Singular dynamic scaling on fractal lattices

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Finite-size scaling calculations are performed for the critical dynamics of the ferromagnetic Ising model on fractal lattices. We confirm the predictions of Henley's heuristic theory of singular dynamic scaling and compare our results with those recently obtained by Kutasov *et al.* using Monte Carlo methods. It is found that fractal lattices which have odd coordination have an exponentially large number of metastable states but that these states do not necessarily play a significant role in the critical dynamics.

I. INTRODUCTION

It is fairly well established that there is a violation of standard dynamic scaling theory[1] in the Glauber[2] dynamics of Ising spin models on critical percolation clusters and regular fractals. The first work suggesting the breakdown of standard dynamic scaling was a simple heuristic theory by Henley[3] which predicted a new "singular" dynamic scaling in which $\tau \propto \xi^z$ as in standard dynamic scaling but with $z \sim 1/T$ instead of $z = \text{const}$. Here, τ is the relaxation time, ξ the thermal correlation length, and z the dynamic critical exponent. His calculations were more intuitive than exact, but relied on the convincing idea that the dynamics of systems with zero temperature phase transitions could be described by energy barrier activation processes using the Arrhenius law. Subsequent Monte Carlo (MC) simulations by Kutasov *et al.*[4] demonstrated that Henley's theory was qualitatively correct but seemed to be incorrect in some of the qualitative details. An approximate real-space rescaling study by Harris and Stinchcombe[5] on dilute Ising systems near the percolation threshold also confirmed the essential correctness of the new "singular" dynamic scaling form of the relaxation time. However, a more formal real-space rescaling calculation by Achiam[6] on regular fractals predicted standard dynamic scaling. Experiments have been performed on dilute magnets near the percolation threshold but the situation is unclear as low enough temperatures to detect the $1/T$ dependence of z have not yet been used. Presently available data do not exclude the new "singular" dynamic scaling.

Our approach to this interesting problem is to use a form of finite-size scaling based on an idea of Kinzel.[7] The starting point of these calculations is the Glauber master equation for the probability $P(\{s_j\})$ of the spin configurations and can be written as

$$\frac{dP(\{s_j\})}{dt} = -\sum_{j=1}^{j=N} [P(\{s_j\})W_j(\{s_j\}) - P(\{-s_j\})W_j(\{-s_j\})] , \quad (1)$$

where $W_j(\{s_j\})$ is the spin flip probability per unit time for the jth spin which depends on the configuration of the surrounding spins. We shall use the conventional Glauber form $[1-s_j \tanh(E_j/kT)]/2\alpha$ for this transition rate where E_j is the local field on site j and α is the relaxation time of an isolated spin. N is the number of lattice sites while $\{-s_j\}$ denotes the spin configuration with the jth spin flipped. Taking these probabilities to be the components of a vector and assuming an exponential time dependence, this equation becomes a $2^N \times 2^N$ matrix eigenvalue problem for the relaxation times. By explicitly diagonalizing this matrix we can find the temperature dependence of the slow modes (i.e., those modes which have a divergent relaxation time at the phase transition and hence are responsible for the critical slowing down.) By repeating the calculation for a larger system the length scaling of the relaxation time can be found and therefore z. We confirm in detail Henley's result for the length dependence of τ in the regime $L \ll \xi$. In addition we investigate the relationship between the number of slow modes and the number of metastable states. The basic advantage of our approach over, say, Monte Carlo simulation is that we can probe the regime $L \ll \xi$ without excessive computer time. On the other hand, our method is limited to smaller clusters than MC simulation because of storage problems.

In the next section we review the work of Henley and Kutasov *et al.* in more detail for the case of the Sierpiński gasket (SG) and the Mandelbrot-Given (MG) fractals. Section III introduces Kinzel's dynamic real-space renormalization transformation and we apply it in lowest order to the SG and MG fractals to recover the results of Henley's theory. We then go on to present our numerical results, which can be regarded as an application of the Kinzel idea to higher order. The role of metastable states is then discussed with reference to odd coordinated fractal lattices. We finally present a conclusion.

II. SINGULAR DYNAMIC SCALING

Dynamic scaling is a generalization of static scaling familiar in thermal critical phenomena. Here, it will be taken to mean that the time-dependent magnetization obeys the following functional equation:

$$M(L,K,t)=M(L/b,K',t/\Omega)\ , \tag{2}$$

where a renormalization-group (RG) transformation with scale factor b has been carried out and both the thermal variable, K, and time have been appropriately scaled. The above equation can be shown to imply both the usual thermal scaling properties and also $\tau \propto \xi^z$ where $\Omega=b^z$. Conventional dynamic scaling occurs when Ω is a finite constant at the critical point while singular dynamic scaling occurs when the eigenvalue, Ω, is divergent at the phase transition. Henley discovered a simple intuitive argument for $\tau \propto \xi^{A/T+B}$ which we shall now describe, as applied to Ising dynamics on the SG and MG fractals. The construction of these fractals is shown in Fig. 1.

The nearest-neighbor ferromagnetic Ising model has a zero temperature phase transition on both the MG and SG fractals. This suggests that the most divergent part of the relaxation time τ is due to thermal activation over energy barriers. In particular, since at very low T the system will prefer to lie in phase space somewhere near one of the two degenerate ground states C and $\overline{C}$, $\min\{E_{\max}(C\rightarrow\overline{C})\}$, the minimal maximum energy on the phase-space trajectory connecting the two ground states will be the crucial energy scale in the problem. On the two fractal lattices which we consider here

$$E_{\max}/2J=Z\ \ln L+\text{const}\ , \tag{3}$$

with $Z=1/\ln 3$ for the MG and $Z=2/\ln 2$ for the SG. It is this logarithmic scaling of the energy barrier which leads to singular dynamic scaling. The origin of this logarithmic scaling is particularly easy to see in the case of the MG fractal by considering the motion of a Bloch wall through the lattice. The energy required to progress through a part of the lattice which is merely a scaled down version of the whole lattice is just $2J$. This can be seen by realizing that the energy to flip the entire scaled down-version is simply zero and thus the energy required to reverse its magnetization is related to the energy of flipping the connecting spin. Hence, for the MG, the Bloch wall needs $2J$ for each new level of the hierarchy which it enters, so that $\Delta E/2J=n+1$. But $L=3^n$ giving

$$\Delta E/2J=\ln L/\ln 3+1\ , \tag{4}$$

and $Z=1/\ln 3$. In addition to this logarithmic scaling, Henley assumes that $\tau(L,T)=\tau_0\exp(\Delta E/kT)$ which is the Arrhenius law for thermally activated processes. By substitution we obtain

$$\tau(L,T)=\tau_0 L^{\beta K},\quad (L \ll \xi)\ , \tag{5}$$

where $\beta=2Z$. The above form is only valid in the low-temperature region. At higher temperatures it can be shown[3] by using the dynamic scaling assumption and the thermal scaling equation that

$$\tau(L,K)\propto \xi^{\beta K/2},\quad (L \gg \xi)\ . \tag{6}$$

So, by the combination of two convincing assumptions, a temperature-dependent critical exponent is found. A similar result has been derived for critical percolation clusters using a heuristic scaling theory.[5] This theory considers the dynamics to be controlled by the motion of Bloch walls and involves a rate suppression factor originating from branching of Bloch walls reminiscent of the calculation of ΔE for the MG fractal. Recent Monte Carlo simulations of Ising critical dynamics on regular fractals by Kutasov *et al.* confirm the validity of this form and show that Ω has an exponential dependence on K. However, they disagree in their quantitative predictions and obtain a different value of β from that of Henley. In the next section we show that a divergent eigenvalue Ω can be obtained from a simple block-spin analysis.

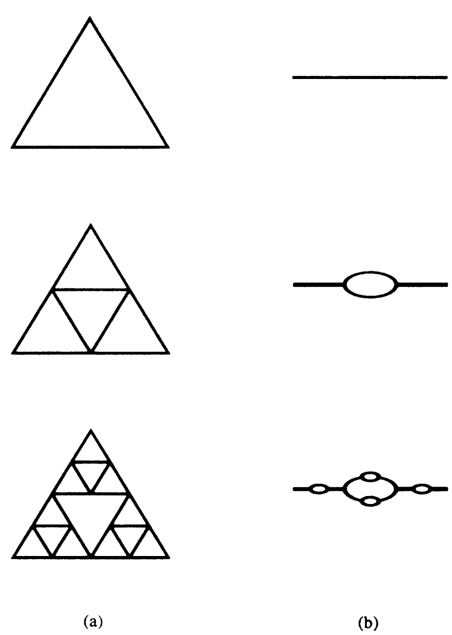

FIG. 1. The first stages in the construction of the (a) Sierpiński gasket and (b) Mandelbrot-Given fractal.

III. FINITE-SIZE SCALING CALCULATIONS

Our approach to Ising spin dynamics is based on an idea by Kinzel.[7] It is a block-spin method similar in spirit to that of Niemeijer and van Leeuven's[8] for statics. Kinzel associates τ', the rescaled relaxation time, with the slowest mode of an isolated block. He applies the method to the triangular lattice and obtains a reasonable (finite) value for z. For the fractal lattices which we consider (MG and SG) his method is straightforward to apply as at their lowest stages of iteration they are small clusters and, by construction, directly related by a scale transformation. To see how this works for the SG consider a triangular cluster, which is its building block. This has eight modes with the slowest having a relaxation time given by $\tau=\alpha/[1-\tanh(2K)]$ where α is the time scale for the relaxation of an isolated spin. This can be written as a scaling equation

$$\tau'\simeq\tfrac{1}{2}e^{4K}\tau,\quad K\gg 1\ . \tag{7}$$

At the (zero temperature) phase transition e^{4K} is divergent. This is the origin of the "singular" dynamic scaling. Writing the scale factor as $\Omega=2^{4K/\ln b-1}$ we see that

Henley's value for β is recovered. We can carry out a similar calculation for the MG fractal where we compare the ratios of the longest relaxation times for the first two structures shown in Fig. 1(b). This gives

$$\tau' \simeq \tfrac{8}{9}3^{2K/\ln 3}\tau, \quad K \gg 1 , \tag{8}$$

so that $\beta = 2/\ln 3 = 2Z$ again in agreement with the Henley prediction. It therefore seems that at the lowest order this rescaling calculation gives a temperature-dependent z in agreement with Henley and Kutasov *et al.* However, it sharply disagrees with a calculation by Achiam[6] for the SG which produces a finite value for Ω at the critical point resulting in $z = 2d_f$ where d_f is the fractal dimension of the lattice. Of course, calculations such as the ones given above are not exact and the result may be an artifact of the approximation. To show that this is not the case we consider scaling with larger clusters.[9]

Kinzel identifies the τ scaling as the ratio of the longest relaxation times of an isolated "block" to that of an isolated spin. We extend this method to higher order by using larger blocks and comparing their longest relaxation times. For an N site lattice this involves diagonalizing the $2^N \times 2^N$ master matrix numerically. Glauber's equations do not specify the spin-flip rate uniquely; it is, however, convenient to use the conventional choice $[1 - s_i \tanh(E_i/kT)]/2\alpha$ as it results in the eigenvalues of the master matrix lying in the bounded range $(-N, 0)$. The longest relaxation time is $-1/\lambda_1$ where λ_1 is the smallest (in magnitude), nonzero eigenvalue of the master matrix, $\mathcal{A}$. Due to the conservation of probability $\mathcal{A}$ always has a trivial zero eigenvalue. Using the up-down symmetry of the Ising model $\mathcal{A}$ can be split into two blocks of order 2^{N-1}. The slowest mode was always found in the antisymmetric subspace which is reasonable as it would be expected to couple strongly to the magnetization. Using this symmetry reduction the eigenvalues for lattices of sizes up to 12 sites could be calculated. By using sparse storage mode and direct iteration it was possible to find the slowest mode of the 15 site SG.

We now present our results for the SG. Figure 2 shows that the slowest relaxation time for the six and fifteen site SG has a simple exponential dependence on inverse temperature. By comparing the slopes of these two graphs a direct check on Henley's prediction $\tau(L,K) = L^{2ZK}$ can be made. It is found to hold to very high accuracy and is presumably exact. A similar plot is shown in Fig. 3 for the MG fractal to confirm that $\tau(L,K)/\tau(L/b,K) = e^{2K}$. Kutasov *et al.* were unable to provide this confirmation because the excessively long relaxation time makes computation with the MC algorithm very expensive.

More interesting is to plot $\Omega(b,K) = \tau(L,K)/\tau(L/b,K')$ where K' is the scaled K found from the static RG. The results for this are shown in Figs. 4 and 5 and correspond to the plots in Kutasov *et al.* Both graphs show curvature near the region $L \simeq \xi$. Taking the slopes of these graphs in the region $\xi \gg L$ we confirm the value of β predicted by Henley, namely, 1.82 for the MG and 5.77 for the SG fractals. It appears to us that the results of Kutasov *et al.* ($\beta = 2.1 \pm 0.1$ for the MG and $\beta = 4.2 \pm 0.2$ for the SG) which are respectively higher and lower than the predicted values, can be explained by the fact that

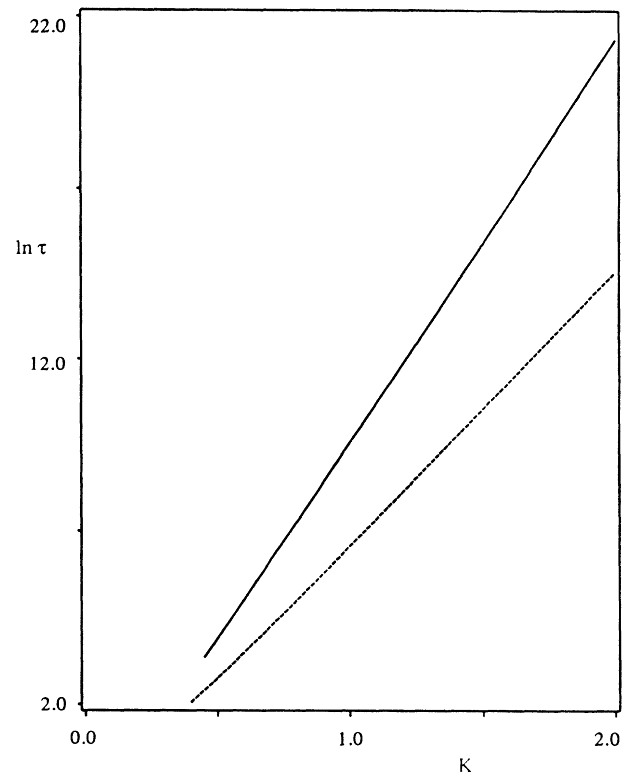

FIG. 2. A plot of $\ln\tau$ against K for the 6 site (dotted curve) and 15 site (solid curve) Sierpiński gasket.

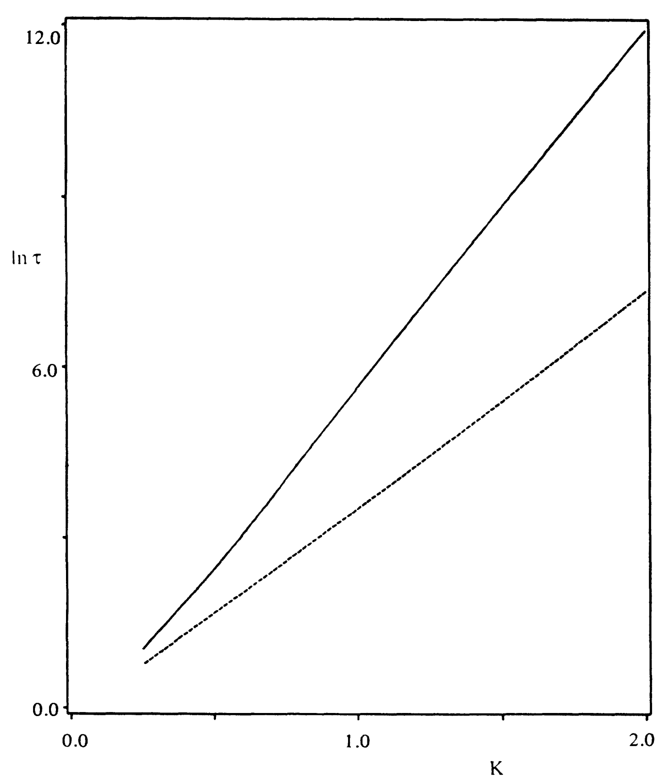

FIG. 3. A plot of $\ln\tau$ against K for the 4 site (dotted curve) and 12 site (solid curve) MG fractal.

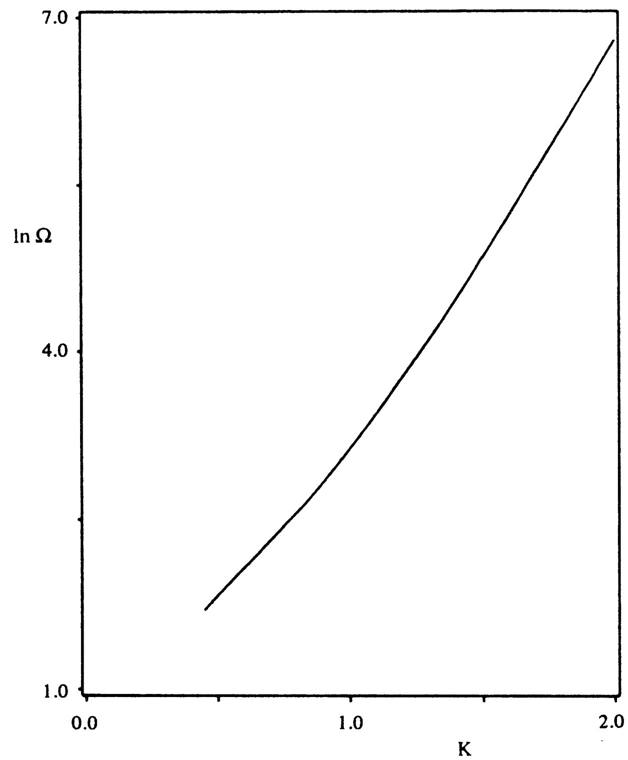

FIG. 4. Dependence of $\ln\Omega$ on K for the SG obtained by comparing the 15 site system with the 6 site system.

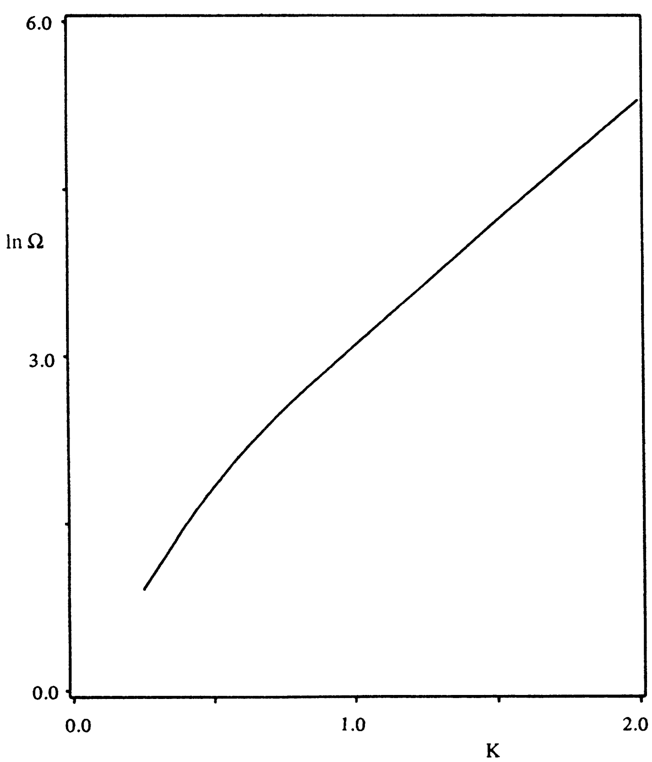

FIG. 5. Dependence of $\ln\Omega$ on K for the MG fractal obtained by comparing the 12 site system with the 4 site system.

their calculations were performed in the regime $L \sim \xi$.

In obtaining the results above only the longest relaxation time was considered, but diagonalization of the master matrix results in a whole spectrum of relaxation times. These additional modes are potentially important if they are slow. It is, in fact, quite easy to show that the number of slow modes is equal to the number of states stable against *any* single spin flip. Intuitively this result makes sense because if the system finds itself in one of these metastable states just above $T=0$, then it will only be able to jump out with some rate depending on $\exp(-\Delta E/kT)$ where ΔE is the energy barrier. A more complete proof can be given if we choose the Glauber weight function. Making the substitution

$$\tilde{P}(\{s_i\})=P(\{s_i\})/\sqrt{P_{\rm eqm}(\{s_i\})} \tag{9}$$

in the master equation, where $P_{\rm eqm}$ is the equilibrium (Boltzmann) distribution, is equivalent to a similarity transformation which symmetrizes the master matrix. Using the detailed balance conditions for the transition rate the off-diagonal elements become

$$\sqrt{W_j(\{-s_j\})W_j(\{s_j\})}=1/\cosh(E_j/kT)\ , \tag{10}$$

which vanish as $T\to 0$ if $E_j\neq 0$ (which is the case for the odd-coordinated lattices but not the even coordinated). The diagonal terms are

$$\sum_{j=1}^{j=N}[1-s_j\tanh(E_j/kT)]\ . \tag{11}$$

But $\tanh(E_j/kT)$ becomes $\mathrm{sgn}E_j$ as $T\to 0$. Now the condition for metastability is $s_j\mathrm{sgn}E_j=1\ \forall i$ so that the diagonal terms are zero for all the metastable states and so the result follows. Because of its even coordination number only the ground states of the SG are stable against single spin flips and so there is only one slow mode. Our

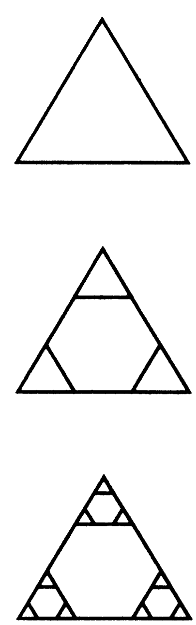

FIG. 6. Construction of the 3-simplex fractal lattice.

approximation of neglecting all the other modes is, then, quite justified. However, the MG fractal has exponentially many metastable states and so there are exponentially many diverging relaxation times near the phase transition. These metastable states have both spins orientated parallel to each other in all of the elementary "bubbles." We find that in the relaxation of the magnetization from the completely aligned state, the additional slow modes do not play a role at low temperature. This is because all of the metastable states have energies far above the ground state in these finite-sized systems.

In order to clarify the role played by metastable states, we have also considered the 3-simplex lattice shown in Fig. 6. This lattice has the same fractal dimension and statics[10] as the SG but in contrast to the SG it has an exponentially large number of metastable states which correspond to the spins in each elementary triangle having the same orientation independent of the other triangles. These metastable states also have a self-similar geometry and this structure suggests that the spectrum of slow modes of the nth order lattice is closely related to the whole spectrum for the $(n-1)$th order lattice. Numerically we found them to be almost identical but with the relaxation times of the modes of the $(n-1)$th order lattice being all scaled by a factor of $\exp(2K)$. Thus even if there are many slow modes, the scaling factor is common to them all and the magnetization will still have the scaling form as in (2) with a temperature-dependent z. The presence of the metastable states may change the functional form of the relaxation from exponential to either power law or possibly stretched exponential but will not change the value of z. The actual form will depend on how the distribution in energy of metastable states accumulates near the ground state.

IV. CONCLUSION

In conclusion, our finite-size scaling calculations agree in both their qualitative and quantitative details with the heuristic theory of Henley and hence provide strong support for the new form of "singular" dynamic scaling. We also argue that although the odd coordinated lattices which we consider have exponentially many metastable states, like a spin glass, they do not play a significant role in the critical dynamics except to possibly change the functional form of the relaxation.

ACKNOWLEDGMENT

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[1]B. I. Halperin and P. C. Hohenberg, Phys. Rev. **117**, 952 (1969).

[2]R. J. Glauber, J. Math. Phys. **4**, 294 (1963).

[3]C. L. Henley, Phys. Rev. Lett. **54**, 2030 (1980).

[4]D. Kutasov, E. Domany, and E. Pytte, Phys. Rev. B **35**, 3354 (1987).

[5]C. K. Harris and R. B. Stinchcombe, Phys. Rev. Lett. **56**, 869 (1986).

[6]Y. Achiam, Phys. Rev. B **31**, 4732 (1985).

[7]W. Kinzel, Z. Phys. B **29**, 361 (1978).

[8]T. Niemeijer and J. M. van Leeuwen, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1976).

[9]For the $1d$ chain, our method would also predict singular dynamic scaling at the first step. However, when larger lengths are considered and the rescaling of K is taken into account, the conventional result $z=2d=2$ is correctly recovered.

[10]B. W. Southern and M. Knežević, Phys. Rev. B **35**, 5036 (1987).