacing a in the sheet and repeat a^{d-1} times until the signs of all the bonds in the sheet have been changed. The probability $\tilde{P}_1(B)$ that any changes in the ground state have occurred a distance $\geq B$ from the sheet is bounded above by $a^{d-1} \tilde{P}_a(B)$ so that

$$\tilde{P}_1(B) \lesssim \frac{a^{(d-1)/2} L^{d-1}}{B^{\theta+(d-1)/2}}. \quad (\text{A11})$$

If we can choose a and B satisfying (A4) so that $\tilde{P}_1(B) \ll 1$, then there is a contradiction, since the new bond configuration could have been equally well reached by first changing the bonds in a parallel sheet a distance $> B$ from the chosen sheet resulting in the changes being localized near that sheet with high probability, and then

doing a gauge transformation in the intervening region to move the changed bonds back to our original sheet. Because of the statistical translational invariance, we should have $\tilde{P}_1(B) = O(1)$ for all B , since the resulting single wall which is the difference between the final and initial ground states will lie at a random position perfectly uncorrelated with which parallel sheet of bonds we changed. In fact we expect that the wall will wander a distance of order L transversely.

We define

$$\alpha = \frac{\theta}{d-1} - \frac{1}{2} \quad (\text{A12})$$

which by assumption is positive so that

$$0 < \alpha \leq \frac{1}{2}, \quad (\text{A13})$$

and choose

$$a \sim L^z \quad (\text{A14})$$

and

$$B \sim L^x, \quad (\text{A15})$$

with

$$0 < z < x < 1. \quad (\text{A16})$$

It is then straightforward to see that we require from Eq. (A8)

$$x < z(2+2\alpha), \quad (\text{A17})$$

and we have $\tilde{P}_1(B) \ll 1$ when

$$x > \frac{1+z/2}{1+\alpha}. \quad (\text{A18})$$

These are only compatible if $\alpha > \alpha_c = (\sqrt{2}-1)/2$, so we have not yet succeeded in demonstrating that any $\alpha > 0$ yields small $\tilde{P}_1(B)$, although we have found a bound on θ well away from $d-1$, namely, $\theta \leq (d-1)/\sqrt{2}$.

In order to improve the bound, we need to perform several intermediate steps on scales between a and B to better control the error $\Delta E_E(B)$. We thus introduce a sequence of lengths b_n with

$$b_0 \equiv a, \quad b_N \equiv B, \quad (\text{A19})$$

and

$$b_n \sim L^{u_n}, \quad (\text{A20})$$

with

$$u_{n-1} < u_n \quad (\text{A21})$$

for all n so that as $L \rightarrow \infty$, $b_n/b_m \rightarrow \infty$ for $n > m$. We now allow successively relaxation in boxes of size b_n^d which form a slab centered on the altered sheet. After allowing relaxation up to scale b_{n-1} we consider the change in energy of a droplet of scale b_n . This will pass through (at most), $(b_n/b_{n-1})^{d-1}$ boxes of size b_{n-1} in which relaxation has already occurred. We want to choose the b_n iteratively so that the random part of the change in energy of droplets of size b_n is larger than the

error term due to the excess energy arising from the confinement of the relaxation in boxes of size b_{n-1} . In each box of size b_{n-1} we then have, by assumption, a random extra energy at most of order $(b_{n-1}/a)^{(d-1)/2}$ which can be picked up by the droplet of scale b_n , with a possible error from the fraction of these boxes

$$\left(\frac{b_{n-1}}{a}\right)^{(d-1)/2} / b_{n-1}^\theta$$

in which the constraint of no relaxation (so far) at scales larger than b_{n-1} causes an error. Thus the random part of the change in energy of the droplets inside the boxes of scales between b_{n-1} and b_n which we now allow to relax will be

$$\begin{aligned} \Delta E_R(b_n, b_{n-1}) &\lesssim \left(\frac{b_n}{b_{n-1}}\right)^{(d-1)/2} \left(\frac{b_{n-1}}{a}\right)^{(d-1)/2} \\ &= \left(\frac{b_n}{a}\right)^{(d-1)/2}, \end{aligned} \quad (\text{A22})$$

while the error term will be

$$\begin{aligned} \Delta E_E(b_n, b_{n-1}) &\sim \left(\frac{b_n}{b_{n-1}}\right)^{d-1} \left(\frac{b_{n-1}}{a}\right)^{(d-1)/2} \\ &\times \left(\frac{\left(\frac{b_{n-1}}{a}\right)^{(d-1)/2}}{b_{n-1}^\theta}\right), \end{aligned} \quad (\text{A23})$$

where the three factors are, respectively, the number of boxes of size b_{n-1} through which a droplet of size b_n could pass, the random energy change in the boxes of size b_{n-1} , and the probability the changes inside a box of size b_{n-1} will result in relaxations outside the box, i.e., the probability of an error. We now choose the u_n so that

$$\Delta E_R(b_n, b_{n-1}) < (b_n/a)^{(d-1)/2}.$$

By the time we get to scale B , we will have improved the original error estimate Eq. (A6) by a factor $(a/B)^\theta$, due to the allowing of the intermediate scale of relaxations. We note that at each stage, the earlier errors will be thrown away once we allow for all the relaxation up to that stage. All that is important is that our estimate for the random energy change at the final step is not invalidated by intermediate scale relaxations.

At stage n we are adding up almost random terms from stage $n-1$, we thus just need overall bounds on the magnitude of these random terms. The errors Eq. (A23) can be kept small compared to the random part (A22) at each stage by choosing

$$u_{n-1} < u_n < u_{n-1}(1+2\alpha) + z \quad (\text{A24})$$

for all n . The overall probability $\tilde{P}_1(B)$ can be made small by choosing

$$0 < z < 2\alpha \quad (\text{A25})$$

and N large enough so that

$$1 > x = u_N > \frac{1+z/2}{1+\alpha} \quad (\text{A26})$$

which can clearly be done for all α since a sequence of u_n 's which increases without bound can satisfy Eq. (A24).

At each stage beyond the first, we have kept the probability that individual droplets flip small, and finally at scales $\geq B$, so small that with high probability no droplets will flip, yielding the desired result; namely, the contradiction occurs for all $\theta > (d-1)/2$. Although these arguments rely on many assumptions of approximate independence, and are, in any case, far from rigorous, we believe that the conclusion will be correct, provided a scaling picture exists for the ground state and its excitations, and it will be left as a future challenge to make tighter arguments for $\theta \leq (d-1)/2$. We note that once $\theta \leq (d-1)/2$, all sorts of problems become apparent in these kinds of arguments and similar reasoning definitely *cannot* be used to yield the natural guess of $\theta = (d-1)/2$ (although, perhaps, by more careful arguments the d in the *inequality* could be replaced by d_s).

APPENDIX B: INSTABILITY OF SPIN-GLASS ORDERING TO A FIELD

Villain¹⁵ has argued that the ordered state of an Ising spin glass may possibly be stable against a small uniform magnetic field. Here we wish to counter some of his arguments. We assume, as argued in Ref. 18, that there are only two pure states in zero field.

First, Villain¹⁵ works within a picture in which only a

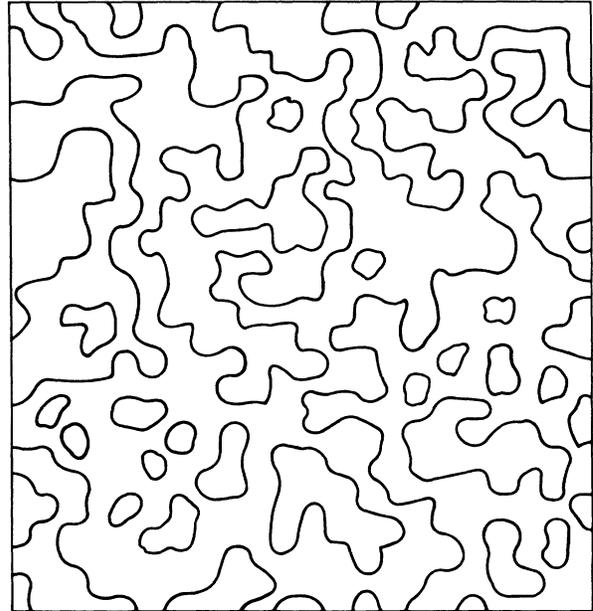


FIG. 4. Schematic picture of the domain walls present in the ground-state configuration of an Ising spin glass in a uniform magnetic field. The domain walls separate regions in which the spins are aligned with one of the zero-field ground states Γ and regions where they are antialigned with Γ . The characteristic length scale of these domains is the correlation length, which diverges (3.24) as a power of the magnetic field H for $H \rightarrow 0$.

fraction of the droplets on each length scale larger than ξ_H align with the field, and these aligned droplets are hierarchically nested as illustrated in his Fig. 1. This contrived scenario is postulated without any justification. Our picture of the equilibrium state in a field is quite different. We argue that most of the droplets of length scale of order ξ_H align with the field, while smaller droplets of size L only align with a smaller probability of order $(l/\xi_H)^{(d/2)-\theta}$. The moments of the droplets of size ξ_H are of random sign; the resulting pattern has no correlations or structure on length scales $\gg \xi_H$, as illustrated in our Fig. 4. The lines in Fig. 4 are domain walls separating regions aligned with a zero-field ground state and regions antialigned with the zero-field ground state. Thus for length scales $\gg \xi_H$ it is inappropriate to discuss the droplets. The former zero-field droplets on these long scales have now broken up into many smaller droplets of size of order ξ_H , each of which is independently aligned with the field.

Villain¹⁵ postulates that a large droplet with $L \gg \xi_H$ will contain many smaller droplets that are frozen into a given alignment, but there remains a connected cluster of free spins with fractal dimensionality 2θ . The number of spins in the clusters is $\sim L^{2\theta}$ and the moment is of order L^θ . Thus the direct interaction with the field can precisely

ly balance on all scales the exchange interaction $\sim \Upsilon L^\theta$ of the large droplet with its environment. To the extent that this scenario makes any sense at all, an important effect has been neglected, namely the exchange interactions of this fractal cluster with the frozen-in smaller droplets inside of it. These exchange interactions add to the effective fields felt by the large fractal cluster of spins, thus making it easier for the field to align the full droplet. In fact, we would argue that these interactions with the frozen-in droplets must cause the fractal cluster to break up into domains of size of order ξ_h , each choosing an alignment that satisfies these interactions as best as possible. This is because the remaining part of the large cluster must, under Villain's hypothesis¹⁵ that it has fractal dimension less than d_s , be very stringy so that it can be cut by domain walls of area $\ll L^{d-1}$, while the interactions with the already frozen-in droplets will occur across a surface of fractal dimension at least d_s . We believe that an internally consistent picture exhibiting Villain's scenario cannot be made.

We note, however, that we cannot completely rule out more complicated scenarios which, for example, might yield more than two pure states in zero field with the destruction of only some fraction of them by a field, the rest mixing together to form new states.

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itive energy. The scaling distribution for the domain wall energies is $\bar{\rho}_w = e^{-x}$, as shown, for example, in Ref. 7. Since the droplet energy is the sum of the two positive domain wall energies, the droplet distribution function has $\bar{\rho}(x) \sim x$ for small x . This is a symptom of the fact that a one-dimensional spin glass with only short-range interactions has no large-scale frustration.

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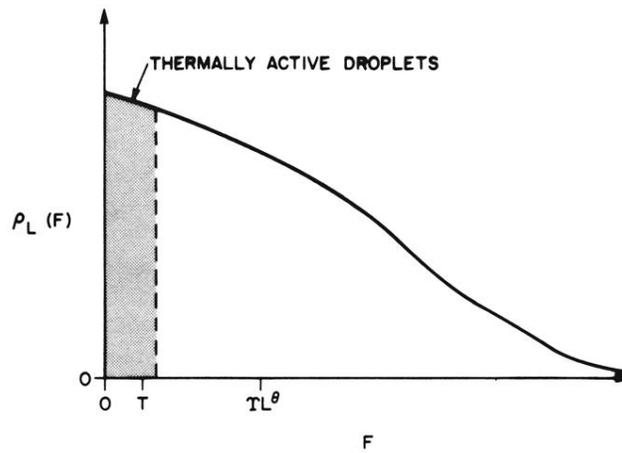


FIG. 2. Schematic distribution $\rho_L(F)$ of droplet free energies F at length scale L . The mean droplet free energy is of order ΥL^θ . The thermally active droplets are those with free energy less than or of order T . For $T \ll \Upsilon L^\theta$ these represent only a small fraction of all droplets. Here we show $\rho_L(F)$ having a finite slope for $F \rightarrow 0$; this corresponds (3.20) to the case $\phi = 1$, which seems likely.