# Nonequilibrium dynamics of spin glasses

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We consider the nonequilibrium behavior of the spin-glass ordered phase within the droplet scaling theory introduced previously. The fundamental long-time nonequilibrium process is assumed to be the thermally activated growth of spin-glass ordered domains. The remanent magnetization, $m(t)$, in zero field is found to decay at long times as $m(t)\sim R_t^{-\lambda}$, where $R_t\sim(\ln t)^{1/\psi}$ is the linear domain size, $\psi$ is the previously introduced barrier exponent describing the growth of activation-barrier heights with length scale, and $\lambda$ is a new nonequilibrium dynamic exponent, satisfying the relation $\lambda\geq d/2$ for $d$-dimensional systems. The effects of waiting for partial equilibration before making a measurement are studied in various regimes. The effects of quenching first to one temperature and then to another are also examined. Such experiments can, in principle, be used to obtain information about the relative rate of dynamic evolution as well as the overlap between the equilibrium states at different temperatures. In particular, the length scale $L_{\Delta T}$, below which equilibrium correlations at temperatures $T$ and $T+\Delta T$ are similar, plays an important role. The decay of $m(t)$ and the growth of spin-glass order after a quench are examined in Monte Carlo simulations of the Sherrington-Kirkpatrick model.

## I. INTRODUCTION

At temperatures below a rather sharp freezing temperature $T_f$, experiments on spin glasses are dominated by history-dependent phenomena which are indicative of nonequilibrium behavior.[1] Indeed, the presence of such behavior is often used to characterize a system as a spin glass. In certain three-dimensional systems, such as Cu:Mn and the Edwards-Anderson model with Ising spins, the weight of experimental and numerical evidence is that in zero magnetic field the onset of nonequilibrium effects coincides reasonably well with the critical temperature $T_c$ at which the nonlinear susceptibility diverges.[1] Below $T_c$, the equilibrium state of the system is thus presumably some kind of ordered spin-glass phase. Based on scaling arguments, a phenomenological picture of the equilibrium properties of the ordered phase below $T_c$ was developed in Ref. 2 and is presented in some detail in the companion paper.[3] In this paper we concentrate on nonequilibrium effects and the approach to equilibrium at temperatures below $T_c$. We focus, for simplicity, on Ising systems with a symmetric distribution of nearest-neighbor exchanges; however, we expect that, qualitatively, many of the results should apply to Heisenberg systems and also in the presence of power-law interactions such as the Ruderman-Kittel-Kasuya-Yosida (RKKY) interactions in metals, as has been argued for the equilibrium behavior in Ref. 3.

For Ising systems with finite-range (i.e., square-integrable) interactions, we have argued[4] that at any given temperature below $T_c$ the system has only two pure equilibrium states which are simply related by global spin reversal. Furthermore, in a nonzero magnetic field $H$, which breaks the global symmetry, the spin-glass phase is destroyed.[2,3] Thus there is no thermodynamic transition in a field $H$, although there is still a reasonably sharp dynamic freezing line $T_f(H,\omega^{-1})$, whose position as measured, say, by ac susceptibility is dependent on frequency $\omega$.[3]

We will see that in spite of this seemingly simple structure of the equilibrium phase diagram [much simpler than the infinitely many states and transitions in a field found in the Parisi solution of the infinite-range Sherrington-Kirkpatrick (SK) model[1,5]], very rich nonequilibrium behavior occurs. Some of these effects are due to an important subtlety of the ordered phase: the correlations in the states at $T<T_c$ are very sensitive to temperature, in that if the temperature is changed, the relative orientation of spins which are far enough apart will change randomly.[2,3,6]

### A. Outline

This paper is organized as follows. In the remainder of the Introduction we summarize the important features of the equilibrium behavior and discuss the onset of hysteretic behavior as the temperature is lowered, then in Sec. II we discuss the approach to equilibrium following a quench from infinite temperature or infinite magnetic field. We find that it is necessary to introduce a new nonequilibrium exponent $\lambda$ which relates the remanent magnetization to the size of the growing spin-glass ordered domains; this exponent is shown to have an analog in pure systems. In Sec. III we discuss a general class of experiments[7-9] in which the system is probed at various frequencies or times after waiting a specified amount of time $t_w$ for the system to equilibrate. The qualitative behavior of the magnetization decay in various field and time regimes is examined. Section IV considers double temperature quenches,[7] where the system is first allowed to equilibrate for a time at one temperature below $T_c$ before moving to another temperature. Here the relative rate of domain growth as well as the differences between the equilibrium states at the two temperatures enter in determining the behavior. In Sec. V we analyze the quasiequilibrium behavior in the frequency regime $\omega t_w \gg 1$, Sec. VI contains a discussion of recent experiments,[7-9] and Sec. VII some conclusions. Results of Monte Carlo simulations of quenches of the SK model[5] and the Mattis spin glass[10] are reported in Appendix A.

### B. Summary of equilibrium behavior

We briefly summarize the salient features of the picture of the equilibrium behavior of the Ising spin-glass ordered phase developed in Ref. 3. At a fixed temperature $T$ below the ordering temperature $T_c$ (which we assume is positive) the global spin reversal symmetry is broken and there are two pure states related by this global symmetry.[4] The dominant low-lying excitations in each state are *droplet excitations* which occur on all length scales $L > \xi_-$, where $\xi_- \sim (T_c - T)^{-\nu}$ is the critical correlation length. The droplet excitations of scale $L$ have a broad distribution of free energies $F_L$ with characteristic magnitude $\Upsilon L^{\theta}$ and weight extending down to zero. The coefficient $\Upsilon(T)$ is a temperature-dependent stiffness modulus which vanishes for $T \to T_c^-$ and the exponent $\theta$ satisfies $0 < \theta \le (d-1)/2$. The droplets' surfaces are fractal with area scaling as $L^{d_s}$ with $(d-1) < d_s < d$. Barriers for annihilation and creation of the droplet excitations also have a broad distribution with characteristic magnitude $\Delta L^{\psi}$, where the *barrier exponent* $\psi$ satisfies $(d-1) \ge \psi \ge \theta$ and $\Delta(T)$ vanishes as $T \to T_c^-$. Much of the behavior is dominated by the *active droplets,* i.e., those with $F_L \lesssim T$; the large active droplets are dilute. As temperature is changed, the states change so that at large separations the relative orientations of spins changes randomly.[2,3,6]

The spin-glass phase is destroyed by a magnetic field but the equilibrium correlation length $\xi_H$ and the logarithm of the correlation time $\ln\tau_H$ both diverge as inverse powers of $H$ for $H \to 0$ at $T < T_c$.

### C. Onset of hysteretic behavior

We now discuss the onset of nonequilibrium behavior as temperature is decreased. In zero field, the characteristic relaxation time $\tau_c$ of the system diverges (as discussed in Sec. V of Ref. 3) as $T \to T_c^+$. The expected form of this divergence is

$$\tau_c \sim t_0 \, |\varepsilon|^{-z\nu} , \tag{1.1}$$

where

$$\varepsilon \equiv \frac{T - T_c}{T_c} , \tag{1.2}$$

and $t_0$ is a microscopic time (which we will often use as our time unit). The onset of marked hysteretic phenomena in zero field occurs just above $T_c$, where $\tau_c$ reaches laboratory time scales.

In a small magnetic field, the relaxation times will also grow very large as $T \to T_c$ but not actually diverge, due to the absence of an equilibrium transition. However, below $T_c$ in fixed field the relaxation times will continue to grow. For temperatures below a crossover temperature $T_x(H)$ given by

$$-\varepsilon_x \equiv \frac{T_c - T_x(H)}{T_c} \sim H^{2/(\beta+\gamma)} , \tag{1.3}$$

where $\beta$ and $\gamma$ are the critical exponents of the zero-field transition, the deviation from criticality will be dominated by the temperature difference $T_c - T$. In this regime, the relaxation times grow exponentially for small $H$ as discussed in Sec. IV of Ref. 3 and have a broad distribution characterized by $\ln\tau_H$, where

$$\ln[\tau_H / \tau_c(\varepsilon)] \sim \left[ \frac{|\varepsilon|^{(\beta+\gamma)/2}}{H} \right]^{2\psi/d-2\theta} . \tag{1.4}$$

Thus for reduced temperatures of a few times $\varepsilon_x(H)$, the relaxation rates will grow extremely rapidly, quickly reaching the macroscopic measuring times of an experiment. Because of the very broad distribution of relaxation times, broad even on a logarithmic time scale, freezing will not occur at a precise temperature but can generally be characterized by a temperature $T_f(H,\omega^{-1})$ below which most of the relaxation requires times longer than the measuring frequency $\omega$ of an experiment.[3,11,12] From Eqs. (1.1)–(1.4) we see that for small $H$,

$$\frac{T_c - T_f(H,\omega^{-1})}{T_c} \sim H^{2/(\gamma+\beta)} \, |\ln\omega t_0|^{(d-2\theta)/\psi(\gamma+\beta)} , \tag{1.5}$$

which is only weakly dependent on the measuring frequency $\omega$ provided $\omega$ is small enough. Except for the logarithmic factor the freezing line as a function of $H$ and $T_c - T$ scales the same way as a de Almeida–Thouless transition would be expected to if it existed.[13,14] Below the freezing temperature, the system will always be substantially out of equilibrium.

We note that above $T_f(H,\omega^{-1})$, indeed even above $T_c$, there will always be some rare regions of the system that are anomalously weakly frustrated, giving rise to very slow relaxation and hence some small amount of nonequilibrium behavior on any experimental time scale.[15] However, only below $T_f$ will these effects become large. It is clear, therefore, that different measurements with varying precisions will find onset of nonequilibrium behavior at different temperatures and $T_f$ thus characterizes the freezing in only a very crude manner.

In large magnetic fields where the Zeeman energy is comparable to the exchange energy $J \sim T_c$, the collective spin-glass aspects of the freezing will not occur and appreciable hysteretic phenomena will only occur at temperatures low enough so that, with a macroscopic measuring frequency,

$$|\ln\omega t_0| \sim J/T \ . \tag{1.6}$$

We will not be concerned with this regime here.

## II. EQUILIBRATION FOLLOWING A QUENCH

We now turn to nonequilibrium phenomena in the ordered spin-glass phase. We first consider an ideal quench of an Ising spin glass in zero magnetic field from infinite temperature to a temperature $T$ below $T_c$. As argued in Ref. 4, we assume the equilibrium states at temperature $T$ and zero field are simply a pair of states related by a global spin flip. As discussed in Ref. 3 each of these states is well approximated on large scales by an effective ground state with active droplet fluctuations which are rare. We call one of these states $\Psi=\Psi_T$ and its spin reversal $\overline{\Psi}=\overline{\Psi}_T$. (We will usually drop the subscript $T$ when only one temperature is being discussed.) Each spin has a preferred direction $\langle S_i \rangle$ in state $\Psi$ (and minus that in $\overline{\Psi}$). We can hence decompose the random configuration immediately after the quench into domains (or individual spins) which have the same orientation as in $\Psi$ and those which have the opposite orientation, and are thus aligned as in $\overline{\Psi}$. At temperatures far below $T_c$, the equilibrium states will be well characterized by the preferred spin directions. However, near to $T_c$, there will be large fluctuation up to the scale of the critical correlation length $\xi_- \sim |\varepsilon|^{-\nu}$. The description in terms of domains of $\Psi$ and $\overline{\Psi}$ is only valid on length scales larger than $\xi_-$, since the domain walls will have width of order $\xi_-$. We generally restrict consideration to this regime of length scales.

After the quench, the system will try to lower its free energy by, on average, decreasing the amount of interface between $\Psi$ and $\overline{\Psi}$, thus growing larger and larger domains of $\Psi$ and $\overline{\Psi}$. This is the same situation as encountered in the growth of ordered domains in a pure Ising system.[16] However, in the spin glass the growth with time $t$ of the characteristic length scale $R_t$ of the domains of the two states will be very slow because of the randomness-induced free energy barriers. As discussed in Ref. 3 we expect that the barriers $B$ that must be surmounted in order to move sections of wall between $\Psi$ and $\overline{\Psi}$ of length scale $L$ will be of order $\Delta L^{\psi}$ for large $L$, where $\Delta(T)$ sets the free-energy scale of the barriers. In a time $t$ after the quench, free-energy barriers of height $B \approx T \ln t$ can be surmounted so the characteristic length scale of domains $R_t$ grows as

$$R_t \sim \left[ \frac{T \ln t}{\Delta(T)} \right]^{1/\psi} , \tag{2.1}$$

where time $t$ is measured in units of a microscopic time $t_0$ for $T \ll T_c$ or the critical correlation time $\tau_c(T)$ which diverges for $T \to T_c^-$ as in Eq. (1.1). We expect that this growth of domains is the fundamental process of the long-time nonequilibrium dynamics of spin glasses below $T_c$.

We use the expression "domains" in a somewhat loose sense: In three dimensions there will be an interpenetrating network of regions of the two states and some closed domains of each of the states (as in, e.g., Fig. 4 of Ref. 3). The characteristic scale of the separation between walls is given by $R_t$. A more precise definition can be given by measuring the overlap between the nonequilibrium configuration of the spins $\{S_i(t)\}$ with their correlations in the equilibrium state: $\{\langle S_i S_j \rangle_\Psi\}$. We define the nonequilibrium overlap

$$\Xi_{\rm NE}(i,j;T,t) \equiv \overline{\langle S_i S_j \rangle_{\Psi_T} S_i(t) S_j(t)} , \tag{2.2}$$

where the overbar denotes averaging over the system with the separation $r_{ij}$ fixed. On scales larger than $R_t$ we expect random relative orientation of $S_i(t)$ and $S_j(t)$ and thus rapid decay with distance of $\Xi_{\rm NE}(r_{ij})$, while for $\xi_- \ll r_{ij} \ll R_t$, $\Xi_{\rm NE}$ will be of order $q_{\rm EA}^2(T)$. An integral over $r_{ij}$ will thus yield contributions primarily from $r_{ij} < R_t$ so that

$$\frac{1}{V}\sum_{ij} \Xi_{\rm NE}(i,j) \sim q_{\rm EA}^2(T) R_t^d \tag{2.3}$$

(with $V$ the volume and the thermodynamic limit presumed).

We note that the logarithmic growth law Eq. (2.1) is of the same form as the growth of domains in other random magnets,[17,18] although the growth in spin glasses will, on the scale of single domains, be less steady. This is because the typical individual domain wall motions occurring in the spin-glass domain growth are displacements by length of order $R_t$; for other random magnets, the typical displacements scale sublinearly in $R_t$ for long times.[17–19]

After the coarsening (domain growth) has proceeded for some time, the fluctuations on length scales $L \ll R_t$ will be in local equilibrium—they will be droplet excitations around the configuration in which the system is stuck at time $t$. Far from domain walls, the droplet excitations will be just those of the equilibrium pure states, $\Psi$ and $\overline{\Psi}$, while near the domain walls they will be modified.

In Ref. 3 we argue that the walls of large droplets will be fractal with the surface area of domain wall segments of linear length scale $L$ scaling as $L^{d_s}$ with

$$(d-1) < d_s < d \ . \tag{2.4}$$

The arguments presented there for locally minimum energy walls should also apply here, so that we expect that on length scales shorter than $R_t$ the domain walls at time $t$ will be fractal with dimension $d_s$. (Note that fractal structure on scales much less than $R_t$ does not occur in quenched pure systems. In pure systems, the typical local radius of curvature of the walls at time $t$ is of order $R_t$.) The domain walls thus occupy a fraction

$$\rho_w(t) \sim \frac{1}{R_t^{d-d_s}} \tag{2.5}$$

of the system's volume. Near the domain walls, the distribution of small scale droplet excitations will be modified, leading to corrections to the equilibrium behavior even at frequencies $\omega$ satisfying $|\ln\omega| \gg \ln t$. These corrections are discussed in Sec. V.

Because of the absence of probes which couple to the spin-glass order, e.g., by distinguishing between the equilibrium states $\Psi$ and $\bar{\Psi}$, the growth of domains cannot be directly observed. An indirect method is possible, however, via the nonequilibrium decay of magnetization. If the system is quenched from infinite magnetic field rather than infinite temperature, the behavior of the growth of the ordered domains will be statistically similar, since the fully aligned configuration that is obtained in infinite field is, as far as the equilibrium states are concerned, no different from any other random configuration. (For real experimental spin glasses with *short-range* ferro- or antiferromagnetic order this is not precisely correct, but the differences between quenches from infinite field and infinite temperature should be significant only at short times when the domain sizes are on the same scale as this short-range order.)

### A. Quench from infinite field

After time $t$ following the quench from infinite field, the magnetization density $m(t)$ is proportional to $[P_s(R_t)-\frac{1}{2}]$, where $P_s(R_t)$ is the probability that a spin is in the same domain in which it started, i.e., that it has flipped an even number of times. This probability can be calculated exactly for a pure one-dimensional Ising Glauber model[20] and numerically for the SK (Ref. 5) model and pure Ising models (and thus the trivially related Mattis[10] spin glass) in higher dimensions, as is discussed in Appendix A. A power-law dependence of $P_s-\frac{1}{2}$ on $R_t$ is found in each case; we thus make the natural conjecture that this will be the case for finite-dimensional spin glasses. We thus have

$$m(t)\sim\frac{1}{R_t^{\lambda}}\sim\left[\frac{\Delta}{T\ln t}\right]^{\lambda/\psi}, \tag{2.6}$$

where we have introduced the nonequilibrium exponent $\lambda$. The exponent $\lambda$ is, as far as we understand, an independent dynamic exponent unrelated to equilibrium exponents, although further investigation may yield a scaling relation between $\lambda$ and other exponents.

An upper bound on $m(t)$ and thus a lower bound on $\lambda$ is obtained by considering the maximum possible $m(t)$ which can arise from domains of size $R_t$. The typical total moment per spin of a spin-glass domain of size $R_t$ is of order $R_t^{-d/2}$, because the relative orientations of the spins are random within the domain. Thus the maximum possible value of $m(t)$ is of order $R_t^{-d/2}$ and we obtain an essentially rigorous lower bound for the exponent $\lambda$:

$$\lambda\geq(d/2)\ . \tag{2.7}$$

An upper bound for $\lambda$ is suggested by the following scaling argument: Consider a region of linear size $L$. Let $\frac{1}{2}+f$ be the fraction of spins in the initial configuration of this region which are in state $\Psi$. Let us then look at the configuration after a time $t$ such that $R_t=L$. The average fraction of the spins in the region that are in state $\Psi$ at $t$ is $P_\Psi(L,f)$. The natural scaling conjecture is that for large $L$

$$P_\Psi(L,f)-\tfrac{1}{2}\approx g(fL^a)\ , \tag{2.8}$$

for some exponent $a$, with $g$ a nondecreasing odd function which we will assume is smooth and satisfies $g(x\to\infty)=O(1)$. Let us first consider a fixed positive $f$, so that the initial configuration is biased a finite amount towards $\Psi$. If $a>0$, then this bias grows as the domains grow, which appears reasonable. If $a<0$, on the other hand, this bias is forgotten as the domains grow. We can now obtain a lower bound on $m(t)$ by assuming that which spins are in state $\Psi$ at time $t$ is completely uncorrelated with which spins are initially in state $\Psi$. This gives a lower bound, because positive correlations which enhance $m(t)$ are expected. The initially random distribution after the quench implies that $f$ is normally distributed with variance $L^{-d}$. Thus we obtain

$$m(t)\geq 4\int df\, e^{-(1/2)f^2L^d}\frac{L^{d/2}}{\sqrt{2\pi}}f\,g(fL^a)\ . \tag{2.9}$$

We now have two cases: (i) if $a>d/2$, then $g$ acts likes a step function on the relevant scale of the normal distribution of $f$ and we obtain $m(t)\geq O(L^{-d/2})$ and thus $\lambda\leq d/2$ which, together with the bound (2.7), would imply $\lambda=d/2$; and (ii) $a<d/2$ in which case the small-argument limit of $g$ dominates (2.9). In the latter case we make two natural assumptions: first $g(x)\sim x$ for $x\to 0$, which yields $m(t)\geq O(L^{a-d})$ and thus $\lambda\leq d-a$. We now suppose that $a$ is non-negative since, at least naively, the forgetting of an initial finite bias (implied by $a<0$) appears unlikely. We thus obtain

$$\lambda\leq d\ . \tag{2.10}$$

Unlike Eq. (2.7), however, this must be viewed as only a suggestive bound, since many assumptions went into its derivation. The inequalitites (2.7) and (2.10) appear to be satisfied for the SK[5] model and for pure Ising models (and thus Mattis[10] spin glasses) in two and three dimensions as is discussed in Appendix A. For the pure Ising model in $d=1$ at $T=0$ the exact Glauber solution[20] gives $a=0$ and $\lambda=d-a=1$, which again satisfy our inequalities. We now turn to various consequences of the picture of the equilibration processes below $T_c$ in terms of the growth of ordered domains.

## III. WAITING-TIME EFFECTS AND MAGNETIZATION DECAY

In the preceding section we have argued that at time $t$ after a quench from infinite temperature or magnetic field, there is a length scale $R_t$ which characterizes the scale on which equilibrium has been established. If we probe the system on scales much smaller than $R_t$, quasiequilibrium results should be found which, for spatially averaged quantities, will be close to those in true equilibrium.

In this section we consider a class of experiments in which the system is cooled rapidly from above $T_c$ to a temperature $T<T_c$, then a time $t_w$ is waited before probing the system. The prototypical example[7–9] is decay of thermal remanent magnetization (TRM) after cooling in a small-magnetic field $H$, which is turned off after the waiting time and then the magnetization is measured at times $t_w+t$. In the small-field regime, we will show that for a wide range of times the magnetization is linear in the magnetic field. Other regimes of magnetic field are also considered.

We first consider *short times* $t \lll t_w$. In this regime, the process which will occur are on length scales

$$L_t \sim \left[\frac{T \ln t}{\Delta(T)}\right]^{1/\psi} . \quad (3.1)$$

This will be much less than the spacing,

$$R_w \equiv R_{t_w} \sim \left[\frac{T \ln t_w}{\Delta(T)}\right]^{1/\psi} , \quad (3.2)$$

between the domain walls which remain after waiting time $t_w$, provided

$$\ln t \ll \ln t_w . \quad (3.3)$$

This condition (3.3) defines the *early epochs*. We use *epoch* to mean an interval of, say, a factor of $2^\psi$ in $\ln t$, which corresponds to a factor of 2 in $L_t$ by (3.1). It is, of course, rather difficult to satisfy (3.3); at this stage, however, we are interested in the asymptotic regimes, and we will return later to questions concerning the nature of the crossovers between asymptotic regimes.

Since in early epochs the processes which occur in time $t$ are on scales small compared to the domain wall spacing, they will be characteristic of one of the two equilibrium states $\Psi_T$ and $\overline{\Psi}_T$ except near the frozen-in domain walls. Only a small fraction of the system is near a domain wall, so that to first approximation, the relaxational processes are characteristic of equilibrium. From the companion paper,[3] we know that the processes which relax a small magnetization in equilibrium at time $t$ are primarily thermally active droplets of size $\sim L_t$. These will give rise to a quasiequilibrium (qe) *component* of the magnetization decaying as

$$m_{\rm qe}(t) \sim \frac{H}{(\ln t)^{\theta/\psi}} \quad (3.4)$$

in the short-time regime, where $\theta$ is the stiffness exponent discussed in Ref. 3. The decay (3.4) is due to the depolarization of independent small droplets of size $\sim L_t$ whose magnetic moment $\mu_L$ is such that their magnetic free energy satisfies

$$\mu_L H \ll T , \quad (3.5)$$

and whose free energy in zero field satisfies

$$F_L \lesssim T , \quad (3.6)$$

which is just the condition for them to be active. In this regime, their response is simply linear in $H$ and characteristic of equilibrium. Since the magnetic moment of a droplet of size $L$ is

$$\mu_L \sim L^{d/2} , \quad (3.7)$$

the condition Eq. (3.5) will require smaller fields as $L_t$ grows.

### A. Weak fields

We must thus distinguish several regimes of magnetic field strength. The simplest regime is weak field for which condition Eq. (3.5) remains valid up to and beyond scale $R_w$, i.e.,

$$q_M^{1/2} H R_w^{d/2} \ll T , \quad (3.8)$$

where we have inserted the magnetic moment factor $q_M(T)$ from Eq. (3.13) of Ref. 3; $q_M(T) = q_{\rm EA}(T)$ for an ideal spin glass. In this limit, the first nonequilibrium effects which will occur as $t$ increases are associated with the finite-waiting time and the continuing growth of the domains rather than the nonlinear effects of the magnetic field.

For times $t \gg t_w$, the domains will continue to grow, reaching a size $R_{t+t_w}$. Each of these domains will have a residual magnetization which has not yet decayed away and which cannot decay further by depolarization of droplets of size $< R_w$, i.e., inside each domain, the size is $\Psi_T$ or its inverse. Thus we now have a situation which is similar to the decay of the magnetization following a quench from infinite magnetic field as discussed in the previous section, except that the average magnetization at time $t_n$ such that, say, $R_{t_n+t_w} = 2R_w$ will be determined by the cooling field $H$ and the relaxation processes at earlier times. Thus we expect from Eq. (2.6) that for $\ln t \gg \ln t_w$, which defines the *late epochs*,

$$\frac{m(t)}{m(t_n)} \sim \left[\frac{R_{t_w+t_n}}{R_{t_w+t}}\right]^\lambda , \quad (3.9)$$

so that at very long times

$$m(t) \sim m(t_n)\left[\frac{R_w}{R_t}\right]^\lambda \sim m(t_n)\left[\frac{\ln t_w}{\ln t}\right]^{\lambda/\psi} , \quad (3.10)$$

where we have used $R_{t+t_w} \approx R_t$ for $t \ggg t_w$, and we expect $m(t_n) \propto H$ in the weak-field regime. Thus we see that for small fields, the logarithmic decay rate of the magnetization as a function of $\ln t$ is *faster* in the late epochs (3.10) than in the early epochs (3.4), because

$$\lambda \ge d/2 > (d-1)/2 \ge \theta . \quad (3.11)$$

In order to obtain the magnetization at time $t_n$, it is necessary to understand the crossover between the early and late epochs. We will return to this crossover at the end of this section. However, we next consider the behavior in stronger fields, in particular when condition (3.5) is not satisfied.

### B. Intermediate fields

For fields in the range

$$T \lesssim q_M^{1/2} H R_w^{d/2} \ll \Upsilon R_w^\theta , \quad (3.12)$$

the fields can give rise to nonlinear polarizations of the active droplets but they are not large enough to flip a significant fraction of all the droplets at scale $R_w$ which typically would cost a free energy $F_L \sim \Upsilon R_w^\theta$. This latter condition is just the condition that

$$R_w \ll \xi_H , \quad (3.13)$$

where the magnetic correlation length $\xi_H(T)$, given by

Eq. (3.24) of Ref. 3, is the length scale at which the order is destroyed by a field for $T < T_c$. In this intermediate-field regime, the magnetization which is established by taking an equilibrium zero-field state and turning on a field for a time $t_w$ is given by the nonlinear response of the droplets with sizes less than or of order $R_w$. This nonlinear response should also affect the contribution of droplets of these sizes to the remanent magnetization decay. It might be expected that the nonlinear response of the droplets would be large since the nonlinear susceptibility of a single active droplet grows rapidly with increasing size and decreasing temperature as discussed in Sec. III of Ref. 3. However, averaging over the distribution of excitation free energies $F_L$ of the droplets reduces the cumulative effect dramatically, due to the assumed smoothness of the distribution $\rho_L$ of excitation free energies $F_L$ for $F_L \to 0$ defined in Sec. III of Ref. 3. This is the same effect which reduces the nonlinear susceptibility divergence from its naive form (see Sec. III of Ref. 3). Physically, the mechanism is quite simple: For a uniform distribution of droplet free energies, there is a zero-temperature regime of linear response with a *total* susceptibility which is the same as that at positive temperatures even though the regime of validity of the positive temperature linear response of an *individual* droplet vanishes as $T \to 0$.

The remaining effects of the nonlinear response of droplets are controlled by the *correction* to the distribution $\rho_L(F_L)$ for small $F_L$

$$\rho_L(F_L) \approx \frac{1}{\Upsilon L^\theta}\left[1 - c\left[\frac{F_L}{\Upsilon L^\theta}\right]^\phi\right] \tag{3.14}$$

from Eq. (2.5) of Ref. 3. These corrections will affect the initial magnetization and its decay due to droplets of scale $L \lesssim R_w$ by amounts of relative order $(Hq_M^{1/2}L^{d/2}/\Upsilon L^\theta)^\phi$ which is still small in the intermediate-field regime by Eq. (3.12). One might have naively expected larger corrections of relative magnitude $Hq_M^{1/2}L^{d/2}/T$ to occur in this intermediate-field regime, but they do not appear in any straightforward fashion, due to the above-mentioned cancellations.

It is quite possible, however, that other more subtle effects may come into play in the intermediate-field regime. For example, the field dependence, and therefore history dependence, of the barriers for droplet annihilation and creation and the barriers for the growth of domains during the waiting time may well have significant consequences. These will be left for future investigation.

### C. Strong fields

When the field is large enough that its nonlinear effects on scale $R_w$ are appreciable, i.e., when

$$q_M^{1/2}R_w^{d/2}H \gtrsim \Upsilon L^\theta , \tag{3.15}$$

then the behavior should change dramatically. In this regime, the effect of the field on the difference between the $H=0$ equilibrium states $\Psi$ and $\overline{\Psi}$ and the state of the system just before the field is turned off is at least as large as the effect of the finite-waiting time.

When Eq. (3.15) is strongly satisfied, the system is disordered by the field on scales larger than the magnetic correlation length $\xi_H$ and can equilibrate in the epoch

$$\ln t \sim \ln\tau_H \sim \frac{\Delta}{T}\xi_H^\psi \ll \ln t_w . \tag{3.16}$$

Thus in this regime the waiting-time effects will be negligible, since the system is in a paramagnetic state that can equilibrate well in times less than $t_w$.[21] Equation (3.15) just corresponds to the condition that the field $H$ is above the dynamic freezing line at time $t_w$ (frequency $\omega \sim t_w^{-1}$) discussed in Sec. I. There will still be some small waiting-time effects in this regime (as mentioned in Sec. I), since some regions will have anomalously large barriers which yield local relaxation on time scales longer than $t_w$.[15] These effects will be swamped by the dominant time dependence, which we now discuss.

In the large-field regime, the short-time behavior at epochs

$$\ln t \ll \ln\tau_H \tag{3.17}$$

will be the near equilibrium response. At late epochs,

$$\ln t \gg \ln\tau_H , \tag{3.18}$$

on the other hand, the nonequilibrium growth of ordered domains will dominate yielding Eq. (2.6). The crossover between these regimes will occur for $\ln t \sim \ln\tau_H$ and be roughly independent of $t_w$.

In not so large fields, where (3.15) is close to an equality so that

$$\ln\tau_H \sim \ln t_w , \tag{3.19}$$

the crossover between the short- and long-time regimes will be affected both by $H$ and $t_w$. Because of the expected broad distribution of barriers, the effects of the field will start to come in when $H$ is still considerably less than, but a significant fraction of, the freezing field $H_f(T,t_w)$ [i.e., where $T_f(H_f,t_w)=T$], so that there is likely to be a range of fields for which the crossover from short-time to long-time regimes is affected by both $H$ and $t_w$.

### D. Crossover regime

We have found that the basic behavior of the magnetization decay for all fields is an *early-epochs* regime with relaxation similar to that in equilibrium and a *late-epochs* regime with relaxation similar to that following an infinite-field quench. Ignoring possible intermediate-field effects, the crossover between these regimes occur at an epoch $\ln t$ which is roughly the minimum of $\ln t_w$ and $\ln\tau_H$. The natural guess is that much of the crossover between early and late-epoch behavior occurs on a logarithmic scale in time so that it is very broad.

It is quite possible, however, that there are also crossover effects at *times* $t \sim t_w$ due to, for example, lack of equilibrium of excitations with relaxation times $\tau$ of order $t_w$ (rather than just $\ln\tau \sim \ln t_w$). It is natural to hypothesize a scaling form, for, for example, the magnetization in the weak-field regime as a function of $t$ and $t_w$:

$$\frac{m(t,t_w)}{H} \sim \frac{1}{(\ln t)^{\theta/\psi}} \Sigma \left[ \frac{\ln t}{\ln t_w} \right] . \tag{3.20}$$

However it could well be that the scaling function $\Sigma(x)$ is not smooth for $x=1$, reflecting the suddenness on a logarithmic time scale of the processes with $t \sim t_w$. The scaling function $\Sigma(x)$ should go to a constant for $x \ll 1$ yielding Eq. (3.4) and decay as $x^{(\theta-\lambda)/\psi}$ for $x \gg 1$ to match (3.10). This suggests that the magnetization near crossover is

$$m(t_w,t_w) \sim m(t_n,t_w) \sim H(\ln t_w)^{-\theta/\psi} , \tag{3.21}$$

which can be substituted into Eq. (3.10) to yield the behavior for late epochs.

At this stage, it is not at all clear how to make further progress in understanding the crossover regime by, for example, establishing the validity of Eqs. (3.20) or (3.21), or investigating the effects of relaxation for $t \sim t_w$. It is possible that some toy models which contain some of the essential features of the dynamics may be useful; we leave pursuit of this for future research.

## IV. EFFECTS OF CHANGING TEMPERATURE

In addition to the simple "quench and wait" experiments discussed above, other system histories provide useful additional information about the spin-glass ordered phase. We consider here one such class of experiments which provides information on the temperature dependence of the spin-glass phase.[7–9]

We consider quenching to a temperature $T_1 < T_c$, waiting for a time $t_{w1}$, then changing rapidly to another temperature $T_2 < T_c$, waiting a time $t_{w2}$ and finally probing the system. For simplicity we consider only linear response measurements: either remanent magnetization decay in the weak-field regime a time $t$ after the second waiting period, or ac susceptibility measurements at frequency $\omega$.

During the wait at $T_1$, the system equilibrates towards the state at that temperature achieving a domain size $R_{w1}$. As discussed in Sec. VII of Ref. 3 the states at $T_2$ will differ on long length scales from those at $T_1$. In particular, the overlap correlation function introduced in Ref. 3

$$\Xi(i,j,T_1,T_2) \equiv \overline{\langle S_i S_j \rangle_{T_1} \langle S_i S_j \rangle_{T_2}} \tag{4.1}$$

will decay with increasing distance $|i-j|$ with a characteristic length scale $L_{\Delta T_{12}}$ which for a not too large $\Delta T_{12} \equiv |T_1 - T_2|$ satisfying

$$\Delta T_{12} < [T_c - \max(T_1,T_2)]$$

is given by[3,6]

$$L_{\Delta T_{12}} \sim |T_1 - T_2|^{-[(1/2)d_s - \theta]^{-1}} . \tag{4.2}$$

Thus immediately after the quench to $T_2$, the size of the domains of states $\Psi_{T_2}$ and $\overline{\Psi}_{T_2}$ cannot be larger than $L_{\Delta T_{12}}$. Their size will generally be

$$R_{q2} \approx \min(L_{\Delta T_{12}}, R_{w1}) , \tag{4.3}$$

although for $L_{\Delta T_{12}} \simeq R_{w1}$ they will probably be smaller than this by a factor of order unity. The domains will continue to grow during the second waiting period but this growth will not be appreciable unless $R_{w2}$ [from Eq. (3.2) with $t_{w2}, T_2$] is $\gtrsim R_{q2}$. The final domain size at the end of the waiting period will be

$$R_w \approx \max(R_{w2}, R_{q2}) , \tag{4.4}$$

again, being slightly larger for $R_{w2} \simeq R_{q2}$. At time $t$ after the end of the waiting periods, we will have behavior similar to the single-quench case but the crossover from early to late epochs will occur when

$$L_t \sim R_w , \tag{4.5}$$

which could be much later than $t_{w2}$ if $R_{q2} > R_{w2}$. We thus have three regimes depending on the relative magnitude of $R_{w2}$, $L_{\Delta T_{12}}$, and $R_{w1}$.

### A. Small temperature change

If $(T_1 - T_2)$, $t_{w1}$, and $t_{w2}$ are small enough then $L_{\Delta T_{12}}$ will be the greatest length and then

$$R_w = \max(R_{w1}, R_{w2}) , \tag{4.6}$$

with, if $R_{w1} \approx R_{w2}$, a small extra increase (which is negligible asymptotically) due to the extra evolution possible in time $t_{w1}$ plus $t_{w2}$. In this regime, observing the crossover from $t_{w1}$-dependent to $t_{w1}$-independent regimes as $t_{w1}$ is varied provides a measure of the relative barrier heights at the two temperatures. This crossover will occur at

$$\ln t_{w1}^c \approx \frac{T_2}{T_1} \frac{\Delta(T_1)}{\Delta(T_2)} \ln t_{w2} , \tag{4.7}$$

where $\Delta(T)$ is the amplitude of the scale dependence of the barriers. Note that near $T_c$, the dominant temperature dependence in Eq. (4.7) will be that of $\Delta(T)$ providing a means of extracting the exponent $\psi\nu$ with which $\Delta(T)$ vanishes as $T \to T_c^-$, as in Eq. (5.18) of Ref. 3.

### B. Large temperature change

If $L_{\Delta T_{12}} \ll R_{w2}$, then $R_w = R_{w2}$ and the waiting time at $T_1$ is ineffectual: The results will be the same as for a quench directly to $T_2$ from above $T_c$ and be essentially completely independent of $t_{w1}$.

### C. Intermediate temperature change

If

$$R_{w2} \ll L_{\Delta T_{12}} \ll R_{w1} , \tag{4.8}$$

which is most easily achieved with $T_1 > T_2$, then

$$R_w = L_{\Delta T_{12}} , \tag{4.9}$$

and the behavior will be independent of both $t_{w1}$ and $t_{w2}$.

This provides a possible way of obtaining the overlap length $L_{\Delta T_{12}}$, by measuring the crossover between early and late epochs in this regime as a function of $\Delta T_{12}$.

### D. Other effects

There will certainly be effects of temperature changes other than those discussed above. For example, even small temperature changes might affect barrier heights enough to yield more subtle behavior. In addition, since the system will tend to evolve at long times by almost always going over the smallest possible barriers, memory effects might result from such pseudodeterministic behavior, which could be at least partially reversible.

## V. APPROACH TO EQUILIBRIUM

We now consider corrections to the true equilibrium behavior which occur due to the finite domain size $R_w$ which is obtained after equilibrating for a time $t_w$ at temperature $T<T_c$. As mentioned earlier, we expect that for measurements at frequency $\omega$, the results should approach equilibrium as $|\ln\omega|/\ln t_w \rightarrow 0$; we are here interested in the form of the corrections to this asymptotic limit which is, of course, difficult to attain. We thus consider a measurement which probes length scales of order $L$; for an ac susceptibility measurement the length scale probed is

$$L_\omega \sim \left[ \frac{T\,|\ln\omega|}{\Delta} \right]^{1/\psi} . \tag{5.1}$$

We will assume that $L_\omega \gg \xi_-$, so that the walls are sharp on scale $L_\omega$; this is just the condition that $\omega\tau_c \lll 1$ with $\tau_c(T)$ the critical correlation time given, near $T_c$, by Eq. (1.1).

We thus focus on regions of size $\xi_- \ll L \ll R_w$. Since the distance between frozen-in domain walls is $R_w$, and they have fractal dimension $d_s$, we expect that in a volume $R_w^d$ there will be of order $(R_w/L)^{d_s}$ regions of size $L$ which the domain walls pass through. Thus, the frozen-in domain walls will affect a fraction

$$n_D(L) \sim \left[ \frac{L}{R_w} \right]^{d-d_s} \tag{5.2}$$

of the regions of size $L$.

Now, in a region of size $L$ which includes a frozen-in wall, the free energy of that section of wall will have an average value of order $(L/R)^{d_s}\Upsilon R_w^\theta$ and a random part of order $\Upsilon L^\theta$. The former arises since the section of wall has a fraction $(L/R)^{d_s}$ of the whole excess free energy on scale $R_w$ which is $\Upsilon R_w^\theta$. Since we are interested in the dynamics of excitations at scale $L$, we must ask what the effect of the frozen-in wall is on local excitations—especially droplets.

If the wall of a droplet coincides for a region of scale $L$ with the frozen-in wall, then it is likely to have, on average, a lower excitation free energy $F_L$ than it would have had if the frozen-in wall were not present. Thus the average excitation free energy of droplets of size $L$ is reduced by

$$\overline{\Delta F_L(R_w)} \sim p_0(L,R_w) \left[ \frac{L}{R_w} \right]^{d_s} \Upsilon R_w^\theta \tag{5.3}$$

due to the frozen-in walls, where $p_0(L,R_w)$ is the probability that a droplet of scale $L$ appreciably coincides with the frozen-in wall. This probability is related to the frozen-in wall density at scale $L$, $n_D(L)$, and the conditional probability, $\hat{p}_0(L)$ that a droplet on scale $L$ coincides appreciably with a section of a much larger wall which is within a distance $L$ of the droplet:

$$p_0(L,R_w) \approx n_D(L,R_w)\hat{p}_0(L) . \tag{5.4}$$

From the general scale invariance of probabilities of certain events,[4] and since two droplet excitations around spins separated by $\sim L$ will coincide with probability independent of $L$ (Sec. II of Ref. 3), we conjecture that

$$\hat{p}_0(L) \rightarrow \text{constant} \tag{5.5}$$

for large $L$.

We can now estimate the correction to the density of active droplets due to the changes in free energy as represented by Eq. (5.3): We expect that this will be parametrized roughly by a reduction in the effective stiffness $\Upsilon$ by an amount

$$\Delta\Upsilon \sim \left[ \frac{L}{R_w} \right]^{d-\theta} \Upsilon . \tag{5.6}$$

This yields results for, for example, the out-of-phase ac susceptibility at frequency $\omega$ measured after waiting time $t_w$:

$$\chi''(\omega,t_w) \approx \chi''(\omega,\infty) \left[ 1 - c_\chi \left[ \frac{|\ln\omega|}{\ln t_w} \right]^{(d-\theta)/\psi} \right] \tag{5.7}$$

for $|\ln\omega| \ll \ln t_w$, where we have used Eqs. (3.2) and (5.1) for $R_w$ and $L_\omega$. We thus expect that the corrections to physical quantities at frequency $\omega$ decay *very* slowly, so that, unless for some reason the constant $c_\chi$ happens to be anomalously small, it will be very difficult to extract the true equilibrium behavior $\chi''(\omega,\infty)$.

It is natural to guess that as long as $\omega t_w \ll 1$, $\chi''(\omega,t_w)/\chi''(\omega,\infty)$ will be a scaling function of $|\ln\omega|/\ln t_w$, the expression in the large brackets in (5.7) being the small-argument behavior of that scaling function.

## VI. DISCUSSION OF EXPERIMENTAL RESULTS

The picture developed in this paper and its companion[3] should enable some comparisons to be made between the theory and experiments on Ising spin glasses. Unfortunately, well-characterized Ising spin glasses have not been found and we are thus faced with the complications of Heisenberg systems with various kinds of anisotropy whose properties below $T_c$ have not yet been thoroughly investigated theoretically. We can, however, hope that many of the qualitative and some of the quantitative aspects of the picture developed here will apply to the more realistic systems and attempt comparison with experiment. Even a cursory discussion of the large body of experimental literature[1] on nonequilibrium phenomena below the freezing temperature in spin glasses is far beyond the scope of this paper, so we will, instead, restrict ourselves to a brief discussion of some aspects of a particular set of experiments.[7-9] We will focus on linear response measurements: ac susceptibilities and magnetization decay in the weak-field regime.

We first consider observation of the expected near-equilibrium behavior at measuring times $t$ or inverse frequencies $\omega^{-1}$ much shorter than the waiting time $t_w$ for which the system has been equilibrated. Experiments in this regime indicate behavior which is apparently independent of the waiting time for long enough $t_w$, and it has been shown that the expected relations between fluctuations and responses of different kinds hold in this regime.[7,8] From Eq. (3.4) we expect the quasiequilibrium component of the magnetization to decay as

$$m(t)/H \sim [\ln(t/\tau_c)]^{-\theta/\psi}$$

with $\tau_c$ a microscopic time for $T \sim \frac{1}{2}T_c$ and the critical correlation time for $T \to T_c^-$. Similarly the dissipative part of the susceptibility is expected from Eq. (4.18b) of Ref. 3 to behavior as

$$\chi''(\omega) \sim |\ln\omega\tau_c|^{-(1+\theta/\psi)} . \tag{6.1}$$

A useful way to parametrize data is via effective exponents:

$$\alpha_m(t) \equiv -\frac{d\ln m(t)}{d\ln t} \tag{6.2}$$

and

$$\alpha_{\chi''}(\omega) \equiv \frac{d\ln\chi''(\omega)}{d\ln\omega} , \tag{6.3}$$

which should be roughly constant over wide ranges of time or frequency, as found in experiments. From the theoretical predictions we expect

$$\alpha_m(t) \approx \frac{\theta/\psi}{\ln(t/\tau_c)} \tag{6.4}$$

and

$$\alpha_{\chi''}(\omega) \approx \frac{1+\theta/\psi}{|\ln\omega\tau_c|} \tag{6.5}$$

when the logarithmic factors are large. The observed values of $\alpha_m$ for $1s < t < 100s$ and $\alpha_{\chi''}$ for $\omega \sim 10$ Hz at $T/T_c \approx 0.7$ in $CdIn_{0.3}Cr_{1.7}S_4$ are 0.02 and 0.06, respectively.[8] For $T/T_c \sim 0.7$, $\tau_c$ should be $\sim t_0 \sim 10^{-12}$ so with the simplest possibility $\theta/\psi = 1$, we expect $\alpha_m \approx 0.03$ and $\alpha_{\chi''} \approx 0.07$ in not unreasonable agreement with experiment. The apparent exponents observed are far smaller than those of order $\frac{1}{2}$ found for the SK model,[22] yet it should be noted they are not much smaller than the power-law decays at the critical point where, for example, we expect $\alpha_m = (d-2+\eta)/2z \approx 0.07$.[23]

A rather surprising feature of some of the experiments is that the approach of $\chi''(\omega, t_w)$ to apparent equilibrium for long $t_w$ appears rather faster than expected from Eq. (5.7)[7]: Indeed the condition $|\ln\omega| \ll \ln t_w$ is never satisfied experimentally. Thus considerable caution should be exercised in assuming that the observed long-$t_w$ behavior is really equilibrium, especially since the fluctuation-dissipation theorem and other relations between correlation functions will be obeyed as long as quasiequilibrium exists with $\omega t_w \gg 1$.

We now turn to aspects of the dynamics further from apparent equilibrium. We first note that, qualitatively, the crossover behavior found in Sec. III for the decay of the magnetization after waiting time $t_w$ in the small-field regime, is also found experimentally: Crossover is observed for $\ln t \sim \ln t_w$, in at least some experiments.[7-9] Moreover, the effective exponent $\alpha_m(t)$ shows a marked increase in this regime consistent with the expectation from Sec. III. In three dimensions it is likely that $\lambda/\theta$ is quite large, perhaps as big as 10, since $\theta$ is small. This implies that if the crossover occurs quickly on a logarithmic time scale, then $\alpha_m$ will increase markedly for $t \gtrsim t_w$, perhaps in a way consistent with the experiments. What is somewhat surprising, however, is that at least some crossover appears quite rapid experimentally,[8] certainly considerably more rapid than $\ln t_w$. Unfortunately, obtaining enough decades in $t_w$ at the *shorter* $t_w$ end [where $\ln(t_w/\tau_c)$ varies most rapidly] to test scaling predictions such as Eq. (3.20) seems unlikely. More theoretical work on the crossover regime is definitely needed in order to make real comparisons with experiments.

Finally, we comment on one "double-quench" experiment in which the system was equilibrated for a time $t_{w1}$ at $0.96T_c$ then cooled to $0.72T_c$.[7] It was found that the behavior at $0.72T_c$ did not depend on the waiting time $t_{w1}$, suggesting agreement with the large-temperature-change regime of Sec. IV, an identification that is reasonable quantitatively. Further investigation of the effects of varying temperature differences would certainly be instructive. The reduced effects of equilibrating at a slightly lower temperature $T_2$ on the behavior when returned to $T_1$, found in the same series of experiments,[7] is also qualitatively consistent with the growth of barriers with decreasing temperature as in Eq. (4.7). At this stage, many of the other results from this rich set of experiments remain a mystery.

## VII. CONCLUSIONS

In this paper we have shown how a rich variety of nonequilibrium effects arise from a picture of the spin-glass phase with only two albeit temperature-dependent states. Much further work on various crossover regimes, the effects of temperature changes on barriers, and the consequences of more complicated Heisenberg systems needs to be done in order to make detailed comparisons with experiments. However, we hope that it is now apparent that the complicated features of the Parisi solution to the SK model,[1] particularly those involving many pure states, are not needed to explain much of the equilibrium and nonequilibrium behavior of spin glasses.

We end with a comment on why it is impossible to equilibrate spin glasses on long length scales, in contrast to the case of random-exchange ferromagnets which also exhibit logarithmic growth of domains following a quench to below $T_c$.

When random-exchange magnets and spin glasses are cooled slowly through $T_c$, the domain sizes will grow large just below $T_c$ since the large barriers are present only on scales larger than the correlation length $\xi_-$, and thus affect only times larger than the critical correlation time $\tau_c$. From scaling near $T_c$ we expect schematically

$$\frac{dR_t}{dt} \sim \frac{\xi_-}{\tau_c} \exp[-(R_t/\xi_-)^{\psi}] \tag{7.1}$$

so that at a *fixed* $T \lesssim T_c$

$$R_t \sim \xi_- [\ln(t/\tau_c)]^{1/\psi} , \tag{7.2}$$

implying that large-domain sizes are possible just below $T_c$. In random-exchange magnets, these domains will be qualitatively similar to the up and down domains far below $T_c$ so that once large domains are formed on slow cooling through $T_c$, they will persist for all $T$ (although they will grow only extremely slowly at lower temperature). For spin glasses, by contrast, the sensitivity of the states to temperature means that the large domains formed near $T_c$ will become small domains separated by networks of domain walls once temperature is decreased much further. Thus the underlying cause for the lack of large scale equilibrium in spin glasses below $T_c$ is the sensitivity of the states to temperature, a feature that might at first have seemed rather esoteric.

*Note added.* While the writing of this paper was being completed, we received a paper by Koper and Hilhorst[24] which presents a somewhat similar domain picture of aging effects in spin glasses. However, they proceed more phenomenologically, assuming a power-law growth of domains, in contrast to our logarithmic growth law (2.1), which is based on activated dynamic scaling. They also have an exponential relationship between the long-time remanent magnetization and the domain size, while we find a power-law relationship (2.6).

Recent work by one of us shows that, in the presence of a remanent magnetization, the domain size $R_t$ can, in principle, be measured by neutron scattering.[28]

## APPENDIX A: DECAY OF REMANENT MAGNETIZATION IN THE SHERRINGTON-KIRKPATRICK MODEL AND THE MATTIS SPIN GLASS

In order to estimate the new nonequilibrium dynamic exponent $\lambda$ introduced to Eq. (2.6), we have performed Monte Carlo simulations of the Sherrington-Kirkpatrick[5] (SK) model and the pure Ising model which is equivalent, via a random gauge transformation, to the Mattis spin glass.[10] The results are presented and discussed in this appendix.

Let us consider a general quench of an Ising spin system in which the system is prepared in the initial random configuration $\{S_i(0)=\pm 1\}$ and then its evolution at temperature $T$ is watched. The overlap with its initial configuration is

$$q_0(t) = \frac{1}{N} \sum_i \overline{\langle S_i(0) S_i(t) \rangle} , \tag{A1}$$

where the average here is over initial configurations, thermal histories, and, for random systems, realizations of the disorder. The sum in (A1) is over all $N$ spins of the system. The remanent magnetization $m(t)$ after a quench from infinite-magnetic field is the special case where the initial configuration is fully magnetized with $S_i(0)=+1$. For a spin glass with no ferro- or antiferromagnetic bias in the distribution of the exchanges $\{J_{ij}\}$ we should have $m(t)=q_0(t)$, because the fully magnetized configuration is not special in any way, the system being statistically invariant under random gauge transformations. Thus we can introduce the exponent $\lambda$ for *any* random or nonrandom system via

$$q_0(t) \sim \frac{1}{R^{\lambda}(t)} , \tag{A2}$$

where $R(t)$ is the domain size at time $t$.

For the infinite-range spin glass (the SK model) we must use a slightly different definition of $\lambda$, since a linear domain size $R(t)$ cannot be simply defined. However, we might hope to define something analogous to the number of spins in a domain which scales as $R^d(t)$ and then we would expect

$$q_0(t) \sim \left[ \frac{1}{R^d(t)} \right]^{\lambda/d} . \tag{A3}$$

We can then define $(\lambda/d)$ for the SK model which can presumably be viewed as some kind of realization of the limit $d \to \infty$. We define $R^d(t)$ as follows: The Hamiltonian is

$$H = \frac{1}{\sqrt{N}} \sum_{i<j} J_{ij} S_i S_j , \tag{A4}$$

where the exchanges are independently distributed with mean zero and variance one. ($J=\pm 1$ was used in the simulations.) Let us make two replicas $\alpha$ and $\beta$ of the same system (with the same $\{J_{ij}\}$). Start the two replicas in different random initial configurations $\{S_i^{\alpha}(0)\}$ and $\{S_i^{\beta}(0)\}$. Then define

$$\tilde{\chi}_{SG}(t) = \frac{1}{N} \overline{\left\langle \left[ \sum_i S_i^{\alpha}(t) S_i^{\beta}(t) \right]^2 \right\rangle} . \tag{A5}$$

For $t \to \infty$ (and finite $N$) the two replicas are at equilibrium and $\tilde{\chi}_{SG}(t)$ becomes the (untruncated) Edwards-Anderson or spin-glass susceptibility $\tilde{\chi}_{SG}$. For $T<T_c$ and large $N$, $\tilde{\chi}_{SG} \sim N$. For finite times $\tilde{\chi}_{SG}(t)$ is a measure of the number of correlated spins in each "domain," thus

$$\tilde{\chi}_{SG} \sim q_{EA}^2 R^d(t) . \tag{A6}$$

Thus we will define $(\lambda/d)$ for the SK model via

$$q_0(t) \sim \tilde{\chi}_{SG}(t)^{-(\lambda/d)} . \tag{A7}$$

Numerical results for $q_0(t)$ versus $\tilde{\chi}_{SG}(t)$ at temperature $T=T_c/2$ are presented in Fig. 1. For each realization of the $\{J_{ij}\}$ we have made six replicas and run them

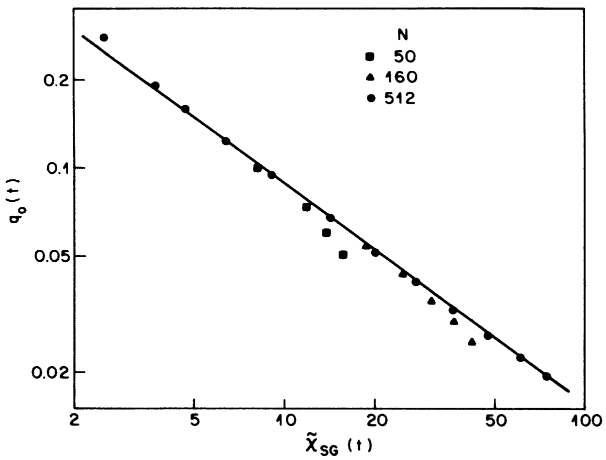


FIG. 1. Monte Carlo simulation results for the overlap with the initial spin configuration $q_0(t)$ defined in Eq. (A1) vs the "domain size" $\tilde{\chi}_{SG}(t)$ defined in Eq. (A5) for SK models of $N$ spins at $T=0.5T_c$. The straight line has slope $\lambda/d=0.75$ [Eq. (A6)]. The finite-size effects enter in the smaller lattices at $\tilde{\chi}_{SG}\simeq N/5$.

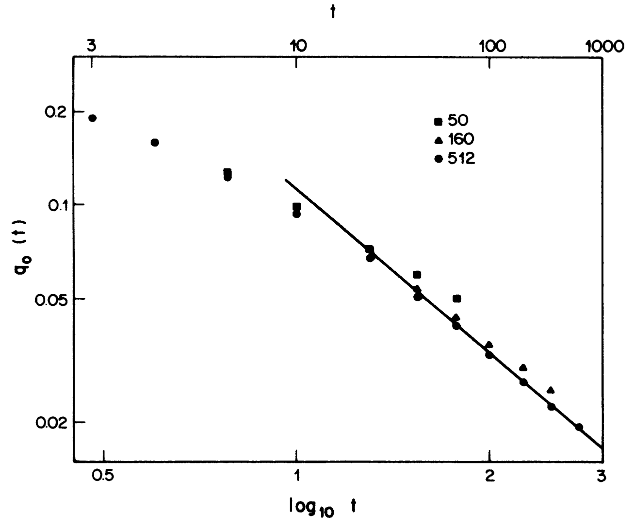


FIG. 2. The time dependence of $q_0(t)$ for SK models of $N$ spins at $T=0.5T_c$. The straight line has slope $(\lambda/\psi)_{\text{eff}}=1.75$ [Eq. (2.6)]. Time is measured in Monte Carlo steps per spin (MCS). The curvature and finite-size effects suggest that the true asymptotic slope is significantly larger: $\lambda/\psi>1.75$.

in parallel, measuring $q_0(t)$ for each replica and $\tilde{\chi}_{SG}(t)$ from the various overlaps. For $N=50$ we have run 5400 realizations, the final data point in Fig. 1 being an average over 50–70 Monte Carlo steps per spin (MCS) at intervals of 10 MCS. For $N=160$ we ran 1000 realizations, and the final data point is an average over 240–420 MCS at 10 MCS intervals. For $N=512$ we ran 600 realizations and the final data point is for 430–740 MCS. The earliest data point in Fig. 1 is a measurement at 2 MCS. We have used the Metropolis algorithm and selected spins to flip at random. The measured statistical errors range up to 3% for $q_0(t)$ and 2% for $\tilde{\chi}_{SG}(t)$ and are thus always less than or of order the size of the points in Fig. 1. The straight line drawn on the log-log plot in Fig. 1 has slope $(\lambda/d)=0.75$. The data for $N<512$ show that the finite-size effect becomes significant around $\tilde{\chi}_{SG}(t)\simeq N/5$ so the data for $N=512$ is probably not finite-size affected. Note that the result for the SK model

$$(\lambda/d)\simeq 0.75 \tag{A8}$$

is precisely midway between the bounds (2.7) and (2.10). The *statistical* error on this estimate is about $\pm 0.02$; of course systematic errors could be larger.

It has been suggested that the *equilibrium* spin autocorrelations in the SK model decay with a temperature-dependent exponent.[22] The nonequilibrium exponent we are measuring here may also be temperature dependent, in which case (A8) applies only to the particular temperature we simulated, namely $T=T_c/2$. Preliminary simulations at $T=0.8T_c$ indicate a possibly larger $(\lambda/d)$; more careful simulations at various temperatures are under way in order to examine this question.

The time dependence of $q_0(t)$ for the SK model is shown in Fig. 2. The prediction (2.6) for short-range systems is that $q_0(t)\sim(\ln t)^{-(\lambda/\psi)}$ so we have plotted $q_0(t)$ versus $\log_{10}t$ on a double-logarithmic graph where this would yield a straight line. [Note this prediction is for finite-dimensional spin glasses and need not apply to the SK model.] The times are measured in MCS so our microscopic time is implicitly assumed to be 1 MCS. The plot shows significant curvature and stronger finite-size effects than Fig. 1. The slope of the longest time data for $N=512$ yields effective exponents $(\lambda/\psi)_{\text{eff}}\simeq 1.75$. In view of the observation that the slope of the data increases both with time and system size this estimate should be viewed as a lower bound on the asymptotic exponents: $\lambda/\psi\geq 1.75$. Kinzel[25] has instead fit $q_0(t)$ with a power-law decay to a lattice-size dependent constant: $q_0\approx q_\infty+At^{-a}$. Our data also fit this well with $q_\infty\simeq 0.012$ and $a\simeq 0.6$ for $N=512$ (Kinzel[25] obtained $a\simeq 0.4$ at $T=0.4T_c$.) Note that for larger times and finite $N$ Kinzel's form must fail because $q_0(t)\to 0$ for $t\to\infty$. We do not feel we can draw any conclusion about the form of the asymptotic time decay of $q_0(t)$ in the $N\to\infty$ SK model from these data; they represent too small $N$ and $t$. Indeed, for $N=\infty$ it is by no means clear, either theoretically or numerically, whether $q_0(t)$ decays to zero or to a positive residual value for $t\to\infty$.

### 1. Mattis spin glasses

We have also simulated $q_0(t)$ for pure nearest-neighbor Ising models on square ($d=2$) and simple cubic ($d=3$) lattices. A random gauge transformation on these systems produces the Mattis spin glass.[10] Our simulations are done at zero temperature, again using the Metropolis algorithm and random selection of spins to update. At zero temperature in these nonrandom systems, the total amount of domain wall left per unit volume (area) is simply proportional to the excess energy per spin (above the ground state) remaining in this system, $\Delta e(t)$. This provides a measure of the domain size via

$$R(t)\sim[\Delta e(t)]^{-1} . \tag{A9}$$

Our results for $q_0(t)$ versus $\Delta e(t)$ are presented in Fig. 3. The measurements shown are made at 0 MCS, 1 MCS,

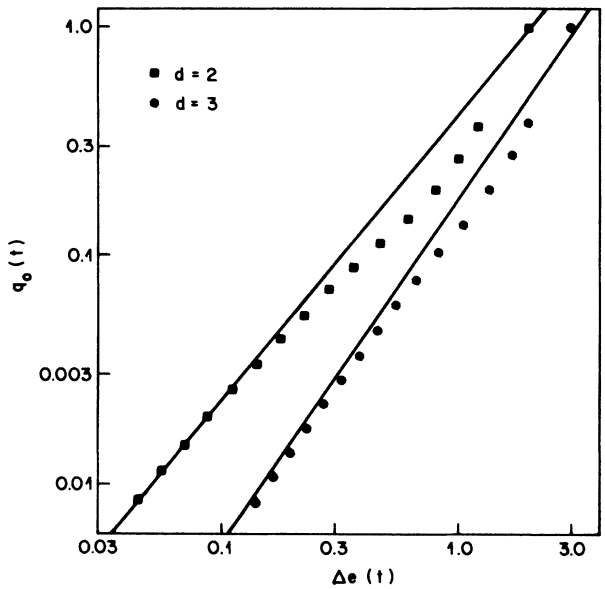


FIG. 3. Results for the Mattis spin glass on square ($d=2$) and simple cubic ($d=3$) lattices at zero temperature. The straight lines have slopes $\lambda=1.25$ and $\lambda=1.50$. The linear domain size is proportional to $[\Delta e(t)]^{-1}$.

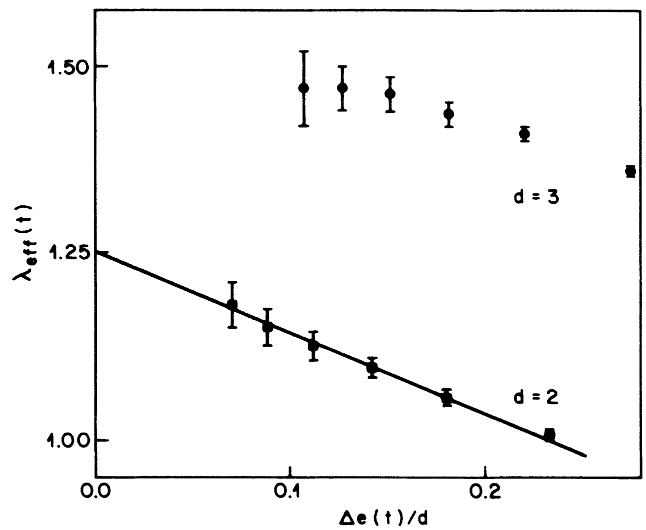


FIG. 4. The effective exponent $\lambda_{\text{eff}}(t)$ [Eq. (A9)] vs $\Delta e(t)/d$ for the Mattis spin-glass data shown in Fig. 3. For $d=2$ the extrapolation to our conjectured result $\lambda=1.25$ is shown. For $d=3$ the data are consistent with our bound [Eq. (2.7)], $\lambda \geq 1.50$.

and every $\frac{1}{5}$ of a decade thereafter until the final measurement at $10^{2.8} \cong 631$ MCS. These data represent 80 histories of a $N=400^2$ sample for $d=2$ and 41 histories of a $N=80^3$ sample for $d=3$. Smaller size lattices were also run to verify that these data are not affected by the finite size of the lattices. The measured statistical errors range up to 3.5% for both $q_0(t)$ and $\Delta e(t)$ at the latest time in $d=3$, and are thus smaller than the points in Fig. 3. The straight lines in Fig. 3 represent $\lambda=\frac{5}{4}$, which we conjecture (see below) may be the exact asymptotic exponent for this system in $d=2$, and $\lambda=\frac{3}{2}$, which is the lower bound (2.7) on $\lambda$ for $d=3$. The time dependence of $\Delta e(t)$ is the expected[16] $\Delta e \sim t^{-1/2}$ for $d=2$, while it is somewhat slower for $d=3$, presumably due to some zero-temperature freezing effect. In $d=1$ the result $\lambda=1$ can be obtained straightforwardly at $T=0$ from the exact solution of Glauber.[20]

There is significant curvature in the data on the log-log plot in Fig. 3. In order to analyze this data, we follow the procedure recently introduced for studies of spinodal decomposition[26] and define an effective exponent for a decade in time via

$$\lambda_{\text{eff}}(t)=\frac{\log_{10}[q_0(t)/q_0(10t)]}{\log_{10}[\Delta e(t)/\Delta e(10t)]} . \tag{A10}$$

This effective exponent is shown versus $\Delta e(t)$ in Fig. 4. It is natural to guess that the deviation of $\lambda_{\text{eff}}(t)$ from the asymptotic $\lambda$ is due to effects that vanish as the surface-to-volume ratio of the domains. Thus one expects that the deviations may vanish as

$$\lambda_{\text{eff}}(t)-\lambda \sim \Delta e(t) . \tag{A11}$$

For $d=2$ the straight line in Fig. 4 represents a reasonable extrapolation, showing the data are quite consistent with the conjecture $\lambda=1.25$. For $d=3$ we see the effective exponent for the times measured is always less than or equal to our lower bound of 1.50. However, the trend is such that any asymptotic exponent in the range $1.50 \leq \lambda \leq 1.65$ would be quite consistent with both Eq. (2.7) and our data.

For $d=2$ this system has the property that only one of the two Ising domains ("up" and "down") can pércolate. When viewed as a continuum system, both domains are at their percolation threshold, each occupying 50% of the area after the random quench. Thus it is natural to guess that the exponent $a$ in our scaling argument Eq. (2.8) is just that of percolation. In (2.8) $f$ is the deviation from 50% of, say, "up" spins, so we may make the identification $f=(p-p_c)$, where $p_c=50\%$ is the percolation threshold for the domain. The percolation correlation length scales[27] as $\xi_p \sim (p-p_c)^{4/3}$ for $d=2$ so the natural scaling variable to put in (2.8) is $fL^{3/4}$ or $a=\frac{3}{4}$. If we then go through the argument following Eq. (2.8) and assume $\lambda$ is equal to the bound obtained, we have $\lambda=d-a=\frac{5}{4}$. Considerable caution is in order here since the process which produces the two incipient percolation clusters certainly has different statistics from independent (Bernoulli) percolation and one must assume that these correlations are irrelevant for the above argument to work. The agreement with the simulation results suggests however that this assumption may be valid.

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