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Heisenberg and Potts spin glasses: A renormalization-group study

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The Migdal-Kadanoff renormalization-group scheme is used to investigate the chaotic behavior of a Heisenberg spin glass at zero temperature in two, three, and four dimensions. The Lyapunov exponent is found to be independent of the spatial dimensionality, d. In the case of Potts models, the predictions of the Migdal-Kadanoff scheme are sensitive to the choice of the length scale factor, b. While no evidence is found for conventional spin-glass behavior, the couplings wander randomly in a basin of attraction for $d=5$ and $b=2$, possibly suggesting a glassy phase.

I. INTRODUCTION

There has been considerable recent interest in the study of random magnetic systems in general and their low-temperature ordering, in particular.[1] In this paper, we study two distinct models using renormalization-group techniques.[2,3] The first of these is the Heisenberg spin glass.[4-9] This model's complexity arises due to the many low-lying local minima produced due to competing exchange interactions (frustration).[10] The second model studied is the q-state Potts model.[11] While the ferromagnetic model is well studied and understood,[11] the antiferromagnetic case is richer and leads to many novel results.[12-22] The complexity, in this case, is due to the residual ground-state entropy.

A. Heisenberg spin glass

The Hamiltonian of the Heisenberg spin glass is given by

$$H=\sum_{\langle i,j\rangle} J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j ,$$

where the $\mathbf{S}_i$ are three-component unit vectors on a lattice, J_{ij} is the exchange interaction between spins S_i, and S_j, and $\langle ij\rangle$ denotes pairs of nearest neighbors. This model has been studied extensively and is thought to have a lower critical dimensionality (LCD) bigger than three.[4,6-8] Unlike the Heisenberg model, the Ising spin glass is thought to have a finite-temperature transition, even in three dimensions.[23-28] For the Ising spin glass, using phenomenological scaling, it has been shown that the low-temperature phase is chaotic in that the spin order at long enough length scales is sensitive to minute changes in the exchange couplings.[29-31] This arises physically as follows: Let J be a measure of the width of the unperturbed bond distribution. The energy cost of a domain wall excitation of dimension L (measured in units of the lattice parameter) is $\sim JL^y$. The exponent y characterizes the scaling behavior of the bond distribution at the $T=0$ fixed point. When the perturbation is applied, such an excitation may in fact become favorable because there is an additional contribution to the energy $\sim \pm\epsilon JL^{d_s/2}$ where εJ is the measure of a small random (equal likely $+$ or $-$) perturbation on the exchange constants and $(d_s/2)$ is a characteristic exponent. For Ising spins d_s is the "fractal dimension" of the interface. The physical meaning of d_s for a Heisenberg system is somewhat less clear. In any case, when $\zeta=d_s/2-y>0$, the ground state becomes unstable against the perturbation on length scales $L>L^*\sim 1/\epsilon^{1/\zeta}$. This Imry-Ma argument[32] has been explicitly verified using a renormalization group scheme for Ising spin glasses. In Sec. II, this analysis is extended to a Heisenberg spin glass at zero temperature and we obtain the exponents y, d_s, and ζ.

B. Potts models

The q-state ferromagnetic Potts model in d dimensions is now well understood.[11] The system is paramagnetic at high temperatures and undergoes a phase transition to a ferromagnetically ordered state at low temperatures for all $q>1$ and $d>1$. The qualitative results for the ferromagnet are independent of the scale parameter b. The case of the pure antiferromagnetic model was first studied by Berker and Kadanoff[12] using a Migdal-Kadanoff renormalization-group scheme.[2,3] Their analysis was for the specific case of the scale parameter $b=3$. They argued for an algebraically ordered low-temperature phase when $q>2$ and $d>d_l(q)$, the lower critical dimension. Phenomenological,[13] and Monte Carlo renormalization-group studies[14] and a subsequent exact analysis[15] have shown that $d_l(3)=2$. Monte Carlo studies[16,17] in $d=3$ show that the low-temperature phase is in fact ordered in a sublattice fashion. A mean-field analysis[22] was found to yield results that were in accord with simulations in $d=3$. The situation with Potts models with random exchange interactions is less clear. While mean-field studies of Potts spin glasses[33] have been carried out, little is known about their behavior in lower dimensions.

In Sec. III, we extend the work of Berker and Kadanoff[12] to arbitrary values of b. In particular, the predictions for $b=2$ are found to be quite distinct from those for $b=3$. The behavior for all even b is qualitatively similar to that of $b=2$, whereas all odd values of b behave like $b=3$. We then discuss the behavior of random exchange Potts models by numerically iterating the Migdal-Kadanoff recursion relations and studying the evolution of the flows. While we do not find any evidence for conventional spin-glass behavior, we do observe that the couplings wander randomly in a basin of attraction for $d=5$ and $b=2$ possibly suggesting a glassy phase.

The Migdal-Kadanoff approximation works well qualitatively for simple pure systems like ferromagnets and even for strongly disordered systems like Ising spin glasses. It's reliability is somewhat more suspect for Potts models. For this case, the results are more appropriately thought of as being valid for hierarchical lattices.

II. CHAOS IN HEISENBERG SPIN GLASSES

It has been shown recently that the ordered phase in Ising spin glass is "chaotic" in nature in that the relative orientation of spins is increasingly sensitive, as the spin separation is increased, to small perturbations in externally controlled variables, such as temperature.[29–31] In this section, we extend these results to a Heisenberg spin glass. We first present numerical results obtained using a Migdal-Kadanoff (MK) transformation and determine the value of the exponents y and ζ, the zero-temperature scaling exponent and the Lyapunov exponent characterizing the chaotic behavior, respectively. Our numerical analysis indicates that for $d=2$, 3, and 4 at zero temperature, the fixed shape probability distribution of the exchange couplings is a Gaussian, to a very good approximation (Fig. 1). Assuming that this is indeed the case, we derive analytic estimates for y and η in excellent agreement with the numerical results.

A. Numerical results

The MK transformation for $d=3$ corresponding to length scale factor $b=2$ is illustrated in Fig. 2. (In d dimensions there are 2^{d-1} parallel paths.) The recursion relation is[4,5]

$$J'_{16}=\sum_{i=2}^{5} J_{16}(i) , \tag{1}$$

with

$$\frac{1}{J_{16}(i)}=\mathrm{sgn}(J_{1i}J_{i6})\left[\frac{1}{|J_{1i}|}+\frac{1}{|J_{i6}|}\right] . \tag{2}$$

This recursion relation, derived in the zero-temperature limit, only approximately describes the Heisenberg model since the decimation step does not preserve the $\mathbf{S}_i \cdot \mathbf{S}_j$ form of the Hamiltonian exactly. The results that follow have all been derived using Eq. (1), nevertheless.[5]

We numerically follow the evolution of the bond probability distribution. We find that two neighboring bond pools, initially differing infinitesimally, diverge exponentially under iteration. The rate of divergence, of course, is characterized by the Lyapunov exponent. We construct a pool of 40 000 exchange couplings from a Gaussian distribution of zero mean and unit variance. Eight bonds from the pool are selected at random and combined as in Eq. (1) to obtain a member of a new renormalized pool. The process is repeated until a new pool of 40 000 renormalized couplings is produced. This corresponds to one iteration of Eq. (1). Upon successive iteration, the probability distribution of the J's, $P(J)$, approaches a fixed shape, but with a width that changes by 2^y at each iteration. The values of y are summarized in Table I. Note that $y<0$ in $d=2$ and 3, whereas $y>0$ for $d=4$ implying that the lower critical dimensionality (LCD) of this model is between 3 and 4. $P(J)$ is plotted in Fig. 1. The data are fitted to a Gaussian very well for all the values of d shown. Following Ref. 31, we take two pools $\{J_i\},\{J'_i\}$ where $J'_i=J_i+\epsilon x_i$ with $\epsilon=10^{-6}$ and J_i and x_i are independent Gaussian random variables with zero mean and unit variance and follow the evolution of the quantity

$$d^2_{(n)}=\sum_i (J_i^{(n)}-J_i'^{(n)})^2\Big/\sum_i [(J_i^{(n)})^2+(J_i'^{(n)})^2] \tag{3}$$

as a function of the iteration number, n. (When eight members of one pool are selected to obtain a renormal-

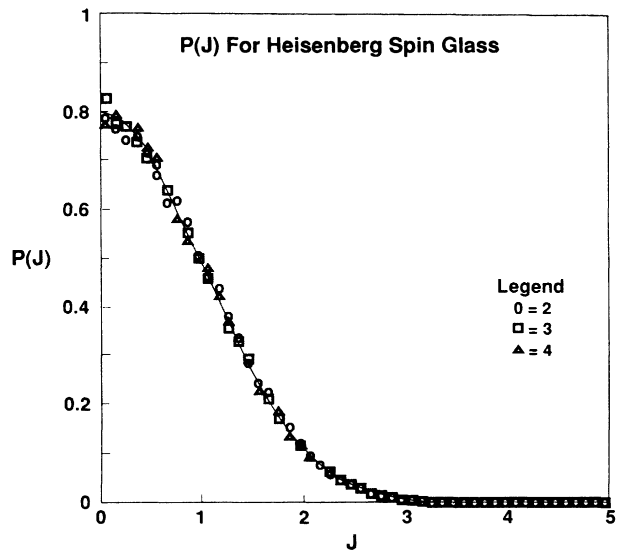

FIG. 1. The shape of the fixed probability distribution $P(J)$ of the exchange couplings, J, for the Heisenberg spin glass in $d=2$, 3, and 4. The solid line is a Gaussian fit.

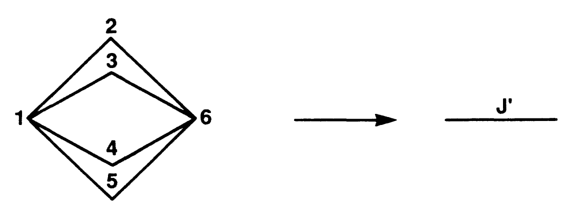

FIG. 2. MK transformation for $d=3$ corresponding to length scale factor $b=2$.

TABLE I. The exponents y, d_s, and ζ obtained analytically for the Heisenberg spin glass at $T=0$. The numerical results agree with the analytic results within 2%.

d	y	d_s	ζ
2	-0.94	-0.52	0.68
3	-0.44	0.48	0.68
4	0.07	1.48	0.68

ized coupling, the corresponding eight members from the neighboring pool are selected to form the corresponding renormalized coupling.) A plot of $d^2_{(n)}$ versus n is shown in Fig. 3. The growth of $d^2_{(n)}$ is initially exponential corresponding to a power law dependence of $d^2_{(n)}$ on the length scale $L=2^n$, $d^2_{(n)}\sim L^{2\zeta}$. The values of ζ are presented in Table I. After a large number of iterations, n^*, $d^2_{(n)}$ saturates at unity corresponding to completely uncorrelated pools. This signals an instability of the ground state to the perturbation on length scales greater than $L^*\equiv 2^{n^*}$.

B. Analytic results

Consider spins 1, 2, and 6 (Fig. 2) and the effect of decimating spin 2. Using Eq. (1), the effective coupling $J_{16}(2)$ is given by

$$\frac{1}{|J_{16}(2)|}=\frac{1}{|J_{12}|}+\frac{1}{|J_{26}|} \ . \tag{4}$$

Consider the effect of perturbing the couplings by random perturbations,

$$J_{ij}\to J_{ij}+x_{ij} \ . \tag{5}$$

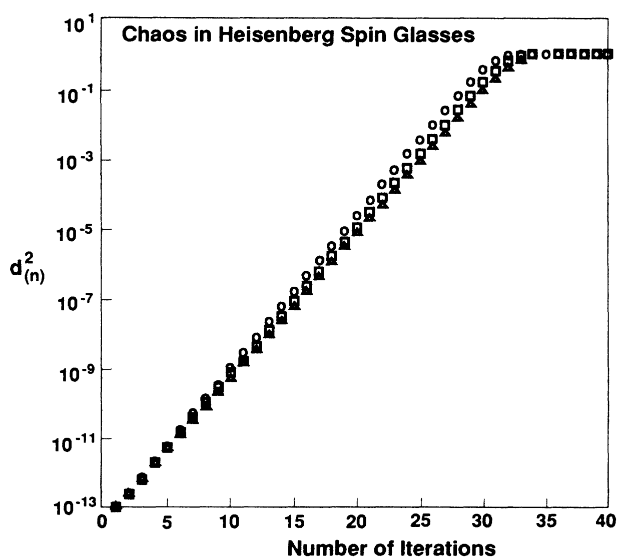

FIG. 3. Square of the normalized distance between bond pools $d^2_{(n)}$ as a function of MK iteration number n for spatial dimensionality $d=2$ (○), $d=3$ (□), and $d=4$ (△).

From (4)

$$\frac{\delta J_{16}(2)}{[J_{16}(2)]^2}=\pm\frac{x_{12}}{J_{12}^2}\pm\frac{x_{26}}{J_{26}^2} \tag{6}$$

which on substituting for $[J_{16}(2)]^2$ from Eq. (4) becomes

$$\delta J_{16}(2)=\pm\frac{J_{26}^2x_{12}\pm J_{12}^2x_{26}}{(|J_{12}|+|J_{26}|)^2} \ . \tag{7}$$

Summing over the 2^{d-1} parallel paths, we find, for the renormalized coupling J',

$$\langle(\delta J')^2\rangle=2^{d-1}\left\langle\frac{J_{12}^4+J_{26}^4}{(|J_{12}|+|J_{26}|)^4}\right\rangle\epsilon^2\equiv 2^{d_s}\epsilon^2 \ , \tag{8}$$

where $\epsilon^2=\langle x^2\rangle$ and the latter equation is the definition of the exponent d_s. Thus defining

$$\lambda_B=\left\langle\frac{J_{12}^4+J_{26}^4}{(|J_{12}|+|J_{26}|)^4}\right\rangle \ , \tag{9}$$

and using Eq. (8)

$$d_s=d-1+\ln\lambda_B/\ln 2 \ . \tag{10}$$

Also

$$\langle J'^2\rangle=2^{d-1}\langle J_{16}(2)^2\rangle=2^{d-1}\left\langle\frac{J_{12}^2J_{26}^2}{(|J_{12}|+|J_{26}|)^2}\right\rangle \ . \tag{11}$$

Denoting

$$\left\langle\frac{J_{12}^2J_{26}^2}{(|J_{12}|+|J_{26}|)^2}\right\rangle=\lambda_J\langle J_{12}^2\rangle \ , \tag{12}$$

and noting that

$$\langle J'^2\rangle=\langle J_{12}^2\rangle 2^{2y} \tag{13}$$

is the definition of the zero-temperature scaling exponent y, we find that

$$2y=d-1+\ln\lambda_J/\ln 2 \ . \tag{14}$$

Finally, since $\zeta=d_s/2-y$, combining Eqs. (10) and (14),

$$\zeta=\ln(\lambda_B/\lambda_J)/(2\ln 2) \ . \tag{15}$$

To evaluate the exponents d_s, y, and ζ using Eqs. (10), (14), and (15), we need to obtain λ_B and λ_J. Approximating the fixed shape probability distribution by a Gaussian, straightforward, but tedious, algebra yields

$$\lambda_B=8/(3\pi)-\tfrac{1}{2} \tag{16}$$

and

$$\lambda_J=(2/\pi)-\tfrac{1}{2} \ .$$

These results are in excellent agreement with the numerical data and yield an estimate for the LCD ($y=0$) of the Heisenberg model of ~ 3.87. The value of ζ is strikingly independent of the spatial dimension, d.

III. POTTS MODEL

In this section, we study the Potts model using the MK transformation. We consider both the $b=2$ and $b=3$ versions of the model. While the qualitative results are independent of b for the ferromagnetic (F) model, the antiferromagnetic (AF) Potts model shows strikingly distinct behavior for $b=2$ and $b=3$. For $b=3$, there is a low-temperature phase characterized by algebraic order as first shown by Berker and Kadanoff.[12] Computer simulations[16,17] have, however, suggested that AF Potts models order at low temperatures for dimensionality (d) greater than the lower critical dimensionality (LCD). Baxter[15] has also shown that the LCD for $q=3$ is equal to 2. The $b=2$ analysis indeed reproduces both these results. Further, the exponent for the AF model are found to be the same as the F model—unlike the predictions of an earlier renormalization-group analysis.[16,17] It should also be noted that MK analysis does not lead to first-order transitions, in, for example, $d=2$ and $q>4$. We also study Potts models with random exchange couplings with both $b=2$ and $b=3$. The numerical results indicate the existence of a paramagnetic phase, the F phase, the AF phase—there is no evidence for a conventional spin-glass (SG) phase. For $d=5$, $b=2$, however, there is unusual behavior at low temperatures where an initial Gaussian distribution of exchange couplings with zero mean and large enough variance (low enough temperature) wanders in a chaotic way in a region around $\langle K\rangle \simeq -14$ and $\sigma_K \sim 14$ ($\langle K\rangle$ and σ_K denote the average and the standard deviation of the coupling distribution). To our knowledge, such chaotic wandering within a basin of attraction has not been observed heretofore. This glassy phase is found only for $b=2$ and only in $d=5$–the LCD for this phase seems to be bigger than 4 and the glassy phase disappears in $d=6$.

A. Ferro- and antiferromagnetic models

The Hamiltonian for the Potts model[11] is given by

$$H=\sum_{i,j} J_{ij}\delta_{\sigma_i,\sigma_j}\ , \tag{17}$$

where $J_{ij}>0$ for the AF case and $J_{ij}<0$ for the F case. The dimensionless coupling is $K_{ij}=-J_{ij}/k_BT$. The recursion relation for the model for b bonds in series in a given path and b^{d-1} such parallel paths is given by

$$K'=\sum_{i=1}^{b^{d-1}} K'_{is}$$

with

$$K'_{is}=\ln\left[1+\frac{q}{\prod_{i=1}^{b}\left[1+\frac{q}{e^{K_i}-1}\right]-1}\right]. \tag{18}$$

When the bonds are ferromagnetic, there are two possible scenarios. For small values of K (high temperature), the flows are to an infinite temperature sink—one has a paramagnetic phase. For sufficiently large values of K, the flows are to a ferromagnetic sink. The F critical point, $K^*=K_c^{\mathrm{F}}$, is found by setting $K'=K_i=K^*$ in Eq. (18). This qualitative behavior holds for the ferromagnet, independent of the value of b.

The Potts AF, however, has a richer and more varied behavior. We first let $b\to 1+dl$. In this limit, the recursion relation becomes

$$\frac{dK}{dl}=(d-1)K-\frac{(e^{K}+q-1)}{q}(1-e^{-K})\ln\left[1+\frac{q}{e^{K}-1}\right]. \tag{19}$$

For the equation to be physically meaningful in the conventional sense, the argument of the logarithm must be positive, i.e.,

$$1+\frac{q}{e^{K}-1}>0\ . \tag{20}$$

We note that as $K\to 0_-$, this inequality is violated, thus invalidating the general use of Eq. (19).

We study the lowest integral values of b, viz. $b=2$ and 3 for the Potts antiferromagnet. The $b=3$ case has been studied in detail by Berker and Kadanoff.[12] Their primary result was that for $d>d_c$ (the LCD), the low-temperature phase is attracted to a sink at *finite* temperature indicating an algebraically ordered low-temperature phase. Monte Carlo simulations[16,17] show that the AF Potts model in $d=3$ has an ordered low-temperature phase. For example, the $q=3$ model is characterized by a ground state with the one of the states predominantly on one sublattice with the other two states dividing themselves equally on the other sublattice. This conventional sublattice ordering is quite distinct from the algebraic ordering predicted by the $b=3$ analysis. Also, based on symmetry arguments,[16,17] it was suggested that the $q=3$ AF model was in the xy universality class, whereas the $q=4$ model was in the Heisenberg universality class.

We now turn to an analysis of the $b=2$ AF Potts model. The recursion relation is

$$K'=2^{d-1}\ln\left[\frac{(q-1)+e^{2K}}{(q-2)+2e^{K}}\right]. \tag{21}$$

There are three fixed points of the preceding equation, $K=0$, K^*, and ∞. An initial large and negative K, after one iteration becomes positive. In the limit of $K\to-\infty$,

$$K'_1=2^{d-1}\ln\left[\frac{q-1}{q-2}\right]. \tag{22}$$

Two cases may be distinguished—when $K'_1>K^*$, the flows are to the ferromagnetic sink; on the other hand, when $K'_1<K^*$, the flow is to an infinite temperature sink and the behavior is paramagnetic. The critical behavior is therefore the same as the F q-state model. The LCD of the AF model is however different from the F model and is obtained by setting $K'_1=K^*$ where K^* is the nontrivial F fixed point of Eq. (21),

TABLE II. The flows and ferromagnetic sink exponents of the $q=3$ Potts model at very low temperatures. I, infinite temperature; BK, Berker-Kadanoff; F, ferromagnetic.

				Flow sink	
d	b	y_F	y_σ	From symmetric Gaussian	Other sinks
2	2	1	0.29	I	F
2	3	1	0.29	I	F
3	2	2	0.77	I	F
3	3	2	0.77	I	F
4	2	3	1.24	I	F
4	3	3	1.24	BK	F
5	2	4		See text	F
5	3	4		BK	F
6	2	5		F	None
6	3	5		BK	F
7	2	6		F	None
7	3	6		BK	F

$$K^* = 2^{d_c-1}\ln\left[\frac{q-1}{q-2}\right] = 2^{d_c-1}\ln\left[\frac{(q-1+e^{2K^*}}{(q-2)+2e^{K^*}}\right] . \tag{23}$$

Solving Eq. (23), we find

$$d_c = 1 + \frac{\ln\left[1+\dfrac{\ln 2}{\ln[(q-1)/(q-2)]}\right]}{\ln 2} . \tag{24}$$

For $q=3$, $d_c=2$ in agreement with the exact result of Baxter,[15] whereas for $q=4$, $2<d_c\sim 2.44<3$ in agreement with the Monte Carlo simulations[16,17] and for $q\to\infty$, $d_c\sim \ln q/\ln 2$ in accord with the mean-field predictions[22] of $K_c(q\to\infty)\sim q$.

B. Random exchange Potts model

We now generalize to the case where the exchange couplings are randomly positive or negative. Our studies are carried out using the recursion relations given in Eq. (18) using pools of up to 10^5 couplings. Renormalized pools are then created as described in Sec. II. The flows typically are to the infinite temperature sink (I), the ferromagnetic (F) sink and the Berker-Kadanoff[12] sink (BK) (for the $b=3$ case). No flows were observed to a low-temperature spin-glass sink (SG) unlike the Ising behavior.

The results of starting from a low-temperature symmetric Gaussian distribution for the $q=3$ Potts model for $b=2$ and 3 and for $d=2$, 3, 4, 5, 6, 7 are summarized in Table II. The only unusual behavior occurs at $d=5$ for $b=2$. In this case, the mean $\langle K\rangle$ and the standard derivation σ_K of the pool wanders "chaotically" in a smallish region around $\langle K\rangle\sim -14$ and $\sigma_K\sim 14$. This wandering behavior persists for 1000's of iterations and happens for a wide range of starting temperatures. While we have no clear explanation of this behavior, it presumably indicates a glassy behavior even at very long length scales. The LCD for this phase is >4 and the phase seems to be nonexistent in $d=6$, even on making the distribution asymmetric.

How do we qualitatively understand and account for the basin of attraction? Consider three states a, b, c populating the hierarchical structure. When all the bonds are AF, the dominant entropy ground state consists of populating, say a spins at the ends (F-like ordering) and placing b or c spins randomly in the middle. The arrangement where unlike states (say a and b) (AF-like ordering) are at the ends can also be used to form a ground state except that the middle spin must be in the third state, viz. c. This has the same internal energy as the first arrangement but a low entropy. Now, if one of the bonds is made ferromagnetic, the second arrangement of unlike states at the ends can still form a ground state, whereas the former arrangement is no longer a ground state. On heating up the system, entropic effects start playing a role and it may become favorable for the system to have F-like order to gain entropy at the expense of the internal energy. The situation is an unusual type of glassy behavior, where the glassy AF-like state has a lower entropy compared to the F-like state. We speculate that this delicate balance between energy and entropy causes the wandering phase in $d=5$. In higher dimensions, the number of parallel paths increases so that the effects of entropy play more of a role, destabilizing the low entropy AF-like phase in favor of the high entropy F-like phase.

We have also investigated the effect of changing the mean value of the Gaussian distribution away from zero. The sinks that are accessible from low temperatures are summarized in Table II. In addition, Table I shows zero-temperature F sink exponents $y=d-1$ and y_σ (the exponent that describes the scaling of the width of a ferromagnetic distribution,[34,35] i.e., $\sigma\sim L^{y_\sigma}$; note that the mean of the distribution scales with the exponent y and $y>y_\sigma$). y_σ is known exactly in two dimensions to be $\frac{1}{3}$.[36] Our renormalization-group analysis yields a value of 0.29. The values of y_σ in higher dimensions have not been determined previously.

The zero temperature $d=3$, $b=2$ result is that when the mean of the distribution is sufficiently large ($+ve$ or $-ve$), the behavior is characterized by F ordering with a paramagnetic phase in between. We have verified this result by carrying out Monte Carlo (MC) simulations on a three-dimensional (3D) cubic lattice containing 14^3 spins with periodic boundary conditions. When all the bonds are ferromagnetic, we observe conventional F ordering. On the other hand, when all the bonds are AF, we reproduce the sublattice ordering discussed in Ref. 17. When a fraction f of the bonds are chosen to be F (and strength $|J|$), we find that the ordered phases are supplanted by a paramagnetic (P) phase for intermediate values of f. A noteworthy feature of the MC simulations is that the P phase is characterized by many distinct local minima containing large domains, somewhat reminiscent of the random-field Ising model. The MK analysis does not yield the exotic behavior predicted by the mean-field theories.[23]

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