Fractal pore space and rock permeability implications

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We have used optical microscopy to perform direct measurements of the fractal volume and surface dimensions of sandstone samples on scales from 0.5–200 μm. The dimensions are evaluated by box-counting techniques on the digitized representation of the microscope pictures. The connection between the fractal dimensions and rock permeability is discussed.

The relation between porosity and permeability in rocks is a subject of scientific as well as practical interest. In general, no simple functional relationship has been found between these quantities. This is basically due to the fact that most porous rocks are very complex systems which may consist of a range of irregular grain sizes. Several analytical models have been suggested[1,2] to predict rock permeability from basic statistical quantities connected to the rock. However, a general problem of many of these models is that they are based on quantities (e.g., "mean channel length"), which are difficult to measure in real rocks.

Various studies suggest that the surfaces of rock grains[3] and of whole rock samples[4] are fractal. In particular, Wong *et al.*[5] have used small-angle neutron scattering (SANS) to show that the microscopic geometry of sandstone surfaces can be characterized by a fractal dimension D_s in the range from 2.55 to 2.95. The fractal interpretation of the surface geometry, combined with some other basic rock or fluid properties may turn out to give a fundamental description and understanding of relative permeability of simultaneous two-phase flow in the rock. However, the absolute rock permeability is determined by the structure of the pore space and surface area on "grain-size" scales, which are usually from 1 μm and larger.

At present there are relatively few direct measurements of possible fractal scaling of both pore surface and volume of these scales. Katz and Thompson[6] (KT) used scanning electron microscopy and optical data to suggest that these quantities are given by the same fractal dimension for certain sandstones. However, this claim has later been disputed both in connection with the SANS experiment[5] as well as on theoretical grounds.[7] In fact, it may be easily shown that this can only be true in special cases. A simple example showing that this is not true is a three-dimensional (3D) model porous medium consisting of closed, uniform Koch curves.[8] The surface dimension here will be $D_s \simeq 2.26$, while the pore volume will have $D_v = 3$, the Euclidean dimension.

In this paper we report direct measurements of the geometry of sandstone samples from the Oseberg North Sea oil reservoir by analyzing optical micrographs. The samples were polished thin sections (thickness approximately 30 μm) bound to a glass substrate. By using an inverted optical microscope in transmitted-light illumination mode, it was possible to focus on the two-dimensional (2D) cuts of the grains. The images were digitized using a video frame grabber with 512×512-pixel resolution and 256 grey levels. Five different microscope magnifications were used setting the relevant scales from 0.5 to 200 μm. As the out-of-focus pore space was darker than the grains, it was possible to identify these features employing digital image analysis. A low-pass filter was used to eliminate noise due to features below the resolution at a particular magnification. This established a uniform relative resolution at each level of magnification. A histogram of the relative number of features at each grey level showed two peaks identifying the pore space and grains. From this it was thus possible to set the threshold level to produce a black and white picture representing these two features. Figure 1 shows these digitized representations revealing smaller and smaller details for increasing magnifications from the same position of a typical sample. This is a good example of the extreme complexities which sandstone pores may exhibit. The similar appearance for different magnifications signifies a possible random fractal scaling. One of the most useful methods to probe this is to use the box-counting technique.[9] To our knowledge, this method has so far not been employed for analysis of the pore space in rock.

To determine the area or "volume" fractal dimension D_a the following procedure was used: The digitized picture containing 512×512 lattice points was covered with imaginary square boxes of edge length L. The number of boxes, $N(L)$, containing any part of the pore-space cuts was counted and averaged for different positions of the boxes relative to the lattice. The counting procedure also considered edge corrections by including only the fractional parts of the boxes inside the edges of the whole lattice. For a fractal structure it is expected that $N(L)$ scales with box size L as

$$N(L) \propto L^{-D_v} . \qquad (1)$$

Figure 2 shows the log-log plot of N versus L for the digitized images in Fig. 1. As may be seen, the data for the whole range of magnifications appear to fall on a straight line with slope $D_v = 1.73\pm0.04$.

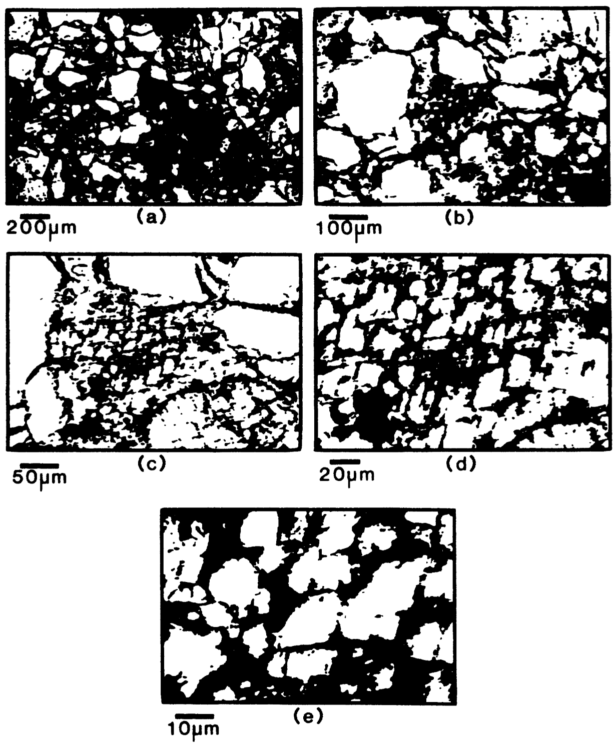

FIG. 1. Digitally displayed pore space (black) in a typical North Sea sandstone for increasing magnifications (a)–(e), respectively.

To probe a possible fractal scaling of the pore-grain interface, the same box-counting technique was used. For this, the number of boxes $N(L)$ needed to cover the contours of the borderlines between the black and white regions in Fig. 1 was counted for different box sizes L. The results are shown in the log-log plot of N versus L in Fig. 3. Again, it may be seen that the data appear to fall on a

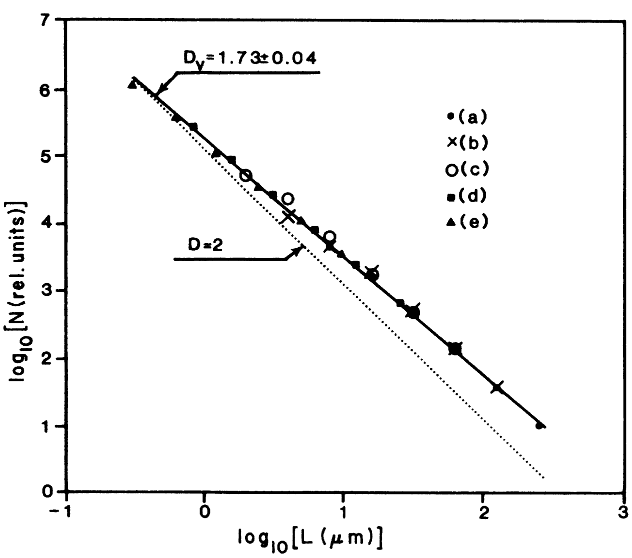

FIG. 2. The fractal dimension D_v of the pore-space area determined from the log-log plot of the relative number of boxes needed to cover the pores vs box size L for the different magnifications (a)–(e) in Fig. 1.

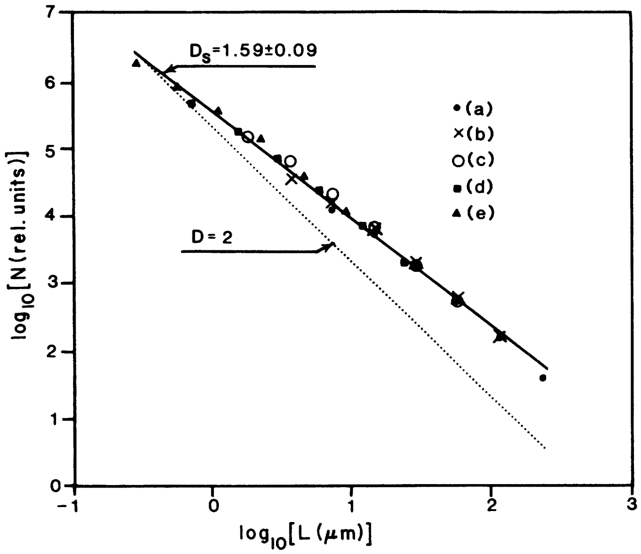

FIG. 3. Same legend as in Fig. 2 for the fractal dimension D_s of the contours.

straight line with slope $D_s = 1.59 \pm 0.09$.

Aside from possible global differences, which will be discussed below, the present box-counting technique has the important feature of being able to average out local statistical fluctuations by choosing a large number of centers for the boxes.[9] This appears to be a distinct advantage compared to the methods of feature counting along one-dimensional sections of the pore-rock interface, perimeter-to-area analysis, or using a yardstick of variable length to measure the coastline.

Tests for systematic errors in the present technique showed that the results were relatively insensitive to filtering and precise setting of the threshold level. However, in order to get sensible data over a wide range of length scales using increasing magnification and not moving the sample, the positions had to be chosen with some care. Clearly, with random choice one could otherwise sometimes end up in the middle of a large solid grain at the highest magnification.

In order to probe the statistical fluctuations between different samples and positions, the following procedure was used: One thin section was chosen at random from each of four core samples within the same geophysical formation. Within each thin section, various positions were chosen at random, but allowing sensible scaling for different magnifications as discussed above. A total of 30 digitized pictures for each of the pore-space and pore-grain interfaces were analyzed using the same box-counting method discussed above resulting in the following average fractal dimensions: $D_v = 1.69 \pm 0.05$ and $D_s = 1.56 \pm 0.07$. The stated standard deviations thus reflect both local and global uncertainties in using the fractal description as an average statistical property for the whole rock on scales from 0.5 to 200 μm.

According to accepted formalism,[8] the fractal dimensions just found in the 2D cuts are on less than the corresponding fractal dimensions D_s and D_v for the surface and pore volume, respectively, in three dimensions. This means that $D_s = 2.56 \pm 0.07$ and $D_v = 2.69 \pm 0.05$ as the statistical averages for the present samples. The value for D_s is compatible with the value $D_s = 2.55$ found by Wong *et al.*[5] for Portland sandstone. However, their results were obtained on much smaller length scales (5–500 Å), and it is not clear whether the surface structures on that scale and our scales are connected. The measurements by KT discussed earlier[6] suggest that $D_s = 2.78$ for Coconino sandstone for a wide range of length scales (10 Å–100 μm). Their value is thus close to the present results for D_v. However, whereas KT found that $D_s \simeq D_v$ for a limited range of length scales (~ 1–10 μm), the results for the present sample thus indicate that $D_s < D_v$.

The basic reason for a fractal pore volume and surface is a wide distribution of irregular features in the system. A simple model taking this into account, giving a box-counting volume and surface dimension, is the following: Consider a "grain-filling procedure" similar to those analyzed by Omnés.[10] A porous medium with any porosity can be created by introducing a fraction β_1 of grains with characteristic size l_1. The pore volume is then changed from V_0 to $\beta_1 V_0$. Next a fraction β_2 of characteristic size l_2 is introduced and the process repeats itself until a smallest grain size l_n is reached. For each filling the total surface in the system will increase from S_i to $\gamma_{i+1} S_i$. The total volume and surface after the filling is completed is now

$$V = \left[\prod_{i=1}^{n} \beta_i \right] V_0 \; , \qquad S = \left[\prod_{j=1}^{n} \gamma_j \right] S_0 \; . \tag{2}$$

The fractal-box counting dimension associated with the pore volume can now be estimated. Covering the pore volume by E_v-dimensional boxes of length l_i (E_v is the Euclidean dimension), and counting the boxes which contain pore space, one can expect to find the following number of boxes N_i:

$$N_i \simeq \left[\prod_{j=1}^{i} \beta_j \right] V_0 / l_i^{E_v} \; . \tag{3}$$

According to Eq. (1), the fractal box-counting dimension is determined by the slope of the number-length relation in a log-log plot. As an approximation to this slope at neighboring scale sizes (l_i, l_{i+1}), the logarithmic difference is

$$d_{i,i+1} \simeq -\frac{\ln(N_{i+1}) - \ln(N_i)}{\ln(l_{i+1}) - \ln(l_i)} = E_v + \frac{\ln(\beta_{i+1})}{\ln\left[\dfrac{l_i}{l_{i+1}}\right]} \; . \tag{4}$$

Hence, a true self-similar pore volume is obtained when all fractions are equal. If this is not the case, the structure obtained after the filling procedure can be described in the multifractality formalism.[11] However, as a model, to provide understanding of the measured results for real systems, it seems more suitable to put equal weights on all the successive approximations, $d_{i,i+1}$, and use the arithmetic average as the fractal box-counting dimensions of the sample:

$$D_v \simeq E_v + \frac{1}{n} \sum_{i=1}^{n} \frac{\ln(\beta_i)}{\ln(l_i / l_{i+1})} \; . \tag{5}$$

If the fluctuations in β_i are relatively small, the use of Eqs. (5) and (1) serves as an excellent approximation to the number-length relations in Eq. (3). This is demonstrated in Table I, where a random choice of fractions and length scales and the evaluated fractal dimension are used to compare the approximations in Eqs. (1) and (5) to the number-length relation. It follows from Eq. (5) that the dimension D_v is always less than or equal to E_v because all $\beta_i < 1$. By covering the surface with E_s-dimensional squares of the same sizes ($E_s = 2$ the Euclidean dimension), one finds similarly for the surface dimension,

$$D_s \simeq E_s + \frac{1}{n} \sum_{i=1}^{n} \frac{\ln(\gamma_i)}{\ln(l_i / l_{i+1})} \; , \tag{6}$$

which is greater than E_s. Note that the $\{\beta_i\}$ are also uniquely given by the volume of the grain sizes. The $\{\gamma_i\}$ are also a function of the grain structures and the packing of each grain size into the system of larger grain sizes. Hence, the measured results of D_s and D_v may be understood from the structure arising from the discussed grain-filling procedure.

We now turn to the implications a description of the rock structure by fractals may have on the understanding of fluid flow properties of the rock. In the low-velocity flow regime, Darcy's law, which expresses a linear relationship between the flow velocity and pressure gradient for one-phase incompressible flow, is assumed to be valid.

TABLE I. Comparison of Eqs. (1) and (5) for (a) a random choice of fractions β_i and length scales l_i ($i = 1$–5), (b) the number of boxes calculated from Eq. (3), and (c) the number of boxes found from Eq. (1) using $D_v = 1.71$ as calculated from Eq. (5) with $n = 5$.

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
(a)	$\beta_i = 0.8$	0.9	0.7	0.85	0.85
	$l_i = 1$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
(b)	$N_i = N_1$	$3.60N_1$	$10.08N_1$	$34.27N_1$	$116.52N_1$
(c)	$N_i = N_1$	$3.27N_1$	$10.70N_1$	$35.02N_1$	$114.56N_1$

The proportionality constant is usually separated in a pure rock-dependent permeability divided by a pure fluid-dependent viscosity. By modeling the pore space as a collection of cylindrical noninteracting tubes one obtains for rock permeability the well-known Kozeny equation[12] which can be written

$$k_r = c\phi / S^2 \ . \tag{7}$$

Here c is a rock dependent constant, ϕ is the porosity, and S is the internal surface area per unit pore volume.[13] A lot of criticism has been raised as to the validity of Eq. (7) for real systems, and many modifications have been made when applying this equation to real rocks. But the simple and physical transparent form of Eq. (7) makes it an attractive model for studying the impact of different fractal dimensions on permeability. By expressing the specific surface for Eqs. (5) and (6) and assuming that all the fractions l_i / l_{i+1} are equal, one gets

$$S = S_0 (l_1 / l_n)^{(D_s - E_s) + (E_v - D_v)} \tag{8}$$

for the specific surface, and by substituting this into Eq. (7) one gets for the rock permeability

$$k_r = c' \phi (l_1 / l_n)^{2[(E_s - D_s) + (D_v - E_v)]} \ . \tag{9}$$

It may be seen from Eq. (9) that the k_r has a power-law dependence of D_s and D_v, decreasing as D_s is increased and increasing as D_v is increased. This is reasonable as D_s and D_v reflect the tortuosity of the rock structure. It is further seen that the variations in permeability for different fractal dimensions can be large if the relation (l_i / l_n) is also large. A common feature of sandstones is large permeability variations for fixed porosities, which is consistent with Eq. (9). The advantage of Eq. (9) is further that it is based on quantities which can be measured in any real porous medium, and the validity of this equation can thus be investigated by experiments, which is postponed for future work.

In summary we have for the first time measured the volume and surface dimension of a sandstone sample by box-counting techniques, finding the average values $D_v = 2.69 \pm 0.05$ for the volume dimension, and $D_s = 2.56 \pm 0.07$ for the surface dimension. Application of a simple grain-filling procedure to model the pore space makes it possible to express the permeability by a power-law dependence of the fractal dimensions.

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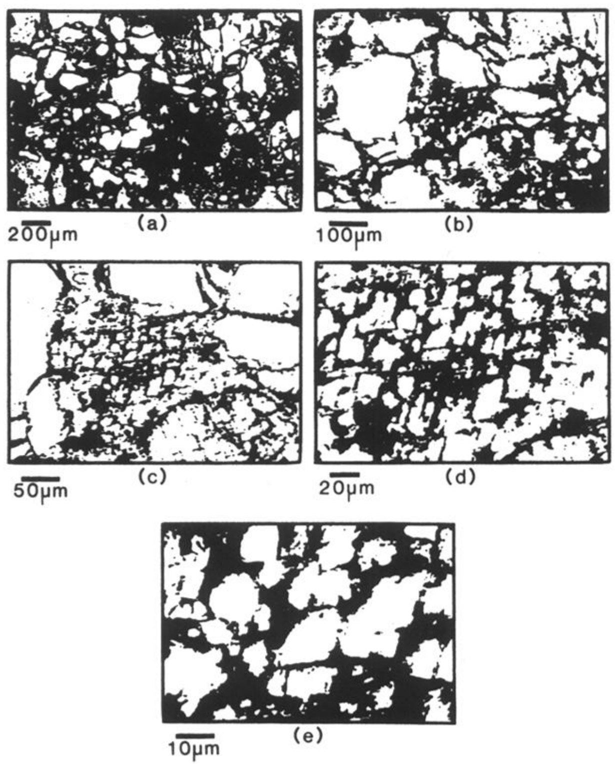
[13]Most commonly S is defined as surface area per unit bulk volume. In that case one gets a ϕ^3 power law in the numerator of Eq. (7).



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