Thermal enhancement of macroscopic quantum tunneling: Derivation from noise theory

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We recompute the thermal enhancement of quantum tunneling using a noise-theory derivation. This calculation is applied to the macroscopic quantum tunneling out of the zero-voltage state of a current-biased Josephson junction. The rate enhancement for T much less than the crossover temperature is shown to be due to the thermal current noise generated at low frequencies by the shunt admittance across the junction. The results of this theory are found to be in perfect agreement with earlier findings based on more formal functional integral techniques. The method is applied to circuits with dissipative elements of arbitrary frequency dependence. The method also gives the first analytical predictions for the effect of the prefactor on the rate enhancement.

I. INTRODUCTION

Even though the phenomenon of quantum tunneling has been known for many years, only recently has the effect of dissipation on tunneling been calculated.[1] This has led to further work on the effect of a finite temperature to the tunneling rate,[2] and finally to an escape-rate prediction encompassing both the quantum tunneling and thermal activation regimes.[3] Experimental confirmation of many of the predictions have been made.[4,5] Thus a complete understanding is emerging for the escape from the metastable state for arbitrary temperature and dissipation.[6–8]

Although predictions of escape rates exist, we believe a clear physical picture is absent that describes the origin of the finite-temperature corrections to the quantum tunneling rate. The main object of this paper will be to rederive this result using a very simple and physical theory. The rate enhancement will be shown to be simply due to low-frequency thermal fluctuations which originate in the dissipative elements. This approach gives in a straightforward manner the contribution from both the exponential and the preexponential factors. It is also generalized quite naturally to give predictions for the escape-rate enhancement for an arbitrary frequency-dependent dissipative term. We hope these calculations will illuminate the physical mechanism for the rate enhancements, as well as eliminate some doubts to the validity of the more elaborate path integral calculations that were originally presented.

An attractive physical system to experimentally test the escape from a metastable state is the current-biased Josephson junction or rf superconducting quantum interference device (SQUID) system. Junction parameters can be obtained so that a measurement of the escape rate from the quantum to thermal limit is possible for both the low and high limit of damping. All of the junction parameters can be measured,[9] thus enabling an accurate test of the theories with no adjustable parameters. For these reasons and for physical clarity, we will concentrate in our derivation on the current-biased Josephson junction system. The generalization of the ideas presented here to an arbitrary metastable system is straightforward.

For the temperature $T=0$ K and for temperatures small compared to the crossover temperature T_0, the escape from a metastable state occurs via quantum tunneling.[1–3] For the current-biased Josephson junction, the tunneling occurs for the macroscopic variable of the phase difference across the junction and is thus referred to as macroscopic quantum tunneling (MQT).[10] Caldeira and Leggett[1] were first to calculate the effect of dissipation on quantum tunneling at $T=0$. They showed that the effective admittance $Y(\omega)$ shunting the Josephson junction leads to an exponentially strong reduction of the zero-temperature MQT rate $\Gamma(0)$ when compared with the rate $\Gamma_0(0)$ of an ideal undamped junction ($Y=0$) with the same Josephson critical current, capacitance, and bias current. The ratio $\ln[\Gamma(0)/\Gamma_0(0)]$ is calculable and can be compared with the experimentally observed suppression of the rate. A natural extension was to calculate the effect of finite temperatures on the rate. Grabert, Weiss, and Hanggi[2] showed that for a junction with admittance $Y(\omega)$, the MQT rate at finite temperatures around $T=0$, $\Gamma(T)$, is enhanced compared with the $T=0$ rate, $\Gamma(0)$, according to a power law $\ln[\Gamma(T)/\Gamma(0)]=\sigma T^2$, where σ is proportional to $Y(\omega=0)$ and calculable. On a log Γ versus T^2 plot the experimentally observed slope can be compared with the predicted value. Indeed, experiments have confirmed these predictions.[5]

In this paper the origin of the reduction of the zero-temperature rate will only be discussed briefly. We shall mainly investigate the thermal enhancement of the rate. We will show that the finite-temperature results around $T=0$ can be easily obtained once a solution to the rate at $T=0$ is known.

The paper is organized as follows. In Sec. II we present the physical ideas for our approach and give a simplified calculation for the thermal enhancement of the escape rate. In Sec. III we present results for Ohmic dissipation, as well as a refinement of the theory to include prefactor effects. Predictions for non-Ohmic dissipation are given in Sec. IV. Section V gives a concluding summary. Finally, two appendexes clarify the relation of the present approach with earlier work.

II. THERMAL ENHANCEMENT OF MQT

As discussed elsewhere[11] we represent the current-biased Josephson junction by the equivalent circuit shown in Fig. 1. The junction critical current is I_0, the junction capacitance is C, and $Y(\omega)$ denotes an admittance in parallel with the junction. $Y(\omega)$ is a linear response function that describes the external loading of the junction by shunt resistors, bias circuity filters, and the quasiparticle tunneling resistance. When the junction is biased by a current I, its *classical* equation of motion reads

$$C\left[\frac{\phi_0}{2\pi}\right]^2\ddot{\delta}(t)+\left[\frac{\phi_0}{2\pi}\right]^2\int_{-\infty}^{t}du\,y(t-u)\dot{\delta}(u)+\frac{\partial U(\delta(t))}{\partial\delta(t)}=\frac{\phi_0}{2\pi}I_N(t)\,, \quad (2.1)$$

where $\delta(t)$ is the Josephson phase difference across the junction, $\phi_0=h/2e$ is the flux quantum, $I_N(t)$ is the classical noise current arising from $Y(\omega)$, and

$$y(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\omega\,Y(\omega)e^{i\omega t} \quad (2.2)$$

is the temporal response of the admittance. The potential

$$U(\delta)=-\frac{I_0\phi_0}{2\pi}\cos(\delta)-\frac{I\phi_0}{2\pi}\delta \quad (2.3)$$

is the tilted cosine potential which describes the effect of the current source and the Josephson coupling between the superconductors.

Generally MQT is observed for values of the bias current I only very slightly below the critical current I_0. Then, the relevant part of the potential $U(\delta)$ is very well approximated by the cubic potential[12]

$$U(\varphi)=\frac{I_0\phi_0}{2\pi}\left[-\frac{\pi}{2}\frac{I}{I_0}+\left[1-\frac{I}{I_0}\right]\varphi-\frac{1}{6}\varphi^3\right]\,, \quad (2.4)$$

where $\varphi=\delta-\pi/2$. This cubic potential has a barrier of height

$$\Delta U=\frac{4\sqrt{2}}{3}\frac{I_0\phi_0}{2\pi}\left[1-\frac{I}{I_0}\right]^{3/2} \quad (2.5)$$

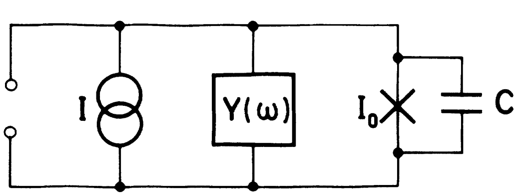

FIG. 1. Equivalent circuit of a current-biased Josephson junction.

and width

$$\Delta\delta=3\sqrt{2}\left[1-\frac{I}{I_0}\right]^{1/2}\,. \quad (2.6)$$

From the curvature of the potential at the local minimum, one obtains for the plasma oscillation frequency at the bottom of the well

$$\omega_p=\left[\frac{2\pi I_0}{\phi_0 C}\right]^{1/2}2^{1/4}\left[1-\frac{I}{I_0}\right]^{1/4}\,. \quad (2.7)$$

The zero-temperature MQT rate $\Gamma(0)$ for the escape from the local potential minimum may be written as

$$\Gamma(0)=Ae^{-B}\,, \quad (2.8)$$

where the exponential factor B and the preexponential factor A can be calculated by the zero-temperature bounce technique[1] or by multidimensional WKB methods.[13] We shall not address here the calculation of A and B, but assume that they are known functions of the junction parameters and the admittance.

At finite temperatures T the environmental modes described by $Y(\omega)$ will become thermally occupied at low frequencies. Using the fluctuation-dissipation theorem, this leads to a thermal noise current $\delta I_{\text{th}}(t)$ with the properties

$$\langle\delta I_{\text{th}}(t)\rangle_T=0 \quad (2.9)$$

and

$$\langle\delta I_{\text{th}}(t)\delta I_{\text{th}}(0)\rangle_T=\int_0^\infty\frac{d\omega}{\pi}\hbar\omega[\coth(\tfrac{1}{2}\beta\hbar\omega)-1]\times\text{Re}[Y(\omega)]\cos(\omega t)\,, \quad (2.10)$$

where $\beta=1/k_BT$ and $\text{Re}[Y(\omega)]$ is the real part of the admittance. Note that the zero-point fluctuations of the noise current are not part of the thermal excess current $\delta I_{\text{th}}(t)$. The effect of zero-point noise is already included in the zero-temperature MQT rate (see also Appendix B). In terms of the Planck occupation number $n_\beta(\omega)$ of a mode with frequency ω which is given by

$$n_\beta(\omega)=\frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}=\tfrac{1}{2}[\coth(\tfrac{1}{2}\beta\hbar\omega)-1] \quad (2.11)$$

we may write

$$\langle\delta I_{\text{th}}^2\rangle_T=\frac{2}{\pi}\int_0^\infty d\omega\,\hbar\omega n_\beta(\omega)\text{Re}[Y(\omega)]\,. \quad (2.12)$$

In Eq. (2.12), $2\hbar\omega n_\beta(\omega)/\pi$ gives the spectral density of the thermal noise. In Fig. 2, we plot $\hbar\omega n_\beta(\omega)$ as a function of the frequency. The magnitude of the spectral density near zero frequency increases as T and is equal to the thermal noise result $2k_BT/\pi$. The characteristic frequency in which the noise decays to zero is $\omega\sim k_BT/\hbar$.

For temperatures T well below $\hbar\omega_P/k_B$ only current modes with frequencies $\omega\ll\omega_P$ will be thermally occupied. These low-frequency modes lead to a quasistationary change of the bias current through the junction.[14] In Fig. 3, we draw the change in the potential well as the bias current is changed by a small amount δI. Hence, at

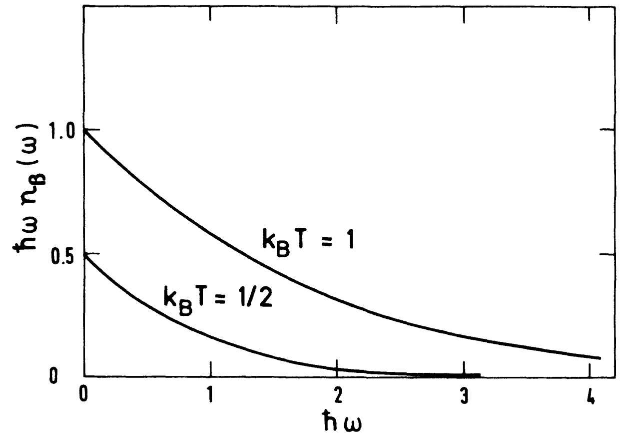

FIG. 2. Plot of $\hbar\omega n_B$ vs $\hbar\omega$ for two values of temperature. Since the ratio $k_B T/\hbar\omega$ determines the functional dependence of the plot, we have chosen for convenience the same arbitrary units for both $k_B T$ and $\hbar\omega$.

low temperatures the system has to penetrate a potential barrier that is modulated slowly by the noise from the shunt admittance. In the remainder of this paper, we assume that the system follows quasistatically and remains in the ground state during the modulation of I. Then a small change δI in the bias current changes the zero-temperature tunneling rate into

$$\Gamma_{I+\delta I}(0)=\Gamma_I(0)\exp\left[-\frac{\partial B}{\partial I}\delta I\right], \tag{2.13}$$

where we have taken into account only the change of the dominant exponential factor of the rate. The current dependence of the preexponential factor gives a correction which will be considered in the following section.

Using (2.13) we see that the thermal noise current $\delta I_{\rm th}$ leads on the average to the finite-temperature decay rate

$$\begin{aligned}\Gamma_I(T)&=\langle \Gamma_{I+\delta I_{\rm th}}(0)\rangle_T\\ &=\Gamma_I(0)\exp\left[\frac{1}{2}\left[\frac{\partial B}{\partial I}\right]^2\langle \delta I_{\rm th}^2\rangle_T\right].\end{aligned} \tag{2.14}$$

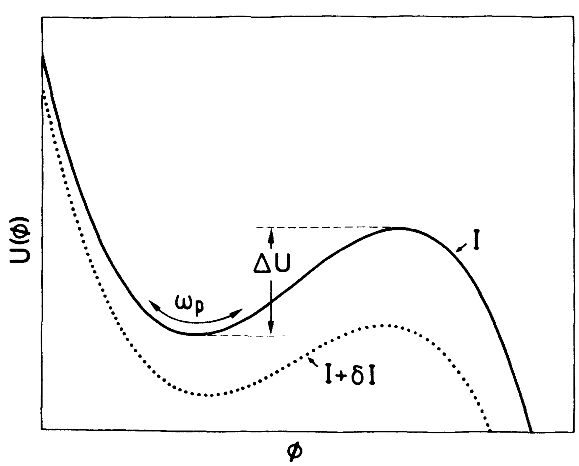

FIG. 3. Potential $U(\phi)$ vs ϕ for bias currents I and $I+\delta I$.

Here we made use of the fact that $\delta I_{\rm th}$ obeys Gaussian statistics, since the circuitry loading the junction is linear. From (2.14) we obtain for the thermal enhancement factor

$$\ln\gamma(T)=\ln[\Gamma(T)/\Gamma(0)]=\frac{1}{2}\left[\frac{\partial B}{\partial I}\right]^2\langle \delta I_{\rm th}^2\rangle_T\,. \tag{2.15}$$

Note that the temperature dependence arises entirely from the thermal fluctuation term $\langle \delta I_{\rm th}^2\rangle_T$. Thus the rate enhancement is proportional to the total current noise $\langle \delta I_{\rm th}^2\rangle$ flowing through the junction. Equation (2.15) combines with (2.12) to yield

$$\ln\gamma(T)=\frac{1}{\pi}\left[\frac{\partial B}{\partial I}\right]^2\int_0^\infty d\omega\,\hbar\omega n_\beta(\omega){\rm Re}[Y(\omega)] \tag{2.16}$$

which in view of (2.11) may also be written as

$$\ln\gamma(T)=\frac{1}{\pi\hbar}\left[\frac{\partial B}{\partial I}k_B T\right]^2\int_0^\infty dx\frac{xe^{-x}}{1-e^{-x}}\times{\rm Re}[Y(k_B T x/\hbar)]\,. \tag{2.17}$$

When the low-frequency admittance has an Ohmic component

$$Y(\omega=0)=\frac{1}{R_0} \tag{2.18}$$

we find that the enhancement factor around zero temperatures follows the power law

$$\ln\gamma(T)=\frac{\pi}{6\hbar R_0}\left[\frac{\partial B}{\partial I}k_B\right]^2 T^2\,. \tag{2.19}$$

Here we made use of

$$\int_0^\infty dx\frac{xe^{-x}}{1-e^{-x}}=\frac{\pi^2}{6}\,. \tag{2.20}$$

Hence, we recover the T^2 law for the thermal enhancement of the low-temperature MQT rate. Our calculation shows that the enhancement arises from the thermally excited fluctuations of the *reservoir* and not from thermal excitations between states in the metastable well. The thermal excitation of the states in the well[15] is exponentially small at temperatures considered in this paper, and thus provides a negligible rate enhancement. As will be shown in Appendix A the formula (2.19) agrees with the result of the finite-temperature bounce calculation.

III. OHMIC DAMPING

In this section we discuss more specifically the case of a Josephson junction shunted by an Ohmic resistance R. The admittance then is frequency independent, i.e.,

$$Y(\omega)=1/R \tag{3.1}$$

and the dissipation caused by it may be characterized by the dimensionless parameter

$$\alpha=1/2Q=\frac{1}{2RC\omega_p}\,, \tag{3.2}$$

where $Q=\omega_p RC$ is the quality factor of the junction. The zero-temperature MQT rate $\Gamma(0)$ was analyzed by Caldeira and Leggett[1] who found

$$B=\frac{\Delta U}{\hbar\omega_p}s(\alpha) \tag{3.3}$$

and

$$A=\frac{\omega_p}{2\pi}\left[\frac{\Delta U}{\hbar\omega_p}\right]^{1/2}\chi(\alpha) , \tag{3.4}$$

where $s(\alpha)$ and $\chi(\alpha)$ are dimensionless functions of the damping parameter α which can be calculated analytically for small and large α, and numerically for intermediate damping. We want to complete the analysis of the thermal enhancement of MQT presented in the preceding section by including the contribution of the preexponential factor A. Extending the expansion (2.13) to the next order, one obtains instead of (2.14) the improved result

$$\Gamma_I(T)=\Gamma_I(0)\exp(\tfrac{1}{2}K\langle\delta I_{\rm th}^2\rangle_T) , \tag{3.5}$$

where

$$K=\left[\frac{\partial B}{\partial I}-\frac{\partial \ln A}{\partial I}\right]^2-\frac{\partial^2 B}{\partial I^2}+\frac{\partial^2 \ln A}{\partial I^2} . \tag{3.6}$$

With the help of (2.15), (2.7), and (3.2) the coefficient K may be evaluated to yield

$$K=\frac{B}{\hbar C\omega_p^3}\kappa , \tag{3.7}$$

where

$$\kappa=\tfrac{1}{6}s(\alpha)\left\{\left[5-\frac{\partial \ln s(\alpha)}{\partial \ln\alpha}\right]^2-\frac{1}{B}\left[40-\frac{\partial \ln s(\alpha)}{\partial\ln\alpha}\left[13-\frac{\partial \ln s(\alpha)}{\partial\ln\alpha}-2\frac{\partial\ln\chi(\alpha)}{\partial\ln\alpha}\right]-10\frac{\ln\chi(\alpha)}{\partial\ln\alpha}+\frac{\partial^2\ln s(\alpha)}{\partial(\ln\alpha)^2}\right]+O(B^{-2})\right\} . \tag{3.8}$$

For the frequency-independent admittance (3.1), we have from (2.12) and (2.20)

$$\langle\delta I_{\rm th}^2\rangle_T=\frac{\pi}{3\hbar R}(k_B T)^2 . \tag{3.9}$$

The thermal enhancement factor $\ln\gamma(T)$ as defined in (2.15) may thus be written as

$$\ln\gamma(T)=\frac{\pi}{3}\alpha B\kappa\left[\frac{k_B T}{\hbar\omega_p}\right]^2 . \tag{3.10}$$

Again, we have recovered the T^2 law for the logarithm of the low-temperature rate. Compared with our earlier result, we have obtained a more specific expression for the slope.

For *weak damping* the functions $s(\alpha)$ and $\chi(\alpha)$ are given by[1]

$$s(\alpha)=\frac{36}{5}\left[1+\frac{45}{\pi^3}\xi(3)\alpha+O(\alpha^2)\right] , \tag{3.11}$$

where $\xi(3)=1.202\ldots$ is a Riemann number, and [16]

$$\chi(\alpha)=12\sqrt{6\pi}[1+C\alpha+O(\alpha^2)] , \tag{3.12}$$

where $C=2.86\ldots.$ When this is inserted into (3.7), the coefficient κ is found to be

$$\kappa=\kappa_0+\kappa_1\alpha+O(\alpha^2) , \tag{3.13}$$

where

$$\kappa_0=30-48B^{-1}+O(B^{-2}) \tag{3.14}$$

and

$$\kappa_1=\frac{810}{\pi^3}\xi(3)-\left[\frac{1512}{\pi^3}\xi(3)-12C\right]B^{-1}+O(B^{-2}) . \tag{3.15}$$

The leading order term of κ, $\kappa=30$, gives when inserted into (3.10) exactly the result derived previously by path integral methods for the exponential factor of the rate. The corrections of order B^{-1}, which include the effect of the change of the prefactor with temperature, have not been derived analytically before. Since under typical experimental conditions B is between 10 and 15, these corrections modify the T^2 slope by about 10%. Inserting (3.13) into (3.10) we find for the enhancement factor in leading order in α

$$\ln\gamma(T)=10\pi\alpha(B-\tfrac{8}{5})(k_B T/\hbar\omega_p)^2+O(\alpha^2) . \tag{3.16}$$

The prefactor correction is consistent with previous numerical calculations.[3] The term of second order in α can also be extracted from the above results.

For *strong damping* the functions $s(\alpha)$ and $\chi(\alpha)$ read[1]

$$s(\alpha)=6\pi\alpha\left[1+\frac{1}{4\alpha^2}+O(\alpha^{-4})\right] \tag{3.17}$$

and[17,3]

$$\chi(\alpha)=16\pi\sqrt{6}\alpha^{7/2}\left[1+\frac{2}{\alpha^2}\ln\alpha+\frac{d}{\alpha^2}+O(\alpha^{-4}\ln\alpha)\right] , \tag{3.18}$$

where $d=1.107\ldots.$ This combines with (3.8) to yield

$$\kappa=16\pi\alpha+\frac{8\pi}{\alpha}[1-(4\ln\alpha+2d-\tfrac{13}{8})B^{-1}]+O(\alpha^{-3}\ln\alpha) . \tag{3.19}$$

Note that the leading-order term proportional to α has no correction of order B^{-1}. This is due to the fact that the preexponential factor A becomes temperature independent for large α.[17] Inserting (3.19) into (3.10) we obtain as the dominant term of the thermal enhancement factor in the strong damping limit

$$\ln\gamma(T)=\frac{16\pi^2}{3}\alpha^2 B(k_B T/\hbar\omega_p)^2[1+O(\alpha^{-2}\ln\alpha)] \tag{3.20}$$

in agreement with the result by Larkin and Ovchinnikov.[18] The next-order correction can also be extracted from the above formulas.

Apart from the previously known results, our method has given the contribution of the preexponential factor to the finite-temperature MQT rate in a rather straightforward manner. We mention that the results for $\ln\gamma(T)$ obtained here are only valid for temperatures well below the crossover temperature T_0, where the thermally excited current fluctuations are slow compared with the dynamics of the phase δ. At higher temperatures there are corrections to $\ln\gamma(T)$ of order $(T/T_0)^4$ arising from fast current fluctuations to which the phase cannot adjust adiabatically.[3] In Appendix A we show that the calculation of the rate enhancement using this method is in agreement with the result of the bounce method for arbitrary frequency-dependent damping and damping strength.

IV. NON-OHMIC DAMPING

For most experimental systems, the actual dissipative circuit is more complicated than the Ohmic model. Under these cases, Eq. (2.16) is used to find the rate enhancement. We illustrate the predictions under these more general circuits for the following two cases of non-Ohmic dissipation.

For simplicity, we will calculate the rate enhancements disregarding the prefactor corrections and considering the underdamped limit, that is where $Q(\omega)=C\omega_p/\mathrm{Re}[Y(\omega)]$ is always much larger than one for all frequencies below a few times ω_p. The exponent B of the zero-temperature MQT rate for vanishing damping is [cf. (3.3) and (3.11)]

$$B=\frac{36}{5}\frac{\Delta U}{\hbar\omega_p} \tag{4.1}$$

which gives

$$\left[\frac{\partial B}{\partial I}\right]^2=30\frac{B}{\hbar C\omega_p^3}\ . \tag{4.2}$$

[This is just the dominant term of (3.7) for $\alpha\ll 1$ and $B\gg 1$.] When this is substituted into (2.16) one finds that the thermal enhancement factor for lightly damped systems can be written as

$$\ln\gamma(T)=\frac{30}{\pi}\frac{B}{\hbar C\omega_p^3}\int_0^\infty d\omega\,\hbar\omega n_\beta(\omega)\mathrm{Re}[Y(\omega)]\ . \tag{4.3}$$

We first study the effect on the temperature dependence of the rate from a low-frequency resonance. Such a resonance might arise from parasitic loading from the current bias circuit. Specifically, we consider the system depicted in Fig. 4(a). Here, an RLC circuit with resistance R_s, inductance L_s and capacitance C_s is placed in parallel with an Ohmic shunt resistance R. The circuit can be reduced to that shown in Fig. 1 with an admittance,

$$Y(\omega)=\frac{1}{R}+\frac{i\omega C_s}{1+i\omega/Q_s\omega_s-(\omega/\omega_s)^2} \tag{4.4}$$

where

$$\omega_s=(L_sC_s)^{-1/2} \tag{4.5}$$

is the resonance frequency of the RLC circuit and

$$Q_s=1/R_sC_s\omega_s \tag{4.6}$$

its quality factor.

We consider the effect of this circuit when $\omega_s\ll\omega_p$ and $Q_s\gg 1$. Inserting the real part of (4.4) into (4.3) we obtain the result

$$\ln\gamma(T)=\frac{5\pi B}{RC\omega_p}\left[\frac{k_BT}{\hbar\omega_p}\right]^2+\frac{15B\omega_s}{L_sC\omega_p^3}n_\beta(\omega_s)\ . \tag{4.7}$$

Here, the first term is just the dominant piece of our earlier result (3.16) arising from the Ohmic component, while the second term comes from the low-frequency resonance. This latter term is proportional to the thermal occupation probability $n_\beta(\omega_s)$ of this mode. For $T\ll\hbar\omega_s/k_B$ there is no contribution to the rate enhancement from this term; for $T\gg\hbar\omega_s/k_B$ the enhancement scales as T. Further low-frequency resonant circuits in parallel with the above circuit would give corresponding contributions.

We see that for an arbitrary frequency-dependent damping term, the temperature dependence of the rate enhancement will obey a T^2 law arising from $Y(0)$ only for very small temperatures. The enhancement will be modified as soon as $k_BT/\hbar$ becomes of the order of the

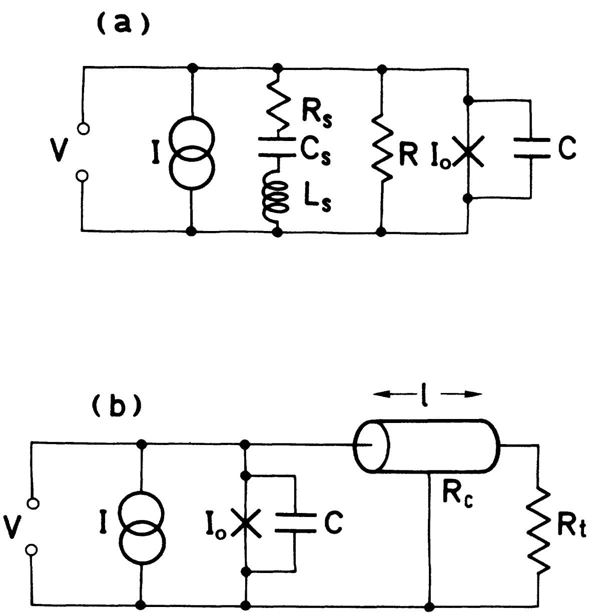

FIG. 4. Junction circuit models for two cases of non-Ohmic dissipation. (a) Admittance loading consisting of a resistor in parallel with RLC resonant circuit. (b) Admittance arising from a transmission line of characteristic impedance R_c, length l, and terminated by resistor R_t.

frequency scale characterizing the frequency dependence of $Y(\omega)$.

As a second non-Ohmic model we consider a junction shunted by an ideal transmission line. This line has impedance R_c and line velocity v, and is terminated after the length l by a resistance R_t [see Fig. 4(b)]. This model[19] is described by the admittance

$$Y(\omega)=\frac{1}{R_c}\frac{1+ae^{-i\omega\tau}}{1-ae^{-i\omega\tau}}\ , \tag{4.8}$$

where

$$a=\frac{R_c-R_t}{R_c+R_t} \tag{4.9}$$

is the reflection factor of the line, and

$$\tau=21/v \tag{4.10}$$

the time after which a reflected signal returns to the junction. The experimental realization of this model is presently under investigation at Saclay.

Inserting (4.8) into (4.3) one finds after some algebra that the thermal enhancement factor may be written as

$$\ln\gamma(T)=\frac{5\pi B}{R_c C\omega_p}\left[\frac{k_B T}{\hbar\omega_p}\right]^2 f\left[a,\frac{k_B T\tau}{\hbar}\right]\ , \tag{4.11}$$

where we have introduced the auxiliary function

$$f(a,x)=1+\frac{12}{\pi^2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{a^n(k^2-x^2n^2)}{(k^2+x^2n^2)^2}\ . \tag{4.12}$$

For small x this function takes the form

$$f(a,x)=\frac{R_c}{R_t}-\frac{2\pi^2}{5}\frac{a(1+a)}{(1-a)^3}x^2+O(x^4) \tag{4.13}$$

while for large x one has

$$f(a,x)=1+\left[\frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{a^n}{n^2}\right]\frac{1}{x^2}+O(e^{-\pi x})\ . \tag{4.14}$$

In Fig. 5 we have plotted $f(a,x)$ versus x^2 for the

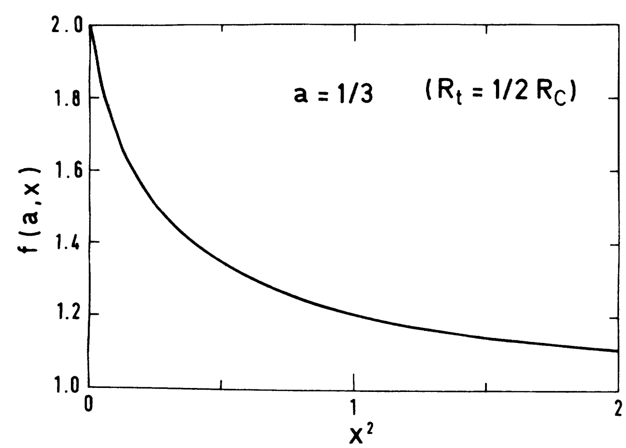

FIG. 5. Plot of $f(a,x)$ vs x^2, where $x=k_B T\tau/\hbar$. This function is used in Eq. (4.11) to predict the thermal enhancement of the rate for the transmission line circuit.

value $a=\frac{1}{3}$ which corresponds to $R_t=R_c/2$. Let us consider the case when $T_L\sim\hbar/k_B\tau=\hbar v/2k_B l$ is well below the crossover temperature T_0. A change from the low- to high-temperature values of the function $f(a,x)$ occurs around the temperature T_L. The enhancement factor $\ln\gamma(T)$ will follow a T^2 law with slope determined by the terminating resistance R_t in the region $T\ll T_L$. A T^2 law is found but with slope determined by R_c for $T_L\ll T\ll T_0$. This result makes physical sense because at temperatures $T\ll T_L$, the wavelength of the thermal radiation is much longer than the transmission line, and thus only the dissipation of the terminating resistor is felt by the junction. At higher temperatures, the typical wavelength of the radiation is shorter than the line, and then only the thermal noise of the transmission line is seen.

V. CONCLUSIONS

We have examined the low-temperature thermal enhancement of macroscopic quantum tunneling in Josephson systems. It was shown that the leading finite-temperature corrections to the tunneling rate arise from the low-frequency thermal noise of the dissipation. To calculate the increase of the rate explicitly, we have assumed that the noise currents lead to an adiabatically slow variation of the potential barrier. This is correct for temperatures well below the crossover temperature T_0 where the frequency of thermally excited current fluctuations is small compared with the plasma frequency. As shown by Pollak[20] tunneling can then be considered as a sudden transition in a quasistatic potential. At higher temperatures the thermal excitation of the states in the well[15] also lead to a further enhancement of the rate. Because of this our approach is limited to temperatures well below T_0.

We remark that the enhancement of the rate by thermal noise could, in principle, be determined independent from an adiabatic approximation if the enhancement of the zero-temperature rate by a sinusoidal microwave current were known as a function of frequency.[21]

We have demonstrated that the predictions of this calculation are in complete agreement with the results of the finite-temperature bounce approach.[2,3] This is as it should be. In the bounce approach one first performs a thermal average over the environment and then calculates the rate. In our approach, we first calculate the rate as a function of the environmental coordinates and then perform the thermal average. This should not make a difference except that the physical origin of the environmental effects is seen more explicitly in the calculation presented here. The bounce method does not involve an adiabatic approximation and can be applied for higher temperatures including the crossover to thermally activated decay.[3]

The rate calculated by the bounce method is a canonical rate. The underlying assumption is that the system decays out of a state of canonical quasiequilibrium in the metastable well. For very weakly damped systems at higher temperatures the observed rate may deviate from the canonical rate due to an underpopulation of the excited states in the well. Such depletion effects are well known from the classical theory of escape.[7] Above the crossover temperature, depletion effects are found to be only important for systems with Q factors large compared with $\Delta U/k_BT$. Below the crossover temperature, the depletion effects are expected to rapidly vanish.[8] This is confirmed by our calculation which gives complete agreement with the results of the bounce approach for arbitrary frequency-dependent damping and damping strength. Hence, this calculation solidifies the earlier results.

Apart from the physical insights provided, our approach is also mathematically quite straightforward. As we have shown, the low-temperature thermal enhancement of MQT is just proportional to the total noise current $\langle \delta I_{\rm th}^2 \rangle_T$ flowing through the junction, a quantity which is often readily determined. The theory presented here has the advantage that refinements, such as prefactor effects and predictions for arbitrary dissipation, are easily worked out. We have given some corresponding new analytical results. While we have presented the calculation specifically for the current biased Josephson junction, the analysis may directly be transferred to other systems with linear dissipation.

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APPENDIX A: EQUIVALENCE TO THE BOUNCE APPROACH FOR ARBITRARY DAMPING

In this appendix we show that the results on the thermal enhancement of MQT obtained here are in agreement with earlier findings based on functional integral techniques. This is shown for arbitrary frequency-dependent damping and damping strength. For a discussion of these latter methods we refer to Ref. 3. In the bounce approach the exponent B of the zero-temperature MQT rate is written as

$$B=\frac{1}{\hbar}S_B\ , \tag{A1}$$

where S_B is the action

$$S(\delta)=\int_{-\infty}^{+\infty}d\tau\left[\frac{1}{2}C\left[\frac{\phi_0}{2\pi}\right]^2(\dot\delta(\tau))^2+U(\delta(\tau))-U_0\right] +\tfrac{1}{2}\int_{-\infty}^{+\infty}d\tau\int_{-\infty}^{+\infty}d\tau'\,k(\tau-\tau')\,\delta(t)\,\delta(t') \tag{A2}$$

evaluated along the "bounce trajectory" $\delta_B(t)$. This trajectory is a saddle point of the action and describes the most probable escape path. U_0 is the value of the potential at the bottom of the metastable minimum and $k(\tau)$ is an influence kernel which is related to the admittance by

$$k(\tau)=\left[\frac{\phi_0}{2\pi}\right]^2\int_0^{\infty}\frac{d\omega}{\pi}\omega Y(-i\omega)\cos(\omega\tau)\ . \tag{A3}$$

We want to determine the change of the exponent arising from a change of the bias current. Using the form (2.3) of of the potential and the fact that $\delta_B(t)$ is a saddlepoint of the action we find

$$\frac{\partial S_B}{\partial I}=-\frac{\phi_0}{2\pi}\int_{-\infty}^{+\infty}d\tau[\delta_B(\tau)-\delta_0]\ , \tag{A4}$$

where δ_0 is the value of the phase at the potential minimum. Note that in most of the work referred to in Ref. 3 units are chosen such that U_0 and δ_0 vanish.

Following Ref. 2 we may introduce a "bounce length" τ_B through

$$\int_{-\infty}^{+\infty}d\tau[\delta_B(\tau)-\delta_0]=\tau_B\,\Delta\delta\ , \tag{A5}$$

where $\Delta\delta$ is the change of phase (2.6) during tunneling. Then, combining (A1) and (A4) we have

$$\left[\frac{\partial B}{\partial I}\right]^2=\frac{1}{\hbar^2}\left[\frac{\phi_0}{2\pi}\right]^2(\Delta\delta)^2\tau_B^2 =\frac{27}{2}\frac{\Delta U}{\hbar^2\omega_p^2C}\tau_B^2\ , \tag{A6}$$

where we have made use of (2.5) and (2.7) to derive the second line. Equation (A6) establishes the relation between $(\partial B/\partial I)$ arising within our method and the bounce length τ_B introduced in earlier work. Equation (A6) may now be inserted into (2.16). Performing the frequency integral for low temperatures and specific forms of $Y(\omega)$ one easily recovers the earlier results obtained by means of the bounce method. For instance, for an admittance with an Ohmic component [cf. (2.18)] one finds

$$\ln\gamma(T)=\frac{9\pi}{4}\frac{1}{R_0C\omega_p}\frac{\Delta U}{\hbar\omega_p}\left[\frac{k_BT\tau_B}{\hbar}\right]^2 \tag{A7}$$

which can easily be shown to be the same as the enhancement factor derived in Ref. 2.

APPENDIX B: ZERO-TEMPERATURE RATE SUPPRESSION AND THERMAL ENHANCEMENT

In this appendix we determine the influence of a low-frequency resonant circuit on both the zero-temperature suppression and the thermal enhancement of MQT. It will be sufficient to consider one single resonant circuit since the generalization to several modes or a continuum of modes is straightforward. The admittance describing an RLC circuit in parallel with the junction reads (cf. Sec. IV)

$$Y(\omega)=\frac{i\omega C_s}{1+i\omega/Q_s\omega_s-(\omega/\omega_s)^2}\ , \tag{B1}$$

where ω_s and Q_s are defined in (4.5) and (4.6). We are interested in a circuit with a narrow resonance, that is $Q_s \gg 1$.

The external circuit affects the tunneling rate by changing the response of the junction to quantum fluctuations, and by providing an additional source of quantum noise from the external circuit. First, we consider how the response of the junction to the external resonant circuit changes the tunneling rate. In the junction, the quantum fluctuations that produce the tunneling occur at frequencies around the plasma frequency. At these frequencies, which are well above ω_s, the response of the external resonant circuit is that of an inductor L_s in parallel with the junction. The equations of motion of the junction and this inductor can be given by a renormalization of the junction potential

$$U'=U+\frac{1}{2L_s}\left[\frac{\phi_0}{2\pi}\right]^2(\delta-\delta_0)^2 \ . \tag{B2}$$

We note that the new term in U' is exactly the "counterterm" that was introduced in the Caldeira and Leggett[1] theory. By using Eqs. (2.5)–(2.7), one can show that ΔU and ω_p are changed by

$$\Delta U'=\Delta U\left[1+\frac{3}{L_s C\omega_p^2}\right] , \tag{B3}$$

$$\omega_p'=\omega_p\left[1+\frac{1}{2L_s C\omega_p^2}\right] \tag{B4}$$

to first order in the effective coupling strength

$$1/L_s C\omega_p^2=C_s\omega_s^2/C\omega_p^2 \ .$$

Since the tunneling exponent $B=7.2\Delta U/\hbar\omega_p$, we find that the change in the tunneling exponent due to the potential renormalization is $1+5/2L_sC\omega_p^2$.

The calculation of the effect of the quantum noise from the external resonant circuit on the tunneling proceeds as in Sec. IV. However, we now have to include the zero-point noise term. This replaces n_β in Eq. (4.3) with $n_\beta+\frac{1}{2}$. Combining the resulting above and the second term of Eq. (4.7), we find the change in the tunneling rate from the external circuit to be

$$\ln[\Gamma'(T)/\Gamma(0)]=-\frac{B}{L_s C\omega_p^2}\left[\frac{5}{2}-15\frac{\omega_s}{\omega_p}[n_\beta(\omega_s)+\tfrac{1}{2}]\right] . \tag{B5}$$

To check this derivation, we recalculate this result at $T=0$ using conventional tunneling theory.[22] The change in the tunneling rate for an arbitrary admittance that loads the junction is given in perturbation theory by[19]

$$\ln[\Gamma'(0)/\Gamma(0)]=\frac{15\pi B}{C\omega_p^5}\int_0^\infty \frac{Y(-i\omega)\omega^3}{\sinh^2(\pi\omega/\omega_p)}d\omega \ . \tag{B6}$$

For the admittance given in (B1) and for $Q_s \gg 1$, one can evaluate this integral in an expansion of ω_s/ω_p. One then finds the result of Eq. (B5) for $T=0$. Conventional tunneling theory[2] can also be shown to reproduce the finite-temperature corrections in (B5) in the perturbative limit.

Because $\omega_s/\omega_p \ll 1$, the contribution of the second term in Eq. (B5) is much smaller in magnitude than the first term. Thus, as expected, one always sees a net depression in the tunneling rate by coupling the junction to a resonant circuit; the change in the tunneling due to the quantum noise is smaller than the change due to the potential renormalization. Equation (B5) also predicts that the increase in the tunneling rate at finite temperatures is given by the thermal noise arising from the resonant circuit. This justifies the subtraction of the zero-point noise in Eq. (2.10).

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