Larmor clock reexamined

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The Larmor clock, a thought experiment designed by Baz' to measure scattering times, is reexamined in the context of one-dimensional tunneling by wave packets, narrow in wave-number space, through arbitrary static barriers. It is shown that the Larmor clock is, in general, inaccurate when used on a single Fourier component, i.e., on a stationary state. A reliable Larmor clock depends for its operation on properly designed wave packets. Precise conditions for its reliability are given. Thus, the apparent conflict between previous Larmor clock results for tunneling times, and results arrived at by other methods, is removed. When properly set, the Larmor clock shows the classic phase times. Stationary-state calculations of Larmor and dwell "times" are shown to represent coherent superpositions of, in general, widely different collision events. The relative merits of the Larmor clock and the center-of-gravity clock are discussed. Finally, it is shown that, for an arbitrary symmetric potential, the reflection and transmission times coincide.

I. INTRODUCTION

With the prospect of fast electronic devices based on the principle of resonant tunneling,[1] the old question of the time needed to complete a tunneling process has acquired new urgency. Reliable answers can, of course, only be found after the circumstances have been more precisely specified. However, even the simplest case conceivable, that of a wave packet, narrow in wave-number space, tunneling coherently through (or being reflected from) a static potential barrier in one spatial dimension, has been controversial in recent years.[2–12] In particular, there is an apparent conflict between results[9–12] that one way or another converge on the classic phase times[13–15] (to be defined in Sec. V) as appropriate for the process at hand and different expressions for tunneling times based on the Larmor clock concept.[3,4,7]

The Larmor clock was first introduced by Baz'[3] and is an appealing thought experiment designed to measure the time associated with scattering events. The mechanism basic to the clock is the constant Larmor precession of a spin in a uniform magnetic field. Shortly after its introduction by Baz' Rybachenko[4] used the Larmor clock to study the case of primary interest to us, that of tunneling in one dimension. Büttiker[7] has recently taken up the Baz-Rybachenko idea and based further developments on it.

In the present paper we reexamine the Larmor clock and remove, for the restricted class of tunneling problems considered here, the apparent conflict alluded to above. One of the reasons for the controversy that has raged over tunneling times is, in our opinion, the proliferation of fast, plausible, and clever arguments. In contrast, we shall be slow, careful, and elementary throughout. This strategy will reveal the source of confusion related to Larmor times, as an incorrect mode of calculations adopted in Refs. 3, 4, and 7. We show that, for the Larmor clock to be properly set, it is necessary to consider wave packets and their evolution in time. The conditions to be put on a reliable Larmor clock will be discussed qualitatively in Sec. II, and substantiated quantitatively in subsequent sections. The elements that go into quantitative calculations are collected in Sec. III. In Sec. IV we check that the Larmor clock is a reliable one. More precisely, we check the conditions discussed qualitatively in Sec. II, in a more quantitative setting. The results of a properly set Larmor clock are read off in Sec. V. In that section our previous work[12] on tunneling times (hereafter referred to as I) will provide a useful backdrop. In fact, we show in Sec. V that a good Larmor clock gives results in essentially perfect agreement with those of I. In particular, the classic phase times[13–15] are read off to lowest order. Section VI is devoted to a closer scrutiny of the stationary-state strategy used in Refs. 3, 4, and 7. We rederive the general results for spin rotations found by Rybachenko,[4] using relations between inverse and direct collisions developed in Appendix A. This rederivation, the details of which are relegated to Appendix B, graphicly demonstrates why the calculations of Refs. 3, 4, and 7, although correct, cannot, in general, be interpreted in terms of tunneling times. The problem of clocks and the relations between the results of the present paper and those of I are, finally, summed up in the discussion of Sec. VII.

II. QUALITATIVE DISCUSSION OF THE LARMOR CLOCK

We first discuss qualitatively the Larmor clock in the context of tunneling through a one-dimensional barrier, as shown in Fig. 1. The static barrier $V(x)$, the form of which is left unspecified, is confined to the x interval (b,a). For simplicity, we assume that the constant potential energy is the *same*, $V=0$, for $x>a$ as for $x<b$. Generalization is straightforward and will be briefly mentioned in Sec. V. An infinitesimal, homogeneous magnetic field (or, rather, magnetic induction) $\mathbf{B}=\hat{\mathbf{z}}B_0$, pointing in the z direction, fills the x interval (x_1,x_2). Outside this

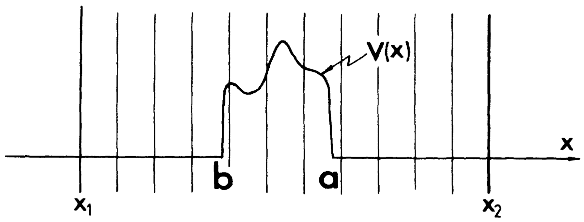

FIG. 1. The Larmor clock configuration for one-dimensional tunneling. The static barrier $V(x)$ is confined to the x interval (b,a). The homogeneous magnetic field $\mathbf{B}=\hat{\mathbf{z}}B_0$ spans the x interval (x_1,x_2).

interval, the magnetic field vanishes. The field interval (x_1,x_2) is always assumed to cover completely the interval (b,a) of the barrier, $V(x)$.

Imagine now a particle impinging upon the field region from the left. Prior to entering this region, the spin (of the spin one-half particle) was polarized in (say) the x direction. Inside the field, the spin should precess at a constant, infinitesimal, Larmor rate, $\omega_L=gqB_0/2m$. [Here g is the gyromagnetic ration, q the charge, and m the effective mass (for simplicity, assumed constant) of the particle.] By assumption, only a fraction of a full rotation of the spin has been completed when the particle, after having tunneled through the barrier with some probability, leaves the field region at $x=x_2$. An ideal measurement of the y component of the spin, s_{yT}, after the particle has moved beyond x_2, should therefore reveal the time, $t_T=\lim_{\omega_L\to 0}(2/\hbar\omega_L)\,|\,s_{yT}\,|$, spent in the field region and, in particular, the transmission time τ_T through the barrier. Basic to this conclusion is the assertion that, in the limit $\omega_L\sim B_0\to 0$ the magnetic field does not disturb the tunneling process itself.

The language of the above simple minded presentation of the Larmor clock is basically classical. When applied to an essentially quantum mechanical problem like tunneling some care must be exercised. First of all, the moving point particle must be replaced by a wave packet.[16] For the notion of time spent in the field region to make sense, the wave packet must be well localized on the scale set by the width, x_2-x_1, of this region. One can only expect the Larmor clock to run smoothly for the period when the entire packet is inside the field. Secondly, for a Larmor clock interpretation to be possible, the quantum mechanical process must also (like the classical one) be reducible to three sequential steps: (1) The packet enters the field, (2) the tunneling process (transmission or reflection) is completed, and (3) the particle leaves the field region. For a Larmor clock calculation, one should *not* solve the stationary problem, in which all three steps are treated as one coherent process. That procedure will, in general, lead to incorrect results, as will be shown in Sec. VI. In other words, the Larmor clock argument does *not* apply to each Fourier component separately!

In order to make contact with our previous work[12] on tunneling times in I we shall sharpen the above conditions somewhat. We have shown in I that, with packets narrow in $\mathbf{k}$ space (i.e., with width, σ, small) the tunneling times for transmission and reflection are, to lowest order, the classic phase times.[13–15] (We also calculated appropriate corrections to those lowest-order results.) Basic to those calculations was the assumption that the entire tunneling process could be treated coherently. Roughly speaking, this requires that the width in x space Δx of the wave packet is much larger than that of the barrier a-b. To prevent interference between the processes of entering and leaving the field region on the one hand, and the tunneling process on the other, we must therefore require that

$$\left.\begin{matrix} b-x_1 \\ x_2-a \end{matrix}\right\} \gg \Delta x \geq \sigma^{-1} \gg a-b \ . \tag{2.1}$$

Fortunately, thought experiments come cheaply, and nothing prevents us from fulfilling conditions (2.1). The qualitative picture given in this section will be basic to the calculations of Secs. IV and V.

III. THE PARTS OF THE CLOCKWORK

In this section we establish the basic elements that, according to the qualitative discussion in Sec. II, are needed for a Larmor clock calculation. First, the dynamical equation is given and the initial packet characterized. Next, the process of entering or leaving the field region is discussed. Finally, the formal description of the tunneling process is established.

A. The dynamical equation

The two-component Schrödinger equation (the Pauli equation) defining the dynamics of our problem reads

$$i\hbar\frac{\partial\Psi}{\partial t}=\left[\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]1-S(x)\frac{\hbar\omega_L}{2}\sigma_z\right]\Psi\ , \tag{3.1}$$

where m is the (effective) mass of the particle, $V(x)$ the (arbitrary) potential barrier confined to the x interval (b,a), 1 is the unit 2×2 matrix, $S(x)$ is unity on the x interval (x_1,x_2) and zero outside (x_1,x_2), $\omega_L=gqB_0/2m$ is the Larmor frequency in the infinitesimal field $\mathbf{B}=\hat{\mathbf{z}}B_0$, and σ_z the Pauli matrix. We write the two-component wave function in the form

$$\Psi(x,t)=\frac{1}{\sqrt{2}}\begin{bmatrix}\psi_+(x,t)\\ \psi_-(x,t)\end{bmatrix}\ . \tag{3.2}$$

Note that (3.1) implies that the effects of the infinitesimal field $\mathbf{B}$ on the orbital motion of the particle have been neglected, reducing the problem to a one-dimensional one.

The expectation value of the spin is

$$\langle \mathbf{s}(t)\rangle=\frac{\hbar}{2N}\int dx\,\Psi^\dagger(x,t)\boldsymbol{\sigma}\Psi(x,t)\ , \tag{3.3}$$

where $\Psi^\dagger$ is the Hermitian conjugate of Ψ, $\boldsymbol{\sigma}=\{\sigma_x,\sigma_y,\sigma_z\}$ are the Pauli matrices, and N the normalization

$$N=\int dx\,\Psi^\dagger(x,t)\Psi(x,t)\ . \tag{3.4}$$

B. The initial wave packet

The initial wave packet is assumed to fulfill the following conditions. (i) It is located entirely to the left of the field region. (ii) The two spin components are described by the same function:

$$\psi_\pm(x,t)=\int \frac{dk}{2\pi}\phi(k)e^{ikx}\equiv\int \frac{dk}{2\pi}\,|\,\phi(k)\,|\,e^{i\xi(k)+ikx}\ . \quad (3.5)$$

(iii) The wave number spectrum contains positive wave numbers only,[17] $\phi(k<0)=0$. (iv) The wave packet can be considered as narrow in **k** space in the context of tunneling (see I, and below). The form (3.5) ensures, when used in (3.3), that the spin initially points in the x direction, i.e., $\langle s_x(0)\rangle=\frac{1}{2}\hbar$ and $\langle s_y(0)\rangle=\langle s_z(0)\rangle=0$. Since the interval (b,a) is, by assumption, covered by (x_1,x_2), the potential energy $V(x)$ vanishes outside the field region. Thus the Fourier component with wave number k in the distribution $|\,\phi(k)\,|^2/2\pi$ carries an energy, $E=\hbar^2k^2/2m$. This total energy is a constant of the motion, and the time factor associated with $\phi(k)$ is thus $\exp(-iEt/\hbar)=\exp(-i\hbar k^2t/2m)$ throughout.

C. Transitions into, and out of, the field region

Entering the field region, the spin-up and spin-down components see different constant potential energies, $\mp\frac{1}{2}\hbar\omega_L$. (The upper sign always refers to the component with the spin in the $+\hat{z}$ direction.) In a Fourier decomposition of the incoming packet, the two components which had the common wave number, k, for $x<x_1$, will be characterized by different wave numbers, $k_\pm$, for $x>x_1$, where

$$k_\pm^2=k^2\pm m\omega_L/\hbar\ . \quad (3.6)$$

The corresponding plane-wave solution is

$$\psi_k(x)=\begin{cases} e^{ikx}+B_1(k,k_\pm)e^{-ikx}, & x<x_1 \\ A_1(k,k_\pm)e^{ik_\pm x}, & x>x_1 \end{cases} \quad (3.7)$$

when the tunneling process further down the line is disregarded. Continuity of $\psi_k(x)$ and $d\psi_k(x)/dx$ at $x=x_1$ gives

$$A_1(k,k_\pm)=\frac{2k}{k+k_\pm}e^{i(k-k_\pm)x_1}\ ,$$
$$B_1(k,k_\pm)=\frac{k-k_\pm}{k+k_\pm}e^{2ikx_1}\ . \quad (3.8)$$

Similarly, the process of leaving the field region at $x=x_2$ is described by

$$\psi_k(x)=\begin{cases} e^{ik_\pm x}+B_2(k_\pm,k)e^{-ik_\pm x}, & x<x_2 \\ A_2(k_\pm,k)e^{ikx}, & x>x_2, \end{cases} \quad (3.9)$$

where A_2 and B_2 follow from A_1 and B_1 of (3.8) by the operations $x_1\to x_2$ and $k\rightleftarrows k_\pm$. Inverse collisions, when the incoming particle impinges on the potential step from the right, rather than from the left, are included in the discussion of general relations in Appendix A.

D. The tunneling process

In the coherent tunneling process, plane waves $e^{ik_\pm x}$ impinge on the static potential, $V(x)$. We treat the stationary solution of the tunneling process as exactly known. When the field steps at x_1 and x_2 are disregarded, we can write the exact stationary solution (with $k>0$) as

$$\chi(x;k)=\begin{cases} e^{ikx}+B(k)e^{-ikx}, & x<b \\ \chi_V(x;k), & b<x<a \\ A(k)e^{ikx}, & x>a\ . \end{cases} \quad (3.10)$$

The complex amplitudes $B(k)=|\,B(k)\,|\,e^{i\beta(k)}$ and $A(k)=|\,A(k)\,|\,e^{i\alpha(k)}$ are taken as known functions of k and so is the wave function $\chi_V(x;k)$ in the region where $v(x)\neq 0$. This implies that the transmission and reflection probabilities, $T(k)=|\,A(k)\,|^2$ and $R(k)=|\,B(k)\,|^2=1-T(k)$, are taken as known functions.

When the field steps at x_1 and x_2 are disregarded, and the definition of $\chi(x;k)$ in (3.10) is extended to the entire x axis these stationary solutions of the tunneling problem obey the orthogonality relation

$$\int dx\,\chi^*(x;k)\chi(x;k')=2\pi\delta(k-k') \quad (3.11)$$

when it is understood that k and k' are both positive.

We can now formulate more precisely what we mean by a packet, narrow in k space, namely,

$$\sigma\equiv[\langle(k-\langle k\rangle)^2\rangle]^{1/2}\ll k_c\equiv\langle k\rangle$$
$$\sigma dT(k_c)/dk\ll T(k_c)\ , \quad (3.12)$$

where $\langle\cdots\rangle$ refers to the initial wave packet.

IV. CHECK ON THE CLOCKWORK

In the next section we shall use the Larmor clock to measure tunneling times. However, confidence in the results requires that the smooth operation of the clockwork has been checked. In particular, the *conditions* under which the clock runs at a constant rate should be clarified. That clarification is the purpose of the present section.

We assume, as discussed in Secs. II and III, that the field region $x_1<x<x_2$ is sufficiently wide that there is a sizable time interval when the transmission past $x=x_1$ has been completed, but no part of the packet has yet reached the far end, $x=x_2$. The two components of the wave function in the field region can then be written as

$$\psi_\pm(x,t)=\int\frac{dk}{2\pi}\phi(k)A_1(k,k_\pm)\times\chi(x;k_\pm)e^{-i\hbar k^2t/2m}\quad (x_1<x<x_2)\ , \quad (4.1)$$

where $\chi(x;k_\pm)$ are the exact eigenfunctions $(k_\pm>0)$ in the field region (with reflections from x_2 and x_1 appropriately neglected), and $A_1(k,k_\pm)$ picks that part of the initial Fourier decomposition, $\phi(k)$, which has been transmitted past $x=x_1$.

The time-dependent spin, $\langle \mathbf{s}(t)\rangle$, is given by (3.3), with $\psi_\pm$ of the form (4.1). We start by calculating the normalization N_B appropriate to this region. This calculation already contains the essential element of our line of thought. From (3.4) one has

$$N_B(t)=\tfrac{1}{2}\int_{x_1}^{x_2}dx[\psi_+^*(x,t)\psi_+(x,t)+\psi_-^*(x,t)\psi_-(x,t)] \; . \tag{4.2}$$

(Subscript B indicates the field region.) In general, $N_B(t)$ depends on time. However, in the time interval when the packet is (for all practical purposes) completely within the field region, we can with impunity let $x_1\to-\infty$ and $x_2\to+\infty$ in (4.2). The resulting N_B is time independent, since we can now appeal to the orthogonality relation (3.11) for $\chi(x;k)$:

$$\begin{aligned} N_B &=\tfrac{1}{2}\int_{-\infty}^{\infty}dx\int\frac{dk'dk}{(2\pi)^2}\phi^*(k')\phi(k)e^{-i\hbar(k^2-k'^2)t/2m} \\ &\quad\times\sum_\pm A_1^*(k',k'_\pm)A_1(k,k_\pm) \\ &\quad\times\chi^*(x;k'_\pm)\chi(x;k_\pm) \\ &=\tfrac{1}{2}\int\frac{dk}{2\pi}\,|\phi(k)|^2\sum_\pm\frac{k_\pm}{k}\,|A_1(k,k_\pm)|^2 \; . \end{aligned}\tag{4.3}$$

In (4.3) we changed integration variables, and used Eq. (3.6) so that $k'dk'=k'_\pm dk'_\pm$. Since $T_1(k,k_\pm)=(k_\pm/k)\,|A_1(k,k_\pm)|^2$ is the transmission probability into the field region at $x=x_1$, (4.3) reads

$$N_B=\tfrac{1}{2}\int\frac{dk}{2\pi}\,|\phi(k)|^2[T_1(k,k_+)+T_1(k,k_-)] \; . \tag{4.4}$$

From (3.8) it follows that both transmission probabilities in (4.4) are unity within corrections of $O(\omega_L^2)$. Normalization of the initial wave packet then gives $N_B=1-O(\omega_L^2)$. Constant terms of $O(\omega_L^2)$ will be consistently neglected henceforth, and we can therefore put $N_B=1$.

The numerator in the expression for $\langle s_z(t)\rangle_B$ has precisely the same form as that for N_B, except that $\sum_\pm\to\frac{1}{2}\hbar\sum_\pm(\pm)$. The resulting spin component can therefore be read off from (4.4) as (to leading order)

$$\langle s_z(t)\rangle_B=\tfrac{1}{2}\hbar\int\frac{dk}{2\pi}\,|\phi(k)|^2[T_1(k,k_+)-T_1(k,k_-)] \; . \tag{4.5}$$

It is easy to see, from (3.8), that $\langle s_z(t)\rangle_B$ is of $O(\omega_L^3)$. This is consistent with $\langle s_z(t)\rangle=0$ being in a constant of the motion: A fraction of $O(1)$ of the initial packet is transmitted into the field region with[7] $\langle s_z(t)\rangle_B=O(\omega_L^3)$. The remaining small fraction of $O(\omega_L^2)$ is reflected at $x=x_1$, with a z-component spin of $O(\omega_L)$. The *total* z component is the weighted sum of the two, and remains zero.

The essential premise for arriving at (4.5) is the statement: For the time interval of interest to us, that part of the initial packet which is transmitted past $x=x_1$ is *entirely* inside the region $x_1<x<x_2$. As a consequence, the orthogonality relation (3.11) could be used. The essential *message* from (4.5) is seen by comparison with (4.10) and (4.11) below, and is the expected one: No time information is contained in $\langle s_z(t)\rangle_B$. (Statements to the contrary are found in Ref. 7.)

Calculations of the x and y components of the spin can be carried out along the same lines. We sketch the derivation of $\langle s_x(t)\rangle_B$, which is given by

$$\langle s_x(t)\rangle_B=\frac{\hbar}{4N_B}\int_{x_1}^{x_2}dx[\psi_+^*(x,t)\psi_-(x,t)+\psi_-^*(x,t)\psi_+(x,t)] \; . \tag{4.6}$$

Based on the same philosophy as before, we introduce (4.1) and extend the x interval to the whole line:

$$\langle s_x(t)\rangle_B=\frac{\hbar}{4N_B}\sum_\pm\int\frac{dk'dk}{(2\pi)^2}\phi^*(k')\phi(k)e^{-i\hbar(k^2-k'^2)t/2m}A_1^*(k',k_\pm')A_1(k,k_\mp)\int dx\,\chi^*(x;k'_\pm)\chi(x;k_\mp) \; . \tag{4.7}$$

The orthogonality relation (3.11), and (3.6), give, after suitable transformations of the integration variables

$$\langle s_x(t)\rangle_B=\frac{\hbar}{4N_B}\sum_\pm\int\frac{dk}{2\pi}\frac{k^2}{k_\mp k_\pm}\phi^*(k_\mp)\phi(k_\pm)e^{-i\hbar(k_\pm^2-k_\mp^2)t/2m}A_1^*(k_\mp,k)A_1(k_\pm,k) \; . \tag{4.8}$$

Drop terms of $O(\omega_L^2)$, write $\phi(k)=|\phi(k)|e^{i\xi(k)}$, and use (3.8) to get

$$\langle s_x(t)\rangle_B=\frac{\hbar}{4}\sum_\pm\int\frac{dk}{2\pi}\,|\phi(k)|^2\exp\{i[-\xi(k_\mp)+\xi(k_\pm)+(k-k_\mp)x_1+(k_\pm-k)x_1\mp\omega_L t]\} \; . \tag{4.9}$$

Keeping only terms of $O(\omega_L)$ in the exponent, and using (3.6) one finally arrives at

$$\langle s_x(t)\rangle_B=\frac{\hbar}{2}\int\frac{dk}{2\pi}\,|\phi(k)|^2\times\cos\left[\omega_L\left[t-\frac{m}{\hbar k}[x_1+\xi'(k)]\right]\right] , \tag{4.10}$$

where $\xi'(k)\equiv d\xi/dk$. Similarly one finds

$$\langle s_y(t)\rangle_B=-\frac{\hbar}{2}\int\frac{dk}{2\pi}\,|\phi(k)|^2\times\sin\left[\omega_L\left[t-\frac{m}{\hbar k}[x_1+\xi'(k)]\right]\right] . \tag{4.11}$$

Strictly speaking, consistency requires that one should replace cos[⋯] by unity in (4.10) and sin[⋯] by its argument in (4.11). However, the time dependence is exact, and even for ω_L small, the parentheses in (4.10) and (4.11) can be arbitrarily large. A good case can therefore be made for retaining the trigonometric functions as they stand.

From (4.10) and (4.11) it is clear that the spin, averaged over the entire field region, precesses at a constant rate, ω_L. Some smearing will result from the averaging over initial phases. In the extreme case, when $|(\phi(k)|^2$ is a sufficiently narrow distribution around k_c that one can replace $(m/\hbar k)$ by $(m/\hbar k_c)\equiv v_c^{-1}$ and $\xi'(k)$ by $\xi'(k_c)\approx -\langle x(0)\rangle$ (see I), (4.11) reduces to

$$\langle s_y(t)\rangle_B \approx -\frac{\hbar}{2}\sin(\omega_L\{t-[x_1-\langle x(0)\rangle]/v_c\}) \tag{4.12}$$

and similarly for (4.10). This shows, as expected, that precession only begins at the time when the free initial packet starting from $x=\langle x(0)\rangle$ has reached the field region at $x=x_1$.

The results (4.10) and (4.11), which apply to the entire wave function in the field region, are still not sufficient to show that the precession rate is constant during the tunneling process. The point is that, in addition to (4.10) and (4.11), we need similar results for the packets tunneling through, and reflected from, the barrier $V(x)$ *separately*. With $V(x)=O(1)$, the transmitted and reflected packets develop z components $\langle s_z(t)\rangle_T$ and $\langle s_z(t)\rangle_R$ that separately[7] are of $O(\omega_L)$ whereas they combine to give (4.5):

$$T\langle s_z(t)\rangle_T + R\langle s_z(t)\rangle_R = \langle s_z(t)\rangle_B = O(\omega_L^3)\approx 0\ . \tag{4.13}$$

One must therefore face the possibility that the precession is disturbed during the tunneling process with the result that the transmitted and reflected packets acquire separate *precession* phase shifts, but in such a way that the *combined* average, $\langle \mathbf{s}(t)\rangle_B$ precesses at a constant rate throughout. This would render the Larmor clock quite unreliable.

We must therefore repeat the arguments leading to (4.10) and (4.11) but this time restricted to that part of the wave packet which is transmitted through the barrier. The same type of reasoning again applies, except that the condition for appeal to the orthogonality relation is now that the (for all intents and purposes) entire *transmitted* packet is located on the line between the far side of $V(x)$, $x=a$, and $x=x_2$ (see Fig. 1). With this restriction, calculations go through as before, with the results

$$\langle s_y(t)\rangle_T = -\frac{\hbar}{2N_T}\int\frac{dk}{2\pi}|\phi(k)|^2 T(k) \times\sin\left[\omega_L\left[t-\frac{m}{\hbar k}[x_1+\xi'(k)]\right]\right],$$

$$\langle s_z(t)\rangle_T = \frac{\hbar}{2N_T}\int\frac{dk}{2\pi}|\phi(k)|^2\frac{m\omega_L}{\hbar k}T'(k)\ , \tag{4.14}$$

$$N_T=\int\frac{dk}{2\pi}|\phi(k)|^2 T(k)$$

and similarly for $\langle s_x(t)\rangle_T$. The normalization N_T is now less than unity, due to the finite transmission probability $T(k)$ for the barrier $V(x)$. The result for $\langle s_z(t)\rangle_T$ shows that, for the time interval when the premise for the calculation is valid, $\langle s_z\rangle_T$ is time independent and is clearly of $O(\omega_L)$, as stated above, in the context of (4.13). Although "large," $\langle s_z\rangle$ does not have the essential property of a clock: uniform motion. The crucial message of (4.14) follows from the fact that the constant part of the phase in sin[⋯] is precisely the *same* as in (4.11). (The x component tells the same story at this point.) The conclusion is immediate: Under the conditions basic to the results (4.10), (4.11), and (4.14), the Larmor clock runs at a constant rate, unperturbed by the tunneling process. Analogous results hold for the reflected packet.

We have thus shown the Larmor clock to be a reliable one, *provided that the field region is chosen sufficiently* wide that it accommodates the incoming packet, the complete tunneling process, *and* the outgoing transmitted and reflected packets. Under this condition, the three processes, transmission into the field, reflection or transmission from the barrier, and transmission out of the field, can be treated sequentially. It is only under sequential conditions that the Larmor clock can be trusted. We return to a completely coherent calculation in Sec. VI where the pitfalls of this approach will be explicitly demonstrated.

V. READING THE LARMOR CLOCK

After having checked that the Larmor clock runs at a constant rate in the field region, we are now in a position to read off the total time spent there. This must be done separately for reflection from, and transmission through, the barrier. We concentrate on transmission here.

What is needed is a calculation of (say) the y component of the spin $\langle s_y\rangle_T$ for the wave packet transmitted through $V(x)$, and subsequently, out of the field region at $x=x_2$. After the packet has passed $x=x_2$ precession stops. The two components of the wave function (sequentially calculated) are in the field free region

$$\psi_\pm(x,t)=\int\frac{dk}{2\pi}\phi(k)A_1(k,k_\pm)\times A(k_\pm)A_2(k_\pm,k)e^{ikx-i\hbar k^2 t/2m}\quad (x>x_2)\ . \tag{5.1}$$

The three sequential processes are represented by the three amplitudes in (5.1). The normalization is

$$N_T(t)=\tfrac{1}{2}\sum_\pm\int_{x_2}^{\infty}dx\,\psi_\pm^*(x,t)\psi_\pm(x,t) \tag{5.2}$$

and asymptotically, one can replace x_2 by $-\infty$, to get the time independent N_T in analogy with (4.3) and (4.4):

$$N_T=\tfrac{1}{2}\sum_\pm\int\frac{dk}{2\pi}|\phi(k)|^2 T_1(k,k_\pm)T(k_\pm)T_2(k_\pm,k)\ . \tag{5.3}$$

In the symmetric combination (5.3) one makes an error of $O(\omega_L^2)$ by deleting all subscripts $\pm$. To $O(1)$, therefore

$$N_T=\int \frac{dk}{2\pi} |\phi(k)|^2 T(k) \tag{5.4}$$

in agreement, up to corrections of $O(\omega_L^2)$, with (4.14).

The y component of the spin is

$$\langle s_y \rangle_T=\frac{-i\hbar}{2N_T}\sum_{\pm}(\pm)\int_{x_2}^{\infty} dx\, \psi_{\pm}^*(x,t)\psi_{\mp}(x,t) \tag{5.5}$$

with $\psi_{\pm}$ given by (5.1). The orthogonality condition of the plane wave in (5.1) which follows from letting $x_2 \to -\infty$ in (5.5) simply gives $\delta(k'-k)$ in this case, in contrast to $\delta(k'_{\pm}-k_{\mp})$ which results when the final state is inside the field region, as in (4.3). Thus

$$\langle s_y \rangle_T=-\frac{i\hbar}{4N_T}\int \frac{dk}{2\pi}|\phi(k)|^2 \sum_{\pm}(\pm)A_1^*(k,k_{\pm})A_1(k,k_{\mp})A^*(k_{\pm})A(k_{\mp})A_2^*(k_{\pm},k)A_2(k_{\mp},k) . \tag{5.6}$$

The absolute value of the product of amplitudes reduces to $|A(k)|^2=T(k)$, within corrections of $O(\omega_L^2)$. The phase picks up a term of $O(\omega_L)$ from every amplitude, with the result [use (3.8), (3.10), and (3.6)]

$$\langle s_y \rangle_T=-\frac{\hbar}{2N_T}\int \frac{dk}{2\pi}|\phi(k)|^2 T(k) \times \sin\left[\frac{\omega_L m}{\hbar k}[x_2-x_1+\alpha'(k)]\right] . \tag{5.7}$$

In (5.7) $\alpha(k)$ is the phase of the transmission amplitude $A(k)$. Since (5.7) is calculated according to the proper Larmor clock prescription, we can interpret the result as an average time of transmission t_T from x_1 to x_2 (divide $\langle s_y \rangle_T$ by $-\frac{1}{2}\hbar\omega_L$ and go to the limit $\omega_L \to 0$):

$$\langle t_T \rangle_T=\frac{\int \frac{dk}{2\pi}|\phi(k)|^2 T(k) v(k)^{-1}[x_2-x_1+\alpha'(k)]}{\int \frac{dk}{2\pi}|\phi(k)|^2 T(k)} . \tag{5.8}$$

Here $v(k)=\hbar k/m$ is the group velocity of the free particle.

The expression (5.8) can be generalized in a straightforward fashion to the case when the constant potential energies differ by ΔE on the two sides of the barrier $V(x)$, i.e., when an incoming wave number k leads to an outgoing one $\tilde{k}$ with $\tilde{k}^2=k^2+2m\Delta E/\hbar^2$. We shall not give the detailed argument, but be content with stating the result: $T(k)=|A(k)|^2$ is replaced by its proper definition for this case, $T(k)=(\tilde{k}/k)|A(k)|^2$ and $x_2 v(k)^{-1}$ is replaced by $x_2\tilde{v}(k)^{-1}$. Here $\tilde{v}=\hbar\tilde{k}/2m$ and $\alpha'(k)$ are both considered as functions of k. As shown in I, the resulting transmission time at given k

$$t_T(k)=x_2/\tilde{v}(k)-x_1/v(k)+\alpha'(k)/v(k) \tag{5.9}$$

is invariant with respect to the choice of x origin.

Similarly, the argument leading to (5.8) is modified to the case of reflection in a straightforward manner. The reflected wave packet to the left of $x=x_1$ is given by

$$\psi_{\pm}(x,t)=\int \frac{dk}{2\pi}\phi(k)A_1(k,k_{\pm})B(k_{\pm}) \times \overline{A}_1(k_{\pm},k)e^{-ikx-i\hbar^2k^2t/2m} \quad (x<x_1) \tag{5.10}$$

with $\overline{A}_1$ the amplitude for the inverse transmission at $x=x_1$ and $\overline{A}_1(k_{\pm},k)=A_1^*(k_{\pm},k)$ (see Appendix A). The resulting reflection time is

$$\langle t_R \rangle_R=\frac{\int \frac{dk}{2\pi}|\phi(k)|^2 R(k) v(k)^{-1}[-2x_1+\beta'(k)]}{\int \frac{dk}{2\pi}|\phi(k)|^2 R(k)} . \tag{5.11}$$

Here $\beta(k)$ is the phase of the reflection amplitude $B(k)$.

The Larmor clock argument leading to (5.8) and (5.11) presupposed a wide field region, i.e., $x_1 \ll b$ and $x_2 \gg a$. However, the results (5.8) and (5.11) imply the obvious fact that the motion undisturbed by the tunneling process is that of a free packet. We can therefore, by *extrapolation*, let $x_1 \to a^-$ and $x_2 \to b^+$ in (5.8), (5.9), and (5.11), and thereby *identify* the times associated with the tunneling process through the barrier $V(x)$ confined to the x interval (b,a). The result of this identification is the transmission and reflection times for tunneling:

$$\langle \tau_T \rangle_T=\frac{1}{N_T}\int \frac{dk}{2\pi}|\phi(k)|^2 T(k)[a-b+\alpha'(x)]/v(k) ,$$
$$\langle \tau_R \rangle_R=\frac{1}{N_R}\int \frac{dk}{2\pi}|\phi(k)|^2 R(k)[-2b+\beta'(k)]/v(k) . \tag{5.12}$$

To the zeroth order in a sharp k distribution around k_c, (5.12) gives (with $k_c \to k$)

$$\tau_T(k)=[a-b+\alpha'(k)]/v(k) ,$$
$$\tau_R(k)=[-2b+\beta'(k)]/v(k) , \tag{5.13}$$

These are nothing else than the classic[13–15] phase times.

It is interesting to note that for an arbitrary *symmetric* potential, $V(-x)=V(x)$, one has $b=-a$ and $\beta'(k)=\alpha'(k)$, as derived in Appendix A. In that case, $\tau_R(k)=\tau_T(k)$. To our knowledge, the general validity of this simple equality for symmetric potentials has not been noted previously.

VI. CONNECTION TO RYBACHENKO'S READING

In this section we consider, like Rybachenko,[4] the stationary problem posed by the configuration in Fig. 1. For an infinitesimal field (i.e., infinitesimal ω_L) we calculate

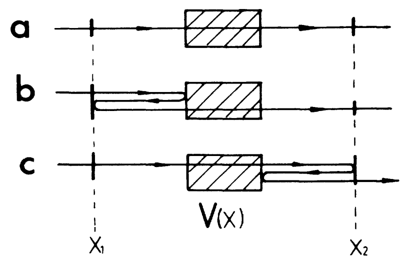

FIG. 2. Graphic representation of the collision sequences (a), (b), and (c) in (6.1).

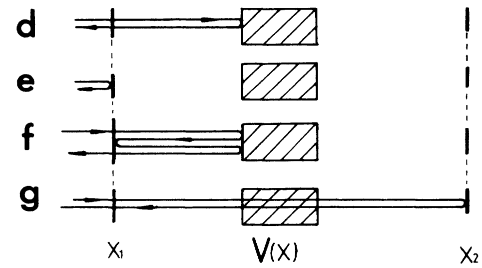

FIG. 3. Graphic representation of the collision sequences (d)–(g) of (6.3).

the corresponding $\langle s_y \rangle_T^c$ and $\langle s_y \rangle_R^c$, where superscript c denotes "coherent." Our alternative way of arriving at the coherent results of Rybachenko illustrates clearly why it does not make sense to interpret them in terms of tunneling times.

As in Sec. V, we shall only need contributions up to linear order in ω_L. Terms of $O(\omega_L^2)$ can be neglected. This was also the strategy used by Rybachenko and, for the problem of relevance here, by Büttiker.[7] Our alternative technique is based on viewing the stationary state as a superposition of different, identifiable collision events. The restriction to $O(\omega_L)$ limits the contributing events to a number which is easily handled explicitly. Details will be relegated to Appendix B.

The $O(\omega_L)$ contributions are found as follows: Whereas the transmission and reflection amplitudes associated with the barrier, $V(x)$, both are of $O(1)$, the reflection amplitudes from the endpoints, x_1 and x_2 of the field region are of $O(\omega_L)$. This is immediately seen from $B_1(k,k_\pm)$ of (3.8) and from the corresponding expressions for $\bar{B}_1(k_\pm,k)$ and $B_2(k_\pm,k)$. [Here $\bar{B}_1(k_\pm,k)$ is the reflection amplitude for the inverse process at $x=x_1$ as discussed in Appendix A.] The terms contributing to $\langle s_y \rangle_T^c$ of $O(\omega_L)$ are then characterized by the following products of amplitudes:

$$\begin{aligned}
&(\text{a})\ A_1(k,k_\pm)A(k_\pm)A_2(k_\pm,k)\ ,\\
&(\text{b})\ A_1(k,k_\pm)B(k_\pm)\bar{B}_1(k_\pm,k)A(k_\pm)A_2(k_\pm,k)\ ,\\
&(\text{c})\ A_1(k,k_\pm)A(k_\pm)B_2(k_\pm,k)\bar{B}(k_\pm)A_2(k_\pm,k)\ ,
\end{aligned} \tag{6.1}$$

pictorially represented in Fig. 2.

The term (a) is that used in Sec. V. It is formally of $O(1)$ but the combination of such terms needed in the calculation of $\langle s_y \rangle_T^c$ is of $O(\omega_L)$. The terms (b) and (c) can be written as correction terms to (a) and need only be retained to leading order, i.e., $O(\omega_L)$. As a consequence of (6.1), the product of amplitudes used in (5.1) for the Larmor clock calculation must, for our purposes, be replaced by

$$A_1(k,k_\pm)A(k_\pm)A_2(k_\pm,k)[1+B(k_\pm)\bar{B}_1(k_\pm,k) + B_2(k_\pm,k)\bar{B}(k_\pm)]\ . \tag{6.2}$$

Similarly, the terms contributing to $\langle s_y \rangle_R^c$ of the reflected particles are

$$\begin{aligned}
&(\text{d})\ A_1(k,k_\pm)B(k_\pm)\bar{A}_1(k_\pm,k)\ ,\\
&(\text{e})\ B_1(k,k_\pm)\ ,\\
&(\text{f})\ A_1(k,k_\pm)B(k_\pm)\bar{B}_1(k_\pm,k)B(k_\pm)\bar{A}_1(k_\pm,k)\ ,\\
&(\text{g})\ A_1(k,k_\pm)A(k_\pm)B_2(k_\pm,k)\bar{A}(k_\pm)\bar{A}_1(k_\pm,k)\ ,
\end{aligned} \tag{6.3}$$

pictorially displayed in Fig. 3.

The product (d) was used in Sec. V. It is formally of $O(1)$ whereas the remaining three terms are of $O(\omega_L)$. The product used in (5.10) for a Larmor clock calculation must now be replaced by

$$A_1(k,k_\pm)B(k_\pm)\bar{A}_1(k_\pm,k)[1+B_1(k,k_\pm)/A_1(k,k_\pm)B(k_\pm)\bar{A}_1(k_\pm,k) + \bar{B}_1(k_\pm,k)B(k_\pm)+A(k_\pm)B_2(k_\pm,k)\bar{A}(k_\pm)/B(k_\pm)]\ . \tag{6.4}$$

Retaining the correction terms of (6.2) to $O(\omega_L)$ and using the relations between inverse and direct collisions discussed in Appendix A, one finds from (5.1), (5.5), and (6.2) the coherent $\langle s_y \rangle_T^c$ to $O(\omega_L)$ as

$$\langle s_y \rangle_T^c = -\frac{\hbar}{2}\omega_L \frac{1}{N_T}\int \frac{dk}{2\pi}\,|\phi|^2 T\left[\frac{1}{v}(x_2-x_1+\alpha')+\frac{\sqrt{R}}{2kv}[\sin(\beta-2kx_1)-\sin(2\alpha-\beta+2kx_2)]\right] \tag{6.5}$$

with N_T given by (5.4) and with the k dependence of the various quantities notationally suppressed.

Similarly, for the reflected particles (6.4) gives

$$\langle s_y \rangle_R^c = -\frac{\hbar}{2}\omega_L \frac{1}{N_R}\int \frac{dk}{2\pi}|\phi|^2 R \left[\frac{1}{v}(-2x_1+\beta') + \frac{\sqrt{R}}{2kv}[\sin(\beta-2kx_1)-\sin(2\alpha-\beta+2kx_2)] + \frac{1}{2kv\sqrt{R}}[\sin(\beta-2kx_1)+\sin(2\alpha-\beta+2kx_2)]\right] . \tag{6.6}$$

Details of calculations leading to (6.5) and (6.6) are found in Appendix B.

Aside from different notation, the results (6.5) and (6.6) are identical to the general formulas derived for the stationary case by Rybachenko. In other words, we are in perfect agreement with Rybachenko and with Büttiker as far as the stationary results are concerned. Where we disagree is on their *interpretation*. They do not, in our opinion, amount to a correct reading of the Larmor clock. In fact, our rederivation shows Rybachenko's results to represent a coherent superposition of quite different physical processes. Clearly it is meaningless, in general, to interpret (6.5) and (6.6) as transmission and reflection times (multiplied by $-\frac{1}{2}\hbar\omega_L$), respectively. There is a trivial exception to this, however. When $V(x)=0$, only contribution (a) of Fig. 2 survives, and consequently, the "transmission" time of a free particle is correctly given by (6.5). Another, less trivial, exception should be noted. When a sharp (in k space) wave packet hits a resonant state in the barrier, the transmission probability is unity. In that case "nothing" is reflected, only the contribution (a) survives and (6.5) gives the phase time (5.8).

If one, nevertheless, formally defines the unphysical times $\theta_T(k)$ and $\theta_R(k)$ (at given k) as the large parentheses in (6.5) and (6.6), respectively, one easily verifies that

$$T\theta_T + R\theta_R = Tt_T + Rt_R + \frac{\sqrt{R}}{kv}\sin(\beta-2kx_1) . \tag{6.7}$$

Here $t_T(k)$ and $t_R(k)$ are given implicitly by (5.8) and (5.11). The combination on the right of (6.7) is recognized, by an identity proven in I, as the "dwell time,"[2] $t_D(k)$, defined[7,18] as

$$t_D(k) = \frac{1}{v(k)}\int_{x_1}^{x_2} dx\, |\chi(x;k)|^2 \tag{6.8}$$

with $\chi(x;k)$ the exact stationary solution (3.10). Thus,

$$t_D = T\theta_T + R\theta_R .$$

(Note that in the symmetric case, $\theta_T = \theta_R = t_D$.)

VII. DISCUSSION

The basic premise for our reexamination of the Larmor clock has been this: Only truly time-dependent calculations (and experiments) can give reliable information on the time aspects of a physical process. Or, conversely: It is not sufficient to study the stationary case, theoretically[3,4,7] or experimentally,[19] point to a plausible quantity with the dimension of time, and thus identify the time taken by the process in question. Not that the result of such a simplified procedure is necessarily wrong. The problem is that it is not necessarily right.

Another simple guideline for our work: Before using a clock one should check that the clockwork runs at a constant rate. This is not quite as trivial as it may seem: It is not clear *a priori* that a (in principle) measurable quantity, increasing at a constant rate, *exists* in the context of interest. In I,[12] we studied the center-of-mass clock. From our present point of view, that clockwork is based on the constant group velocity of free wave packets. As discussed in detail in I, problems arise with packets of finite width, σ, in **k** space, due to the fact that the transmitted and reflected packets move faster or slower than the initial one. In other words, the clockwork is influenced by the tunneling process itself. As a consequence, cross-correlations in the initial packet, which have nothing to do with the tunneling times per se, interfere with the reading of the center-of-mass clock.

In this sense, the Larmor clock is superior. We showed explicitly in Sec. IV that when the field region is sufficiently wide to allow the three processes inherent in the Larmor clock to occur *sequentially* rather than coherently as in a stationary calculation, the Larmor clockwork remains undisturbed by the tunneling process.

This does *not* imply, however, that the Larmor clock is always superior to the center-of-mass clock. In computer studies[9–11] of moving wave packets it is precisely the motion of the center of mass (or, alternatively, the peak) of the wave packet which is monitored. Clearly, the calculations of I, *including* the cross-correlation effects, are of direct relevance here. Those effects of $O(\sigma^2)$ are missed by the Larmor clock.

To $O(\sigma^0)$ the center-of-mass and Larmor clocks agree completely: The transmission and reflection times for tunneling processes are the classic phase times.[13–15] Even to $O(\sigma^2)$, most of the correction terms following from (5.12) agree with those calculated in I. However, aside from the missing cross-correlation terms, which correct for an intrinsic "weakness" in the center-of-mass clock, there are other subtle differences. They reflect the fact that (5.12), roughly speaking, involves $\langle v^{-1}(k)\rangle$, whereas for the center-of-mass clock, one calculates the average position involving $\langle v(k)\rangle$ and subsequently inverts, to find time through $\langle v(k)\rangle^{-1}$.

It is worth noting that a reliable reading of *both* the center-of-mass and the Larmor clock is *asymptotic*. The readings themselves should be made far from the barrier, and the tunneling times inferred from this by linear extrapolation return back to the tunneling barrier (based on the simple motion of free wave packets). In both cases this is necessary in order that the results are not disturbed by the interference effects characteristic of a coherent stationary state.

Finally, a comment on the distribution of tunneling times: For the center-of-mass clock, this distribution is clearly *wide.* The uncertainty principle gives $\Delta\tau \sim \Delta x > \sigma^{-1}$ with σ small. For the Larmor clock, however, one can argue that the width of the τ distribution is limited by σ not by σ^{-1}. In other words, there is no uncertainty principle which forces a wide distribution onto the intrinsic tunneling times of a wave packet narrow in k space. We leave this as an unproven proposition here.

Subtleties aside, the following simple statements sum up the results of this paper: We have considered tunneling times for a restricted class of problems only, that in which wave packets, narrow in k space, impinge on an arbitrary, static barrier in one dimension. For this class of problems, the Larmor clock seemed to give tunneling times in conflict with the classic phase times, and with numerical experiments. We have removed this conflict: When properly set, in wave-packet fashion, the Larmor clock shows the phase time. Stationary calculations of Larmor times correspond to coherent superpositions of widely different scattering events, and cannot therefore be interpreted as tunneling times. Finally, for an arbitrary one-dimensional potential with reflection symmetry, the transmission and reflection phase times coincide.

Note added in proof. We would like to direct the reader's attention to related work by Jaworski and Wardlaw,[21] and thank those authors for illuminating correspondence.

ACKNOWLEDGMENT

We are grateful to Tor A. Fjeldly who introduced us to these problems, and contributed constructive comments throughout.

APPENDIX A

The consequences of time reflection symmetry and particle conservation on the stationary problem defined by Fig. 4 have been considered by Azbel.[20] With the 2×2 matrix M defined by

$$\begin{bmatrix} U \\ W \end{bmatrix} = M \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} \tag{A1}$$

one easily convinces oneself that time reflection invariance implies

$$M_{21}=M_{12}^*, \quad M_{22}=M_{11}^* \tag{A2}$$

and that particle conservation gives

$$\det M = M_{11}M_{22}-M_{12}M_{21}=\tilde{k}/k\ , \tag{A3}$$

where (see Fig. 4)

$$\tilde{k}^2=k^2+2m\Delta E/\hbar^2\ . \tag{A4}$$

Equation (A4) allows one to consider $\tilde{k}$ as a function of k (or vice versa).

From this follows that the transmission amplitude $\bar{A}$

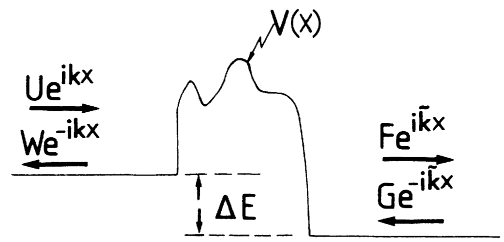

FIG. 4. The general stationary scattering configuration in one dimension.

for a particle coming from the right (the "inverse" process, described by $G=1, F=\bar{B}, U=0, W=\bar{A}$) is related to the transmission amplitude A for a particle coming from the left (the "direct" process, described by $U=1, W=B, F=A, G=0$) as

$$\bar{A}(\tilde{k},k)=\frac{\det M}{M_{11}}=\frac{\tilde{k}/k}{M_{11}}=\frac{\tilde{k}}{k}A(k,\tilde{k})\ . \tag{A5}$$

As a result, the corresponding transmission probabilities are equal, since

$$\bar{T}(\tilde{k})=\frac{k}{\tilde{k}}\,|\,\bar{A}(\tilde{k},k)\,|^2=\frac{\tilde{k}}{k}\,|\,A(k,\tilde{k})\,|^2=T(k) \tag{A6}$$

and so are the phases, in the sense that

$$\bar{\alpha}(\tilde{k},k)=\text{phase}(\bar{A})=\text{phase}(A)=\alpha(\tilde{k},k) \tag{A7}$$

since $\tilde{k}$ and k, related by (A4), are both real.

The reflection amplitudes are similarly connected. With

$$\bar{B}(\tilde{k},k)=-\frac{M_{12}}{M_{11}},\quad B(k,\tilde{k})=\frac{M_{21}}{M_{11}} \tag{A8}$$

one obviously has $\bar{R}(\tilde{k})=|\,\bar{B}(\tilde{k},k)\,|^2=|\,B(k,\tilde{k})\,|^2=R(k)$. Furthermore, from (A8) and (A5),

$$\bar{B}(\tilde{k},k)=-B^*(k,\tilde{k})A(k,\tilde{k})/A^*(k,\tilde{k}) \tag{A9}$$

so that

$$\bar{\beta}(\tilde{k},k)=-\pi-\beta(k,\tilde{k})+2\alpha(k,\tilde{k})\ , \tag{A10}$$

where the check mentioned below has been used to determine the sign of π, which at this stage is arbitrary.

As a simple application of (A5) and (A9) one finds, from (3.8)

$$\begin{aligned} \bar{A}_1(k_\pm,k)&=\frac{2k_\pm}{k+k_\pm}e^{i(k-k_\pm)x_1}=A_1^*(k_\pm,k)\ , \\ \bar{B}_1(k_\pm,k)&=-\frac{k-k_\pm}{k+k_\pm}e^{-2ik_\pm x_1}=B_1^*(k_\pm,k) \end{aligned} \tag{A11}$$

which can be verified directly.

For a potential with reflection symmetry $V(-x)=V(x)$ one must have $\tilde{k}=k$ so that $\alpha(k,\tilde{k})=\alpha(\tilde{k},k)\equiv\alpha(k)$ and $\beta(k,\tilde{k})=\beta(\tilde{k},k)\equiv\beta(k)$. Also, $\bar{\beta}(k)=\beta(k)$, so that (A10) gives

$$\beta(k)=-\pi/2+\alpha(k)\ . \tag{A12}$$

This checks with explicit results for the square barrier. (See, e.g., the Appendix of I, in which one should let $\beta \to \beta - kb$ as a result of shifting the origin to the center of the barrier.) The square barrier calculation also fixes the convenient sign for π in (A10). Note that, as an immediate consequence of (A12), $\beta'(k) = \alpha'(k)$.

APPENDIX B

In this appendix we fill in some of the details of the calculations summarized in Sec. VI.

To $O(\omega_L)$, the square bracket of (6.2) which corrects the product of amplitudes used in Sec. V to calculate $\langle s_y \rangle_T$ reads

$$[\cdots]_T \equiv 1 + B(k_\pm)\bar{B}_1(k_\pm, k) + B_2(k_\pm, k)\bar{B}(k_\pm) = 1 - B(k)\frac{k - k_\pm}{2k} e^{-2ikx_1} + \frac{k_\pm - k}{2k} e^{2ikx_2}\bar{B}(k) , \tag{B1}$$

where we used (3.8), (A11), and the fact that, to (ω_L), we can replace $k_\pm$ by k everywhere, except in the combination $k_\pm - k = \pm m\omega_L / 2\hbar k$ [see (3.6)]. Now use (A9) and (A10) to get

$$[\cdots]_T = 1 \pm \frac{m\omega_L \sqrt{R}}{4\hbar k^2}\left(e^{i(\beta - 2kx_1)} - e^{i(2\alpha - \beta + 2kx_2)}\right) . \tag{B2}$$

To leading order, the real part is 1. Neglect of the $O(\omega_L)$ correction to the real part (which will not affect the results to significant order), allows us to write

$$[\cdots]_T \simeq \exp\left[\pm i\omega_L \frac{\sqrt{R}}{4kv}[\sin(\beta - 2kx_1) - \sin(2\alpha - \beta + 2kx_2)]\right] . \tag{B3}$$

This extra factor combined with the product (a) in (6.1) [i.e., the product used in (5.1)] is responsible for the additional term in (6.5) beyond the phase time of (5.7).

Similarly, the square bracket of (6.4) which corrects the product of amplitudes used in Sec. V to calculate $\langle s_y \rangle_R$ reads, to $O(\omega_L)$

$$\begin{aligned}[\cdots]_R &\equiv 1 + B_1(k, k_\pm)/A_1(k, k_\pm)B(k_\pm)\bar{A}_1(k_\pm, k) + \bar{B}_1(k_\pm, k)B(k_\pm) + A(k_\pm)B_2(k_\pm, k)\bar{A}(k_\pm)/B(k_\pm) \\ &= 1 + \frac{k - k_\pm}{2k} e^{2ikx_1} B^{-1}(k) - \frac{k - k_\pm}{2k} e^{-2ikx_1} B(k) - A^2(k)\frac{k - k_\pm}{2k} e^{2ikx_2} B^{-1}(k) \\ &= 1 \pm \frac{m\omega_L}{2\hbar k^2}\left[\frac{-1}{\sqrt{R}} e^{i(2kx_1 - \beta)} + \sqrt{R}\, e^{i(\beta - 2kx_1)} + \frac{1 - R}{\sqrt{R}} e^{i(2\alpha - \beta + 2kx_2)}\right] \\ &\simeq \exp\left[\pm i\omega_L\left[\frac{\sqrt{R}}{4kv}[\sin(\beta - 2kx_1) - \sin(2\alpha - \beta + 2kx_2)] + \frac{1}{4kv\sqrt{R}}[\sin(\beta - 2kx_1) + \sin(2\alpha - \beta + 2kx_2)]\right]\right] .\end{aligned} \tag{B4}$$

In (B4), as in (B3), the $O(\omega_L)$ correction to the real part, 1, has been neglected. The phase factor (B4) is, when combined with the calculations of Sec. V, responsible for the additional terms in (6.6) beyond the reflection phase time of (5.11).

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