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Thermal conductivity and ultrasonic attenuation in heavy-fermion superconductors

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We calculate the thermal conductivity and ultrasonic attenuation in anisotropic superconductors in which the scattering of quasiparticles by nonmagnetic impurities is the dominant process. The concentration of impurities is assumed to be low enough that broadening of quasiparticle states may be neglected, and hence the quasiparticle Boltzmann equation may be used. Detailed calculations are presented for two limiting choices of the normal-state phase shift, $\delta_N \ll \pi/2$ and $\delta_N = \pi/2$. The superconducting states considered are the axial and polar p-wave states and an axial d-wave state.

I. INTRODUCTION

A fascinating feature of experimentally measured transport properties of heavy-fermion superconductors is that at low temperatures they do not depend exponentially on the temperatures, as one would expect for a standard BCS superconductor. This is best illustrated in the case of UPt_3, for which the thermal conductivity, K, varies as T in the normal state for temperatures between the superconducting transition temperature, T_c, and 550 mK (Ref. 1) and the ultrasonic attenuation is a rather weak function of T for temperatures between just above T_c and about 550 mK.[2] Both these dependences are what one would expect for scattering of electron quasiparticles by nonmagnetic impurities in a normal metal at temperatures well below the Fermi temperature. In the other heavy-fermion superconductors, the normal-state properties are more complicated; even at temperatures close to the superconducting transition temperature they do not behave like Fermi liquids in which electrons scatter from impurities, a fact which gives strong evidence for the importance of other processes, such as electron-electron scattering. In the superconducting state one would expect the thermal conductivity and the ultrasonic attenuation to fall off exponentially with decreasing temperature if the gap had a finite value at all points on the Fermi surface, as it does for the usual BCS state. However, the thermal conductivity of polycrystalline UPt_3 is observed to vary roughly as T^2 between 35 and 100 mK in a field of 1 kG (Ref. 3) and the attenuation of sound in UPt_3 is observed to vary approximately as a power law, with an exponent that depends on the direction of propagation and polarization of the sound. Different groups[2,4,5] find different results for longitudinal sound, but in no case is the decrease in the attenuation with decreasing temperature as rapid as one would expect for a BCS superconductor.

An important question is whether the measurement of transport properties can provide information about the nature of the superconducting state or states of heavy-fermion superconductors. Initially, Bishop *et al.*[4] argued that their ultrasound measurements provided support for the superconducting phase having a gap structure similar to the polar p-wave state. Subsequently, Rodriguez[6] carried out more detailed calculations based on the assumption that, as in a BSC superconductor, the mean free path of a quasiparticle in the superconducting state is the same as in the normal state. He concluded that the gap structure in UPt_3 is similar to the axial (*ABM*) p-wave state of liquid ^{3}He. Following this, calculations of the quasiparticle mean free path by Coffey, Rice, and Ueda[7] and by Pethick and Pines[8] showed that the mean free path in the axial and polar p-wave states diverges at low temperatures if the scattering is treated in the Born approximation. The latter authors also showed that in this approximation, the ultrasonic attenuation does not generally vanish as $T \to 0$, but is typically of the same order of magnitude as in the normal state. Likewise, K/T was shown generally to tend to a finite value as $T \to 0$. Both these results are in sharp contrast to what is found experimentally.

In Ref. 8 it was shown that if, instead of using the Born approximation, one assumes that multiple interactions of a quasiparticle with an impurity are important, which corresponds to the phase shift in the normal state being close to $\pi/2$, the qualitative temperature dependence of the transport coefficients in the superconducting phase is in better agreement with experiment. The physical reason for this is that quasiparticle states with energies comparable to or less than the maximum value of the energy gap on the Fermi surface are strongly affected by superconducting correlations. These enter as intermediate states in the scattering process, and consequently the quasiparticle scattering amplitude is also strongly affected at such energies.

In the calculation by Pethick and Pines the broadening of quasiparticle states by the impurity scattering was neglected, which is a valid approximation for low concentrations of impurities, and subsequently calculations allowing for the width of quasiparticle states were performed by Hirschfeld, Vollhardt, and Wölfle,[9] by Scharnberg *et al.*,[10] and by Schmitt-Rink, Miyake, and Varma.[11] In Refs. 9 and 10 phase shifts other than $\pi/2$ were considered, and in Ref. 10 vertex corrections were also included. These calculations generally confirmed the conclusions of Ref. 8, except that the broadening of quasiparticle states led to qualitatively different behavior at temperatures where the quasiparticle width is comparable to or greater than the thermal energy $k_B T$, and to a very slight rounding of the behavior at the transition temperature T_c. All the above work was for p-wave states, but following this, Monien *et al.*[12] performed calculations for two d-wave states and found results qualitatively similar to that for the p-wave states. Recently, Monien *et al.*[13] have improved the treatment of vertex corrections, and they and Pethick and Pines[14] have pointed to a number of asymmetries in scattering that can arise for phase shifts different from $\pi/2$.

This paper is the first of a series in which we calculate transport properties of a number of different superconducting states starting from the quasiparticle Boltzmann equation. The Boltzmann-equation approach has two virtues compared with the Green's-function approach used in Refs. 9–13: It is computationally much simpler, and its physical content is more transparent. Its shortcoming is that it cannot be used to describe situations where the width of quasiparticle states is important, but, as is clear from the earlier calculations, the effects of the finite width are likely to be important only at temperatures $\lesssim T_c/10$ in practical situations.

In this paper we shall calculate the quasiparticle lifetime, the thermal conductivity, and the ultrasonic attenuation for the axial and polar p-wave states and for the d-wave state consistent with the hexagonal symmetry. Here we shall consider the limiting cases of small phase shifts (Born approximation) and a phase shift of $\pi/2$, corresponding to resonant scattering. The physical motivation for these two cases is provided by noting that in heavy-electron materials the f-electron atoms, such as U and Ce, are thought to be primarily responsible for the heavy-electron behavior. When nonmagnetic impurities are introduced into a heavy-electron material, one can therefore envisage two distinct types: the first where the impurity replaces an f-electron atom, and the second when it replaces one of the other atoms. In the first case, the impurity corresponds to the absence of a magnetic site, and might therefore be expected to give rise to a phase shift close to $\pi/2$, while in the second case, the phase shift is likely small. The calculations for small phase shifts are also useful by way of reference, since in most calculations of transport properties of superconductors to date it has been assumed that the phase shift is small. Novel effects that arise for intermediate values of the phase shift will be discussed in a future paper.

The outline of the paper is as follows. Scattering theory and the quasiparticle lifetime are considered in Sec. II. The thermal conductivity is calculated in Sec. III and the ultrasonic attenuation in Sec. IV. Section V is a brief conclusion.

II. QUASIPARTICLE RELAXATION TIME

In this section we calculate the quasiparticle relaxation rate due to elastic scattering from nonmagnetic impurities. When the density of impurities, n_i, is sufficiently low, the relaxation rate, $1/\tau_{\mathbf{p}\sigma}$, of a quasiparticle of momentum $\mathbf{p}$ and spin σ in the superconductor is given by

$$\frac{1}{\tau_{\mathbf{p}\sigma}}=\frac{2\pi}{\hbar}n_i\sum_{\mathbf{p}',\sigma'}|t^S_{\mathbf{p}'\sigma',\mathbf{p}\sigma}|^2\delta(E_{\mathbf{p}\sigma}-E_{\mathbf{p}'\sigma'})\ , \tag{1}$$

where $t^S_{\mathbf{p}'\sigma',\mathbf{p}\sigma}$ is the amplitude for a single impurity to scatter a quasiparticle from the state $\mathbf{p}\sigma$ to the state $\mathbf{p}'\sigma'$, and $E_{\mathbf{p}\sigma}$ is the quasiparticle energy. In the calculations in this paper we shall restrict our attention to states for which the gap matrix, $\underline{\Delta}_{\mathbf{p}}$, a 2×2 matrix in (pseudo)spin space, is unitary. The quasiparticle spectrum is then independent of the spin index and is given by

$$E_{\mathbf{p}}=(\xi_p^2+\underline{\Delta}_{\mathbf{p}}\underline{\Delta}_{\mathbf{p}}^{\dagger})^{1/2}\ , \tag{2}$$

where $\xi_{\mathbf{p}}\approx v_F(p-p_F)$ is the energy of a quasiparticle in the normal state, relative to its value at the Fermi momentum p_F. Here v_F is the Fermi velocity. The operators $a^{\dagger}$ and a which create and destroy normal-state quasiparticles are related to the corresponding ones $\alpha^{\dagger}$ and α for superconducting-state quasiparticles by a unitary transformation,

$$a_{\mathbf{p}\sigma}=u_{\mathbf{p}\sigma\sigma'}\alpha_{\mathbf{p}\sigma'}+v_{\mathbf{p}\sigma\sigma'}\alpha^{\dagger}_{-\mathbf{p}\sigma'}\ . \tag{3}$$

The scattering amplitude $t^S_{\mathbf{p}'\sigma',\mathbf{p}\sigma}$ in the expression for the relaxation time is obtained by performing the unitary transformation on the corresponding amplitude for scattering of normal-state quasiparticles, and is given by

$$\begin{aligned} t^S_{\mathbf{p}',\mathbf{p}} = {} & u^{\dagger}_{\mathbf{p}'}(t_{11})_{\mathbf{p}'\mathbf{p}}u_{\mathbf{p}}+v^{\dagger}_{\mathbf{p}'}(t_{12})_{\mathbf{p}'\mathbf{p}}u_{\mathbf{p}} \\ & +u^{\dagger}_{\mathbf{p}'}(t_{21})_{\mathbf{p}'\mathbf{p}}v_{\mathbf{p}}+v^{\dagger}_{\mathbf{p}'}(t_{22})_{\mathbf{p}'\mathbf{p}}v_{\mathbf{p}}\ , \end{aligned} \tag{4}$$

where t_{ij} is the amplitude in the superconductor for scattering a normal-state quasiparticle ($j=1$) or quasihole ($j=2$) of momentum $\mathbf{p}$ to a quasiparticle ($i=1$) or quasihole ($i=2$) of momentum $\mathbf{p}'$. For brevity we have suppressed spin indices.

To calculate the amplitudes t_{ij} we follow the treatment of Ref. 8, which was based on earlier work on the mobility of ions in the superfluid phases of liquid ^{3}He.[15] In the Nambu formalism the single-particle Green's function for the superconductor in the absence of impurities has the form of a 4×4 matrix,

$$G(\mathbf{p},E)=\frac{1}{E^2-E_{\mathbf{p}}^2}\begin{bmatrix} E+\xi_{\mathbf{p}} & -\underline{\Delta}_{\mathbf{p}} \\ -\underline{\Delta}_{\mathbf{p}}^{\dagger} & E-\xi_{\mathbf{p}} \end{bmatrix}\ . \tag{5}$$

The scattering amplitude, T, in the Nambu notation satisfies the Lippmann-Schwinger equation

$$T=V+VGT\ , \tag{6}$$

where the scattering potential is $V=v\tau_3$. The quantity v is the potential due to the impurity, which we take to be diagonal in the spin variable, since the impurity is assumed to be nonmagnetic, and τ_3 is the Pauli matrix in

Nambu space. By using in Eq. (6) the expression for the Green's function in the absence of impurities, we have implicitly neglected the fact that when the concentration of impurities is finite, the intermediate states in the scattering from one impurity may be affected by other impurities. We shall discuss in more detail below the conditions under which this condition holds. It is convenient to work with the normal-state reaction or K matrix defined by the equation

$$K^N = V + VG_{\text{off}}^N K^N , \tag{7}$$

where $G_{\text{off}} = G - G_{\text{on}}$. G_{off} and G_{on} are, respectively, the off- and on-energy-shell contributions to the Green's function.[15] Using Eqs. (6) and (7), we obtain

$$T = K^N + K^N(G - G_{\text{off}}^N)T . \tag{8}$$

Since we are interested only in states close to the Fermi surface, it is convenient to eliminate those far away. Far from the Fermi surface G is unaffected by superconducting correlations, and has no on-shell part for energies of order the gap energy, which are the ones of interest. Consequently $G - G_{\text{off}}^N$ vanishes there, and the sum over intermediate states in Eq. (8) may be restricted to states close to the Fermi surface. If one sums over a set of states in an energy range within $\pm E_0$ of the Fermi surface, where $\Delta \ll E_0 \ll E_F$, the normal-state off-shell contribution to Eq. (8) vanishes, and we may write simply

$$T = K^N + K^N G' T , \tag{9}$$

where the prime on G indicates the restriction on the intermediate states. We shall assume that the scattering is of short range, and that it scatters only in s states. The normal state K matrix is then independent of initial and final momenta and is given by

$$K^N = k_N \tau_3 , \tag{10}$$

where $k_N = -\tan\delta_N / \pi N(0)$, with δ_N being the normal-state s-wave phase shift, and $N(0) = m^* p_F/(2\pi^2\hbar^3)$ the density of quasiparticle states at the Fermi surface for a single spin. We assume the Fermi surface to be a sphere of radius p_F, and the mass, m^*, to be isotropic.

The superconducting states we shall consider in this paper are the axial and polar p-wave ones and a d-wave state that is consistent with the hexagonal symmetry of UPt_3. Group theory considerations argue against the polar state,[16,17] so it thus must be regarded as an illustrative example, rather than a serious candidate for the superconducting state.

For singlet pairing, the gap matrix may be written as

$$\underline{\Delta} = \sigma_2 \Delta(\hat{\mathbf{p}}) , \tag{11}$$

where the σ_i are Pauli spin matrices. For the BCS case, $\Delta(\hat{\mathbf{p}})$ is independent of $\mathbf{p}$, while for the d-wave state consistent with hexagonal symmetry, one has

$$\Delta(\hat{\mathbf{p}}) = 2\Delta(T)\cos\theta \sin\theta e^{i\phi} , \tag{12}$$

where $\Delta(T)$ is the maximum value of the gap on the Fermi surface. Here θ and ϕ give the direction of $\mathbf{p}$ in polar coordinates. This state has point nodes at the poles, and a line node along the equator.

In the case of triplet pairing, the gap matrix may be written as

$$\underline{\Delta} = i\sigma_2 \boldsymbol{\sigma}\cdot\boldsymbol{\Delta}(\hat{\mathbf{p}}) . \tag{13}$$

The polar state has

$$\boldsymbol{\Delta}(\hat{\mathbf{p}}) = \Delta(T)\cos\theta \hat{\mathbf{d}} , \tag{14}$$

and the axial (ABM) state has

$$\boldsymbol{\Delta}(\hat{\mathbf{p}}) = \Delta(T)(\hat{\mathbf{i}} + i\hat{\mathbf{j}})\cdot\hat{\mathbf{p}}\hat{\mathbf{d}} = \Delta(T)\sin\theta e^{i\phi}\hat{\mathbf{d}} , \tag{15}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are unit vectors in the 1, 2, and 3 directions of momentum space, and $\hat{\mathbf{d}}$ is a fixed unit vector in spin space.

Since the K matrix is independent of momenta, the T matrix is also. With the intermediate state sums written out explicitly, Eq. (9) has the form

$$T(E) = K^N + K^N \left[\sum_{\mathbf{p}} G'(\mathbf{p}, E) \right] T(E) . \tag{16}$$

This matrix equation simplifies considerably, since the off-diagonal matrix elements in Nambu space, proportional to $\sum_{\mathbf{p}} [\Delta_{\mathbf{p}}/(E^2 - E_{\mathbf{p}}^2)]$, vanish for the three states we consider. T is thus diagonal in spin and in particle-hole space and is given simply by

$$T = k_N \tau_3 [1 + i\pi N(0) k_N \tau_3 g(E)]^{-1} , \tag{17}$$

where τ_3 is the Pauli matrix in the particle-hole part of Nambu space, and

$$g(E) = \frac{i}{\pi} \int \frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi} \int_{-E_0}^{E_0} d\xi_{\mathbf{p}} \frac{E}{E^2 - E_{\mathbf{p}}^2} \tag{18}$$

takes into account the modification of intermediate states by superconducting correlations. In the normal state $g(E)$ is unity, so the matrix elements of T are simply $\mp e^{\pm i\delta_N} \sin\delta_N / [\pi N(0)]$. The function $g(E)$ has a real contribution proportional to the density of states, and an imaginary part which corresponds to a dispersive correction to the quasiparticle self-energy. The latter vanishes if the magnitude of the energy is above the maximum value of the energy gap as a function of angle on the Fermi surface, but is nonzero below. Consequently, it is important only in anisotropic superconductors, since in isotropic ones there are no excitations with energies less than the maximum energy gap.

In this paper we shall confine ourselves to the limiting cases of small phase shifts, and ones close to $\pi/2$. For small phase shifts, the matrix elements of T are simply

$$t_{11} = -t_{22} \simeq t_N = -\delta_N / [\pi N(0)] , \tag{19}$$

where t_N is the normal-state scattering amplitude for particles, while for $\delta_N = \pi/2$ they are given by

$$t_{11} = t_{22} = \frac{1}{i\pi N(0) g(E)} = \frac{t_N}{g(E)} , \tag{20}$$

since $t_N = [i\pi N(0)]^{-1}$ in this case. We shall discuss the case of other phase shifts elsewhere. The amplitude for scattering of superconducting-state quasiparticles, given by Eq. (4), thus has the form

$$t^S_{\mathbf{p}'\mathbf{p}} = t_N(u^\dagger_{\mathbf{p}'}v_{\mathbf{p}} - v^\dagger_{\mathbf{p}'}v_{\mathbf{p}}) \quad \text{for } \delta_N \ll \pi/2 , \tag{21}$$

and

$$t^S_{\mathbf{p}'\mathbf{p}} = \frac{t_N}{g(E)}(u^\dagger_{\mathbf{p}'}v_{\mathbf{p}} + v^\dagger_{\mathbf{p}'}u_{\mathbf{p}}) \quad \text{for } \delta_N = \pi/2 . \tag{22}$$

In this paper we are not interested in spin-dependent properties, so we calculate the quantity

$$\overline{|t^S_{\mathbf{p}'\mathbf{p}}|^2} = \tfrac{1}{2}\sum_{\sigma,\sigma'} |t^S_{\mathbf{p}'\sigma',\mathbf{p}\sigma}|^2 , \tag{23}$$

which is the square of the scattering matrix element summed over final spin states and averaged over initial spin states. Using the usual definitions of the quantities $u_{\mathbf{p}}$ and $v_{\mathbf{p}}$, we find

$$\overline{|t^S_{\mathbf{p}'\mathbf{p}}|^2} = \frac{|t_N|^2}{2}\left[1 + \frac{\xi_{\mathbf{p}}\xi_{\mathbf{p}'}}{E_{\mathbf{p}}E_{\mathbf{p}'}} - \frac{\mathrm{Re}(\Delta_{\mathbf{p}}\Delta^*_{\mathbf{p}'})}{E_{\mathbf{p}}E_{\mathbf{p}'}}\right] \quad \text{(singlet pairing, } \delta_N \ll \pi/2) , \tag{24}$$

$$\overline{|t^S_{\mathbf{p}'\mathbf{p}}|^2} = \frac{|t_N|^2}{2|g(E)|^2}\left[1 + \frac{\xi_{\mathbf{p}}\xi_{\mathbf{p}'}}{E_{\mathbf{p}}E_{\mathbf{p}'}} + \frac{\mathrm{Re}(\Delta_{\mathbf{p}}\Delta^*_{\mathbf{p}'})}{E_{\mathbf{p}}E_{\mathbf{p}'}}\right] \quad \text{(singlet pairing, } \delta_N = \pi/2) , \tag{25}$$

$$\overline{|t^S_{\mathbf{p}'\mathbf{p}}|^2} = \frac{|t_N|^2}{2}\left[1 + \frac{\xi_{\mathbf{p}}\xi_{\mathbf{p}'}}{E_{\mathbf{p}}E_{\mathbf{p}'}} - \frac{\mathrm{Re}(\boldsymbol{\Delta}_{\mathbf{p}}\cdot\boldsymbol{\Delta}^*_{\mathbf{p}'})}{E_{\mathbf{p}}E_{\mathbf{p}'}}\right] \quad \text{(triplet pairing, } \delta_N \ll \pi/2) , \tag{26}$$

and

$$\overline{|t^S_{\mathbf{p}'\mathbf{p}}|^2} = \frac{|t_N|^2}{2|g(E)|^2}\left[1 + \frac{\xi_{\mathbf{p}}\xi_{\mathbf{p}'}}{E_{\mathbf{p}}E_{\mathbf{p}'}} + \frac{\mathrm{Re}(\boldsymbol{\Delta}_{\mathbf{p}}\cdot\boldsymbol{\Delta}^*_{\mathbf{p}'})}{E_{\mathbf{p}}E_{\mathbf{p}'}}\right] \quad \text{(triplet pairing, } \delta_N = \pi/2) . \tag{27}$$

In terms of $\overline{|t^S_{\mathbf{p}'\mathbf{p}}|^2}$, the relaxation rate, Eq. (1), is given by

$$\frac{1}{\tau_{\mathbf{p}}} = \frac{2\pi}{\hbar} n_i \sum_{\mathbf{p}'} \overline{|t^S_{\mathbf{p}'\mathbf{p}}|^2}\,\delta(E_{\mathbf{p}} - E_{\mathbf{p}'}) . \tag{28}$$

When the sum in Eq. (28) is performed, the $\xi_{\mathbf{p}}\xi_{\mathbf{p}'}$ terms in (24)–(27) vanish since contributions for $\mathbf{p}'$ above and below the Fermi surface cancel. The $\Delta_{\mathbf{p}}\Delta^*_{\mathbf{p}'}$ terms also vanish on summation for all the states we consider in this paper—they vanish for *any* p-wave state because of the odd parity of the gap, and for the d-wave state (12) they vanish on integrating over ϕ, since $\Delta(\phi+\pi) = -\Delta(\phi)$. The expression for the relaxation rate is thus simply

$$\frac{1}{\tau_{\mathbf{p}}} = \frac{1}{\tau_N}\frac{N_S(E_{\mathbf{p}})}{N(0)} \quad (\delta_N \ll \pi/2)$$

and (29)

$$\frac{1}{\tau_{\mathbf{p}}} = \frac{1}{\tau_N}\frac{1}{|g(E_{\mathbf{p}})|^2}\frac{N_S(E_{\mathbf{p}})}{N(0)} \quad (\delta_N = \pi/2) ,$$

where τ_N is the normal-state relaxation time, given by

$$\frac{1}{\tau_N} = \frac{2\pi}{\hbar} n_i |t_N|^2 N(0) . \tag{30}$$

The quantity g for the various superconducting states may be determined from Eq. (18), and we find

$$g(E) = \int \frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi}\frac{E}{(E^2 - |\Delta_{\mathbf{p}}|^2)^{1/2}} \tag{31}$$

where the square root is to be interpreted as $+i(|\Delta_{\mathbf{p}}|^2 - E^2)^{1/2}$ for $E < |\Delta_{\mathbf{p}}|$, where $|\Delta_{\mathbf{p}}|$ is the magnitude of the energy gap in the direction $\mathbf{p}$ on the Fermi surface. We find for the axial state

$$\begin{aligned} g(x) &= \frac{x}{2}\ln\left[\frac{1+x}{1-x}\right] - i\frac{\pi x}{2} \quad \text{for } x < 1 \\ &= \frac{x}{2}\ln\left[\frac{x+1}{x-1}\right] \quad \text{for } x > 1 , \end{aligned} \tag{32}$$

for the polar state

$$\begin{aligned} g(x) &= \frac{\pi x}{2} - ix\ln\left[\frac{1+(1-x^2)^{1/2}}{x}\right] \quad \text{for } x < 1 \\ &= x\arcsin\left[\frac{1}{x}\right] \quad \text{for } x > 1 , \end{aligned} \tag{33}$$

and for the d-wave state Eq. (12)

$$\begin{aligned} g(x) &= \frac{x}{2}\left[\int_0^{\mu_1} + \int_{\mu_2}^1\right]\frac{d\mu}{(\mu^4 - \mu^2 + x^2/4)^{1/2}} \\ &\quad - i\frac{x}{2}\int_{\mu_1}^{\mu_2}\frac{d\mu}{(\mu^2 - \mu^4 - x^2/4)^{1/2}} \quad \text{for } x < 1 \\ &= \frac{x}{2}\int_0^1\frac{d\mu}{(\mu^4 - \mu^2 + x^2/4)^{1/2}} \quad \text{for } x > 1 , \end{aligned} \tag{34}$$

where $\mu_1 = ([1-(1-x^2)^{1/2}]/2)^{1/2}$ and $\mu_2 = ([1+(1+x^2)^{1/2}]/2)^{1/2}$ and $x = E/\Delta$, Δ being the maximum value of the gap. The density of states is given for all state and all energies by $N_S(E)/N(0) = \mathrm{Re}\,g(E/\Delta)$.

In Fig. 1 we plot the quasiparticle relaxation time as a function of E/Δ. For $\delta_N \ll \pi/2$, the relaxation time diverges as $E \to 0$ because the density of states tends to zero as $E \to 0$. The leading contributions to τ_S/τ_N at low energies are $(2/\pi)\Delta/E$ for the polar state, Δ^2/E^2 for the axial state, and $(4/\pi)\Delta/E$ for the d-wave state. Close to Δ, the relaxation time tends to $2\tau_N/\pi$ for the polar state, and tends to zero as $1/|\ln(|1-x|)|$ for the axial and d-wave states. In the case of a phase shift $\pi/2$, in the axial state τ_S/τ_N tends to $\pi^2/4$ as $E \to 0$ and diverges as $\frac{1}{2}\ln(1/|1-x|)$ for $E \to \Delta$, while for the polar state, τ_S/τ_N behaves as

$$x\left[\frac{\pi}{2} + \frac{2}{\pi}\ln^2 x\right]$$

for small x, and is $\pi/2$ at $E = \Delta$. For the d-wave state, τ_S/τ_N behave as

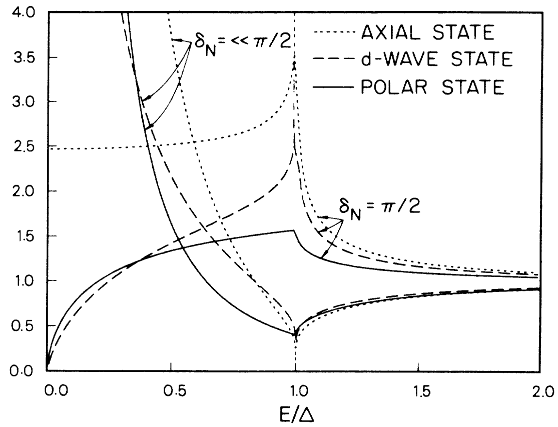

FIG. 1. Plots of the quasiparticle relaxation times in the axial, polar, and *d*-wave states as a function of energy for the cases of small phase shifts ($\delta_N \ll \pi/2$) and of resonant scattering ($\delta_N = \pi/2$).

$$\frac{x}{2}\left[\frac{\pi}{2}+\frac{2}{\pi}\ln^2 x\right]$$

at low energies, and diverges as $\sim\frac{3}{2}\ln(1/|1-x|)$ for $E\rightarrow\Delta$.

A remarkable feature of the relaxation time is its insensitivity to energy for a phase shift of $\pi/2$. In calculations which include effects of impurity scattering on intermediate states in the scattering process, this effect is even more pronounced, because τ_S at $E=0$ remains finite for all states, and the logarithmic singularities at $E=\Delta$ are smoothed out.[11]

III. THERMAL CONDUCTION

In this section we calculate the thermal conductivity starting from the quasiparticle Boltzmann equation. The Boltzmann-equation approach to kinetic phenomena in superconductors has been placed on a firm microscopic footing by Betbeder-Matibet and Nozières,[18] and it is valid provided frequencies are small compared with the gap frequency, length scales are large compared with the temperature-dependent coherence length, and the width of quasiparticle states due to pair-breaking processes is small compared with both the quasiparticle energies of interest and Δ. The Boltzmann equation has the form,

$$\frac{\partial n_{\mathbf{p}}}{\partial t}+\nabla_{\mathbf{r}} n_{\mathbf{p}}\cdot\nabla_{\mathbf{p}}E_{\mathbf{p}}-\nabla_{\mathbf{p}} n_{\mathbf{p}}\cdot\nabla_{\mathbf{r}}E_{\mathbf{p}}=\left[\frac{\partial n_{\mathbf{p}}}{\partial t}\right]_{\text{coll}}, \tag{35}$$

where $n_{\mathbf{p}}$ is the quasiparticle distribution and $(\partial n_{\mathbf{p}}/\partial t)_{\text{coll}}$ is the collision integral. In the hydrodynamic limit one may replace the distribution function on the left-hand side of the equation by its local equilibrium value, and after the standard Chapman-Enskog analysis, the equation reduces to

$$-E_{\mathbf{p}}\mathbf{v}_{\mathbf{p}}\cdot\frac{\nabla T}{T}\frac{\partial n_{\mathbf{p}}^0}{\partial E_{\mathbf{p}}}=\left[\frac{\partial n_{\mathbf{p}}}{\partial t}\right]_{\text{coll}}, \tag{36}$$

where $n_{\mathbf{p}}^0$ is the equilibrium distribution function, T is the temperature, and $\mathbf{v}_{\mathbf{p}}=\partial E_{\mathbf{p}}/\partial\mathbf{p}$ is the quasiparticle velocity. In writing this equation we have neglected the term involving the specific entropy, which is small compared with the term we have retained.

The collision integral is given by

$$\left[\frac{\partial n_{\mathbf{p}}}{\partial t}\right]_{\text{coll}}=-\frac{2\pi}{\hbar}n_i\sum_{\mathbf{p}'}\overline{|t_{\mathbf{p}'\mathbf{p}}^S|^2}\delta(E_{\mathbf{p}}-E_{\mathbf{p}'})(n_{\mathbf{p}}-n_{\mathbf{p}'})\ . \tag{37}$$

We now write

$$n_{\mathbf{p}}=n_{\mathbf{p}}^{\text{LE}}+\frac{\partial n_{\mathbf{p}}^0}{\partial E_{\mathbf{p}}}\Phi_{\mathbf{p}}\ , \tag{38}$$

where $\Phi_{\mathbf{p}}$ is a measure of the deviation of the distribution function from its local equilibrium value, $n_{\mathbf{p}}^{\text{LF}}$. Note that the local equilibrium distribution is the Fermi function evaluated for the local values of the temperature, chemical potential, and quasiparticle energy. The latter quantity is not generally equal to its value in global equilibrium. The transport equation may thus be written in the form

$$\Phi_{\mathbf{p}}-\tau_{\mathbf{p}}\frac{2\pi n_i}{\hbar}\sum_{\mathbf{p}}\overline{|t_{\mathbf{p}'\mathbf{p}}^S|^2}\delta(E_{\mathbf{p}}-E_{\mathbf{p}'})\Phi_{\mathbf{p}'}=\tau_{\mathbf{p}}E_{\mathbf{p}}\mathbf{v}_{\mathbf{p}}\cdot\frac{\nabla T}{T}\ , \tag{39}$$

where we have adopted the simplified notation $\tau_{\mathbf{p}}\equiv\tau_s(E_{\mathbf{p}})$, for the relaxation time, Eq. (28).

The driving term is odd in $\xi_{\mathbf{p}}$, since $v_{\mathbf{p}}=(\xi_{\mathbf{p}}/E_{\mathbf{p}})v_F$, when one neglects effects of order Δ/E_F, where $E_F=p_F^2/2m^*$, v_F is the Fermi velocity, and it has odd parity. Since the collision integral preserves parity and symmetry under the replacement of $\xi_{\mathbf{p}}$ by $-\xi_{\mathbf{p}}$, Φ must be odd in $\xi_{\mathbf{p}}$ and have odd parity. As a result, the terms proportional to 1 and to $\Delta_{\mathbf{p}}\Delta_{\mathbf{p}'}^*$ in Eqs. (24)–(27) vanish on summation because Φ is an odd function of $\xi_{\mathbf{p}}$, and the $\xi_{\mathbf{p}}\xi_{\mathbf{p}'}$ terms vanish because of the odd parity. Thus Φ is given simply by

$$\Phi_{\mathbf{p}}=\tau_{\mathbf{p}}E_{\mathbf{p}}\mathbf{v}_{\mathbf{p}}\cdot\frac{\nabla T}{T}\ , \tag{40}$$

and the heat current is

$$\begin{aligned}\mathbf{j}_E&=\sum_{\mathbf{p},\sigma}E_{\mathbf{p}}\mathbf{v}_{\mathbf{p}}(n_{\mathbf{p}}-n_{\mathbf{p}}^{\text{LE}})\\&=\sum_{\mathbf{p},\sigma}E_{\mathbf{p}}\mathbf{v}_{\mathbf{p}}\frac{\partial n_{\mathbf{p}}^0}{\partial E_{\mathbf{p}}}\Phi_{\mathbf{p}}\\&=\sum_{\mathbf{p},\sigma}E_{\mathbf{p}}^2\mathbf{v}_{\mathbf{p}}\frac{\mathbf{v}_p\cdot\nabla T}{T}\frac{\partial n_{\mathbf{p}}^0}{\partial E_{\mathbf{p}}}\tau_{\mathbf{p}}\ .\end{aligned} \tag{41}$$

From this we find the thermal conductivity tensor K_{ij}, given by $(\mathbf{j}_E)_i=-K_{ij}\nabla_jT$, to be

$$K_{ij} = -\sum_{\mathbf{p},\sigma} \frac{E_{\mathbf{p}}^2}{T} (\mathbf{v}_{\mathbf{p}})_i (\mathbf{v}_{\mathbf{p}})_j \frac{\partial n_{\mathbf{p}}^0}{\partial E_{\mathbf{p}}} \tau_{\mathbf{p}} . \qquad (42)$$

The superconducting states we are considering are uniaxial, and as a consequence transport properties are anisotropic. The thermal conductivity tensor is diagonal if we choose axes along and perpendicular to the anisotropy axis. The components of the thermal conductivity tensor are then given simply by

$$K_{ii} = -\sum_{\mathbf{p},\sigma} \mu_i^2 \frac{(E_{\mathbf{p}} v_{\mathbf{p}})^2}{T} \frac{\partial n_{\mathbf{p}}^0}{\partial E_{\mathbf{p}}} \tau_{\mathbf{p}} , \qquad (43)$$

where the μ_i are the direction cosines of $\hat{\mathbf{p}}$. If one neglects effects of order Δ/E_F, the quasiparticle velocity is

$$v_{\mathbf{p}} = |\partial E_{\mathbf{p}}/\partial_{\mathbf{p}}| = |\xi_{\mathbf{p}}|/E_p v_F \equiv \tilde{v}\, \tilde{v}_F ,$$

where $\tilde{v} = |\xi_{\mathbf{p}}|/E_{\mathbf{p}}$, and therefore the thermal conductivity takes the form

$$K_{ii} = 4N(0)\frac{v_F^2}{T}\int_0^\infty E^2 \left[\frac{-\partial n^0}{\partial E}\right] \tau(E) \langle \tilde{v}^2 \mu_i^2 \rangle , \qquad (44)$$

where we use the notation

$$\langle \cdots \rangle = \int \frac{d\Omega}{4\pi} dE \frac{E}{(E^2 - |\Delta_{\mathbf{p}}|^2)^{1/2}} \cdots . \qquad (45)$$

$E/(E^2 - |\Delta_{\mathbf{p}}|^2)^{1/2}$ is the density of quasiparticle states per unit energy in the direction $\hat{\mathbf{p}}$, normalized to its normal state value. The integral in Eq. (45) is to be taken only over angles for which $|\Delta_{\mathbf{p}}|^2 < E^2$. Since for a given superconducting state τ_S/τ_N and the $\langle \tilde{v}^2 \mu_i^2 \rangle$ are functions only of E/Δ, the result (44) may be written in the form

$$K_{ii} = K_N(T_c)\frac{T}{T_c} F\left[\frac{T}{\Delta}\right] , \qquad (46)$$

where

$$F\left[\frac{T}{\Delta}\right] = \frac{18}{\pi^2}\int_0^\infty dE \frac{E^2}{T^2}\left[\frac{-\partial n^0}{\partial E}\right] \frac{\tau_S(E)}{\tau_N} \langle \tilde{v}^2 \mu_i^2 \rangle , \qquad (47)$$

and

$$K_N(T_c) = \frac{2\pi^2}{9} N(0) v_F^2 \tau_N T_c$$

is the thermal conductivity in the normal state. In the normal state K/T is independent of temperature, and therefore the content of Eq. (46) is that K/T in the superconductor is F times its value in the normal state. The integrals $\langle \tilde{v}^2 \mu_i^2 \rangle$ may be calculated analytically for the axial and polar states, and we give the results in the Appendix. For the d-wave state the integrals may be expressed in terms of elliptic functions, but in practice we found it more convenient to evaluate them numerically. In Figs. 2–4 we plot F for the three states we consider, for thermal conduction along the symmetry axis and perpendicular to it. For the case of small phase shifts, one component of thermal conductivity divided by the temperature tends to a constant, as $T \to 0$, in agreement with the conclusions of Pethick and Pines,[8] and in the particular case of transport along the symmetry axis for the axial state, K/T tends to precisely its normal state value. In directions perpendicular to the nodal directions the components of the thermal conductivity are smaller by a factor $\sim(T/\Delta)$. For $\delta_N = \pi/2$ the largest components of K/T vanish as $T \to 0$. One general feature of the results is that at low temperatures K/T for $\delta_N = \pi/2$ for a given superconducting state and a given direction is smaller than for $\delta_N \ll \pi/2$ while closer to T_c $(T/\Delta \to \infty)$, K/T for $\delta_N = \pi/2$ is greater than for $\delta_N \ll \pi/2$.

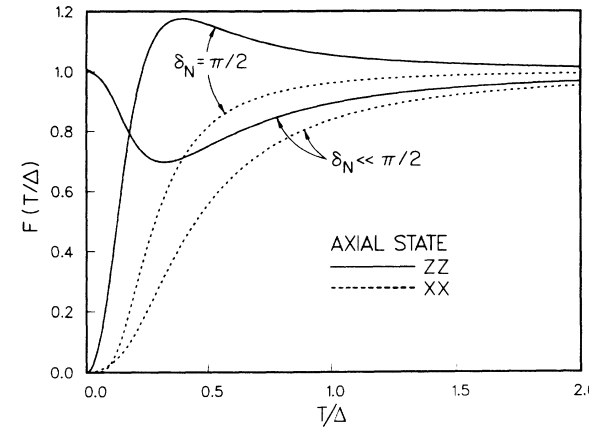

FIG. 2. Plot of the dimensionaless factor F for thermal conduction along and parallel to the symmetry axis as a function of T/Δ for the axial state for the cases of small phase shifts and for resonant scattering.

To predict the thermal conductivity as a function of T we need an expression for the magnitude of the gap as a function of temperature. For the axial state the form we have used is the one proposed by Wölfle and Koch[19]

$$\Delta(T) = \Delta(0)\tanh\left[\pi\frac{T_c}{\Delta(0)}\left[\frac{\Delta C}{C}\frac{T_c - T}{T}\right]^{1/2}\right] , \qquad (48)$$

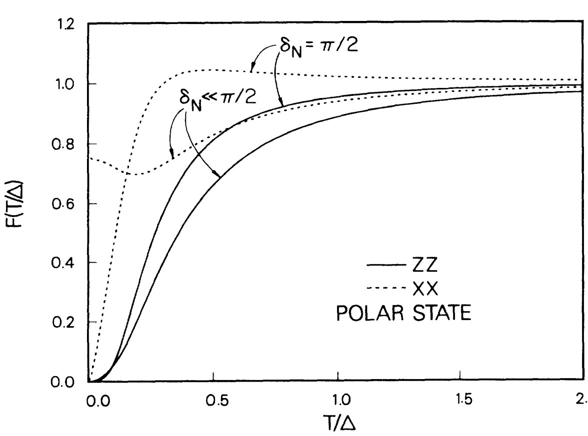

FIG. 3. Same as Fig. 2 but for the polar state.

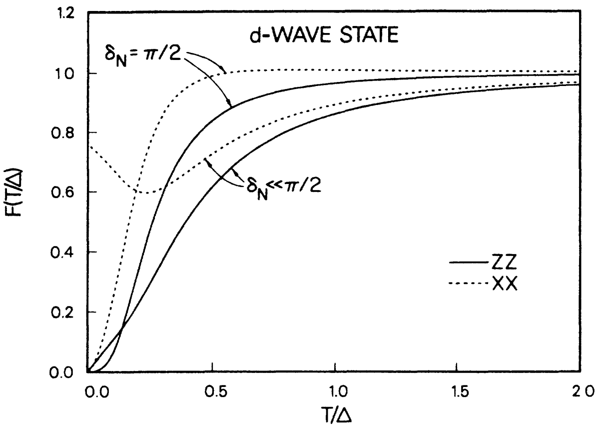

FIG. 4. Same as Fig. 2 but for the d-wave state.

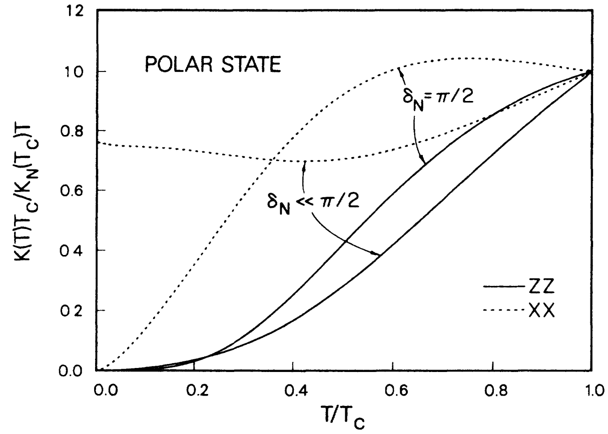

FIG. 6. Same as Fig. 5 but for the polar state.

where $\Delta(0)$ is the maximum gap at $T=0$, and $\Delta C/C$ is the jump in the specific heat at $T=T_c$. This expression allows certain strong coupling corrections to be taken into account by employing the measured value of the specific-heat jump, rather than a purely theoretical one. The generalization of this expression to other superconducting states is

$$\Delta(T)=\Delta(0)\tanh\left[\pi\frac{T_c}{\Delta(0)}\left[\frac{2}{3f}\frac{\Delta C}{C}\frac{T_c-T}{T}\right]^{1/2}\right], \quad (49)$$

where

$$f=\frac{\int\frac{d\Omega_p}{4\pi}\,|\Delta_p|^2}{\Delta^2} \quad (50)$$

is the mean square value of the gap relative to the maximum value. With the standard expression for the condensation energy in terms of the gap, Eq. (49) gives a specific-heat jump equal to the actual one, $\Delta C/C$. For the axial state, f is equal to $\frac{2}{3}$, and therefore Eq. (49) reduces to (48), while for the polar state $f=\frac{1}{3}$, and for the d-wave state, $f=\frac{8}{15}$. The values of $\Delta(0)$ we adopted are those given by solving the weak-coupling gap equation, $\Delta(0)=2.02T_c$ (axial), $\Delta(0)=2.45T_c$ (polar), and $\Delta(0)=2.10T_c$ (d wave). We have carried out calculations for $\Delta C/C=0.86$, the "idealized" value extracted by Sulpice *et al.*[3] from their data. The results are plotted in Figs. 5–7. The choice of $\Delta C/C$ has little effect on the results, since it changes Δ most in the vicinity of T_c, where the thermal conductivity is relatively insensitive to the magnitude of Δ.

Calculations of the thermal conductivity in which the width of quasiparticle states is taken into account have been performed in Refs. 9–12. Our results are in excellent agreement with these, except at temperatures below about $T_c/10$ for the values of the depairing parameter $1/\tau_N\Delta \simeq 10^{-2}$ used in the calculations.

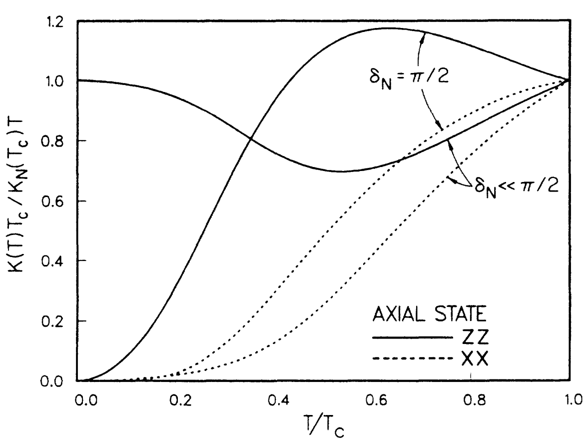

FIG. 5. Thermal conductivity along and perpendicular to the symmetry axis divided by the temperature as a function of temperature for the axial state, and for $\delta_N \ll \pi/2$ and $\delta_N=\pi/2$. The expression used for the gap as a function of temperature is described in the text.

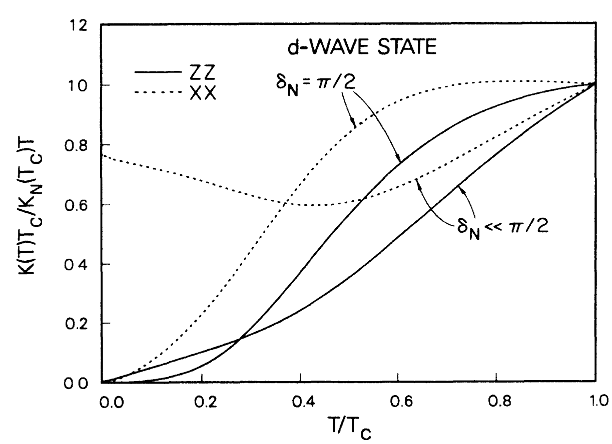

FIG. 7. Same as Fig. 5 but for the d-wave state.

IV. ULTRASONIC ATTENUATION

When a sound wave passes through a metal, quasiparticle energies are modified by the strain associated with the wave. The resulting deviation of the quasiparticle distribution from its local equilibrium value gives rise to a nonequilibrium stress, which leads to dissipation. The basic physics of the process has been discussed at length by Pippard[20,21] and by Kadanoff and Pippard,[22] and we shall follow their approach. A discussion of the microscopic basis of the approach has been given recently by Khan and Allen.[23] All experiments on ultrasonic attenuation in heavy-fermion superconductors that have been performed to date appear to have been in the hydrodynamic regime, since the attenuation varies as ω^2, where ω is the angular frequency of the sound. Consequently, we may adopt the simplified approach used by Kadanoff and Pippard,[22] and consider the response of the metal to a homogeneous strain. For long wavelength sound waves, the normal and superfluid velocities are essentially equal, and therefore one does not have to allow in the kinetic equation for the shift of the quasiparticle energy due to superfluid velocity, which would give rise to viscous effects usually described in terms of the sound viscosity ζ_3.[24,25] Other second viscosities are much smaller than ζ_3 by factor of order $(T/E_F)^2$ or $(T/E_F)^4$, and therefore we have also neglected terms that lead to them. The Boltzmann equation for this situation is $\partial n_{\mathbf{p}}^{\mathrm{LE}}/\partial t=(\partial n_{\mathbf{p}}/\partial t)_{\mathrm{coll}}$. Since the local equilibrium distribution function is simply the Fermi function, the time dependence of $n_{\mathbf{p}}^{\mathrm{LE}}$, has two sources—the variation of the quasiparticle energy and the variation of the temperature. The latter effect is of order T/E_F compared with the former, and we shall neglect it. Therefore, the Boltzmann equation becomes

$$\frac{\partial n_{\mathbf{p}}^0}{\partial E_{\mathbf{p}}}\frac{\partial E_{\mathbf{p}}}{\partial u_{ij}}\dot{u}_{ij}=\left[\frac{\partial n_{\mathbf{p}}}{\partial t}\right]_{\mathrm{coll}}, \tag{51}$$

where $u_{ij}=\partial u_i/\partial x_j$ is the strain tensor, $\mathbf{u}$ being the displacement vector. The quantity $\partial E_{\mathbf{p}}/\partial u_{ij}$ is a deformation potential which describes how the quasiparticle energy in the superconductor changes due to application of a strain field. For systems with inversion symmetry, and in the hydrodynamic limit, possible electric field terms, that might contribute to the sound attenuation, vanish.[22] Consequently, we do not consider them. The energy is given by Eq. (2), and therefore we may write

$$\frac{\partial E_{\mathbf{p}}}{\partial u_{ij}}=\frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}}\frac{\partial \xi_{\mathbf{p}}}{\partial u_{ij}}+\frac{|\Delta_{\mathbf{p}}|}{E_{\mathbf{p}}}\frac{\partial|\Delta_{\mathbf{p}}|}{\partial u_{ij}} . \tag{52}$$

The quantity $\partial\xi_{\mathbf{p}}/\partial u_{ij}$ gives the strain dependence of normal-state quasiparticle energy and is the usual deformation potential D_{ij} which occurs in the standard works on ultrasonic attenuation.[20–23]

The second term in Eq. (52) is due to the modulation of the gap by the strain. Since D_{ij} is typically of order E_F, the second term is of order $(\Delta/E_F)\partial\ln\Delta/\partial u_{ij}$ compared with the first one. It is this term that leads to the "Landau-Khalatnikov damping" discussed in the context of heavy fermion systems by Miyake and Varma.[26] The physics of this process is closely related to that of gap relaxation in ordinary superconductors,[27] and inelastic processes play an important role. We shall consider these processes elsewhere, and shall not dwell on them further here. The deformation potential in the superconductor is, therefore, simply $\partial E_{\mathbf{p}}/\partial u_{ij}=(\xi_{\mathbf{p}}/E_{\mathbf{p}})D_{ij}$. The linearized transport equation analogous to Eq. (39) in the case of thermal conduction is thus

$$\Phi_{\mathbf{p}}-\tau_{\mathbf{p}}\frac{2\pi n_i}{\hbar}\sum_{\mathbf{p}'}\overline{|t^S_{\mathbf{p}'\mathbf{p}}|^2}\delta(E_{\mathbf{p}}-E_{\mathbf{p}'})\Phi_{\mathbf{p}'}=-\tau_{\mathbf{p}}\frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}}D_{ij}\dot{u}_{ij} . \tag{53}$$

The deformation potential is a function of momentum, and it has not yet been estimated in detail for heavy-electron materials. We shall assume that it has the same dependence on the direction of the momentum as it does in an isotropic Fermi liquid, $D_{ij}=d_{ij}\lambda_{ij}$, where d_{ij} is a quantity that is independent of direction on the Fermi surface, and $\lambda_{ij}=\mu_i\mu_j-\frac{1}{3}\delta_{ij}$. We have neglected contributions to D_{ij} proportional to δ_{ij}, which give rise to the second, or bulk viscosity. In a normal Fermi liquid with quasiparticle-quasiparticle scattering the main relaxation mechanism, the second viscosity (which comes from the δ_{ij} term) is of order $(T/E_F)^4$ times the shear viscosity (which comes from the λ_{ij} term).[28,29] In heavy-electron compounds, when both quasiparticle-quasiparticle and quasiparticle-impurity scattering occur, the physics is different, since scattering from impurities has little effect on the bulk viscosity, but a large one on the shear viscosity. However, one would still expect the viscosity due to terms in the deformation potential proportional to δ_{ij} to be small compared with that due to the λ_{ij} term.

The driving term in Eq. (53) is odd in $\xi_{\mathbf{p}}$, and therefore $\Phi_{\mathbf{p}}$ must be. Consequently, the only term in the sum in (52) that does not vanish on performing the sum is the one proportional to $\xi_{\mathbf{p}}\xi_{\mathbf{p}'}$ in Eqs. (24)–(27). Equation (53) may thus be expressed in the form

$$\Phi_{\mathbf{p}}-\frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}}\frac{\langle\bar{v}\Phi\rangle}{\langle 1\rangle}=-\tau_{\mathbf{p}}\frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}}d_{ij}d_{ij}\dot{u}_{ij} . \tag{54}$$

The second term on the left-hand side of this equation corresponds to vertex corrections in the microscopic calculations.[10,12,13] Multiplying this equation by $\xi_{\mathbf{p}}/E_{\mathbf{p}}$ and averaging over angles, we find

$$\langle\bar{v}\Phi\rangle(1-\langle\bar{v}^2\rangle/\langle 1\rangle)=-\tau_{\mathbf{p}}\langle\bar{v}^2\lambda_{ij}\rangle d_{ij}\dot{u}_{ij} . \tag{55}$$

The relaxation time $\tau_{\mathbf{p}}$ is a function only of $E_{\mathbf{p}}$, and therefore it has been taken outside the average in Eq. (55). One finds

$$\Phi_{\mathbf{p}}=-\tau_{\mathbf{p}}\frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}}\left[\lambda_{ij}+\frac{\langle\bar{v}^2\lambda_{ij}\rangle}{\langle 1\rangle-\langle\bar{v}^2\rangle}\right]d_{ij}\dot{u}_{ij} . \tag{56}$$

Next we evaluate the stress resulting from the nonequilibrium part of the quasiparticle distribution. The stress produced by a superconducting quasiparticle is minus the derivative of the quasiparticle energy with respect to the corresponding strain, and it is therefore equal to the negative of the deformation potential in the superconductor.

Thus the nonequilibrium contribution to the stress is

$$\delta\pi_{ij} = -\sum_{\mathbf{p}} \frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}} D_{ij} \frac{\partial n_{\mathbf{p}}^0}{\partial E_{\mathbf{p}}} \Phi_{\mathbf{p}} . \tag{57}$$

On substituting Eq. (56) for $\Phi_{\mathbf{p}}$ in Eq. (57), one finds

$$\delta\pi_{ij} = -\eta_{ij,kl} \dot{u}_{kl} , \tag{58}$$

where the viscosity tensor $\eta_{ij,kl}$ is given by

$$\eta_{ij,kl} = d_{ij} d_{kl} 4N(0) \int_0^\infty dE \left[-\frac{\partial n^0}{\partial E} \right] \tau_s(E) \times \left[\langle \lambda_{ij} \lambda_{kl} \bar{v}^2 \rangle + \frac{\langle \lambda_{ij} \bar{v}^2 \rangle \langle \lambda_{kl} \bar{v}^2 \rangle}{\langle 1 \rangle - \langle \bar{v}^2 \rangle} \right] . \tag{59}$$

In Eq. (59), the factor 4 takes into account the two spin populations and the two branches of the spectrum in the superconductor.

The viscous contribution to the attenuation of a sound wave with wave vector $\mathbf{q}$ and polarization vector $\hat{\epsilon}$ is given by

$$\alpha(\mathbf{q},\epsilon) = \frac{q^2}{\rho c_s} \eta_{ij,kl} \hat{\epsilon}_i \hat{q}_j \hat{\epsilon}_k \hat{q} l , \tag{60}$$

where c_s is the sound velocity of the mode and ρ is the mass density.

In applying our results in UPt_3 we shall assume that the symmetry axis of the superconducting state is parallel to the hexagonal axis of the crystal, a result consistent with the measured anisotropies of the attenuation of transverse sound and of H_{c2}.[6,31] In such a case the superconducting state in the hexagonal crystal still has hexagonal symmetry, while if the axis of the superconducting state were not aligned along the c axis, it would not. There are a total of five independent viscosities for a system with hexagonal symmetry,[30] which we may take to be $\eta_{zz,zz}, \eta_{xx,xx} = \eta_{yy,yy}, \eta_{xy,xy}, \eta_{xz,xz}$, and $\eta_{zz,xx} = \eta_{zz,yy}$. From the hexagonal symmetry it follows that

$$\eta_{xx,yy} = \eta_{xx,xx} - 2\eta_{xy,xy} . \tag{61}$$

The other nonvanishing viscosities are related to them by the symmetry conditions $\eta_{ij,kl} = \eta_{ji,kl} = \eta_{ji,lk} = \eta_{kl,ij}$. Since the deformation potential couples only to the λ_{ij}, one of the five viscosities may be expressed in terms of the others, since $\lambda_{ii} = 0$. As our four basic viscosity components we shall take $\eta_{zz,zz}$, $\eta_{xx,xx} \eta_{xz,xz}$, and $\eta_{xy,xy}$, which determine the attenuation of longitudinal and transverse wave propagating along the symmetry axis of the crystal, and in the basal plane. From the condition $\lambda_{ii} = 0$ and Eq. (59), one sees that

$$\frac{\eta_{xx,xx}}{d_{xx}} + \frac{\eta_{yy,xx}}{d_{yy}} + \frac{\eta_{zz,xx}}{d_{zz}} = 0 , \tag{62}$$

with the help of Eq. (61), the symmetry relations, and the fact the $d_{xx} = d_{yy}$, this may be rewritten as

$$\eta_{zz,xx} = \frac{2d_{zz}}{d_{xx}} (\eta_{xy,xy} - \eta_{xx,xx}) . \tag{63}$$

In view of the fact that no calculations of the d_{ij} exist for heavy-electron compounds, we shall calculate the viscosities relative to their normal state values. These are given by

$$\frac{\eta_{ij,kl}}{\eta_{ij,kl}^N} = H(T/\Delta) , \tag{64}$$

where

$$H(T/\Delta) = \frac{2}{\overline{\lambda_{ij}\lambda_{kl}}} \int_0^\infty dE \left[\frac{-\partial n^0}{\partial E} \right] \frac{\tau_s(E)}{\tau_N} \times \left[\langle \lambda_{ij} \lambda_{kl} \bar{v}^2 \rangle + \frac{\langle \lambda_{ij} \bar{v}^2 \rangle \langle \lambda_{kl} \bar{v}^2 \rangle}{\langle 1 \rangle - \langle \bar{v}^2 \rangle} \right] . \tag{65}$$

Here $n_{ij,kl}^N$ is the viscosity in the normal state, and

$$\overline{\lambda_{ij}\lambda_{kl}} = \int \frac{d\Omega}{4\pi} \lambda_{ij} \lambda_{kl} . \tag{66}$$

If the normal state viscosity corresponded to that for an isotropic solid, one would have

$$\eta_{ij,kl}^N = 15\eta \overline{\lambda_{ij}\lambda_{kl}} , \tag{67}$$

where η is the usual viscosity. Generally in anisotropic solids, the viscosity coefficients will be proportional to $d_{ij} d_{kl}$, and therefore measurement of the viscosity coefficients in the normal state enables one to determine the relative magnitudes of the deformation potential coefficients, d_{ij}.

We turn now to the numerical calculations. In the Appendix we give expression for the various integrals required in the calculations. First we show in Figs. 8–10, the viscosities relative to their normal state values for the same states and for the same phase shifts as we used in the calculation of the thermal conductivity in the previous section. A significant feature is that vertex corrections reflected in the second term in Eq. (65) are important for longitudinal disturbances ($i = j$, $k = l$), but vanish for transverse ones, since $\langle \lambda_{ij} \bar{v}^2 \rangle$ vanishes for $i \neq j$. At small Δ, vertex corrections play little role, since they vanish in the normal state, but for $T \ll \Delta$ they can be significant. For example, for the axial state one finds that vertex corrections increase the viscosities $\eta_{zz,zz}$ and $\eta_{xx,xx}$ by a factor of 1.5 for $T/\Delta \to 0$. The corresponding factors for the polar state are 2 for $\eta_{zz,zz}$ and $\frac{13}{11}$ for $\eta_{xx,xx}$. We now turn to the temperature dependence, and first discuss properties for $T \ll \Delta$. In the axial state, $\eta_{zz,zz} = \frac{5}{2} \eta_{zz,zz}^N$ and $\eta_{xx,xx} = \eta_{zz,zz}/4$ for $\delta_N \ll \pi/2$. The latter result holds also for $\delta_N = \pi/2$, and as we shall show in a later paper, for any other phase shift. At low temperatures the qualitative behaviors of the reduced viscosities, $H(T/\Delta)$, are easily understood in terms of the node structure of the gap and the energy dependence of the relaxation time. For example, in the axial state at low tem-

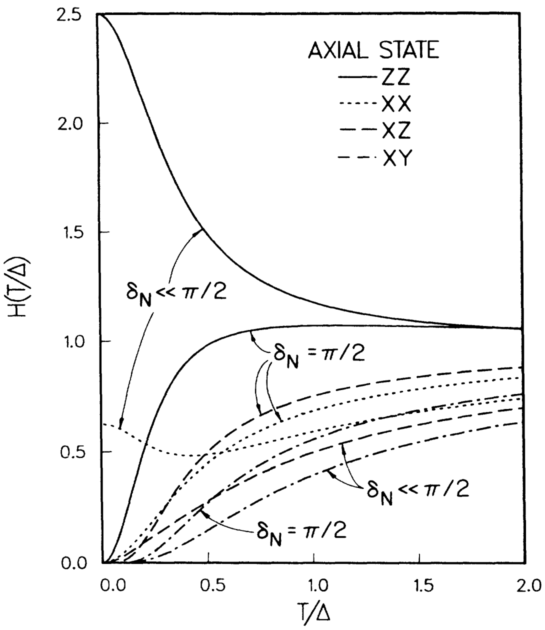

FIG. 8. Components of the viscosity tensor relative to their normal state values for the axial state for the cases of small phase shifts and for resonant scattering. The curves on the figures labeled by ij refer to the viscosity $\eta_{ij,ij}$.

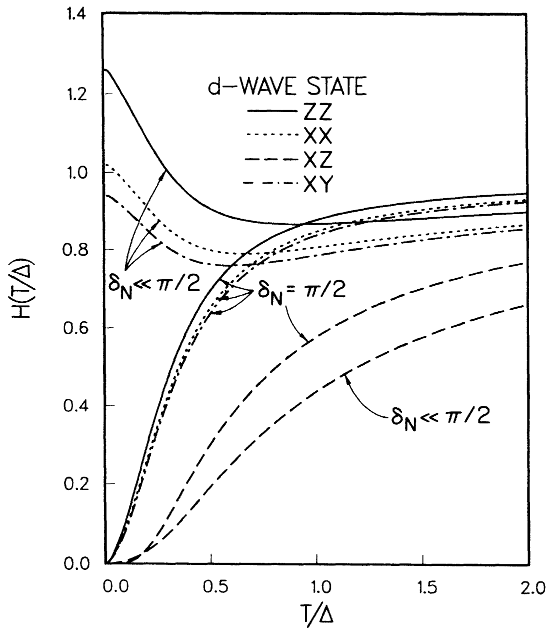

FIG. 10. Same as Fig. 8 but for the d-wave state.

perature, thermally excited quasiparticles have momenta predominantly along the direction of the hexagonal axis, where the nodes of the gap lie, and consequently the largest components of the viscosity are $\eta_{zz,zz}$, $\eta_{xx,xx}$, $\eta_{xx,zz}$, and $\eta_{xx,yy}$. The two latter components are not shown explicitly, but they are related to the ones we show by the symmetry relations discussed earlier. All other components are smaller by a factor at least of order $(T/\Delta)^2$, since for thermally excited quasiparticles $\lambda_{zz} \simeq \frac{2}{3}$ and $\lambda_{xx} = \lambda_{yy} \simeq -\frac{1}{3}$, while all others λ_{ij} are of order T/Δ. From Eq. (65) one then sees that $H_{xx,xx} = H_{zz,zz}/4$, $H_{xx,zz} = H_{zz,zz}/2$, and $H_{xx,yy} = H_{xx,xx}$. The results for $\delta_N = \pi/2$ are always smaller than those for $\delta_N \ll \pi/2$, by a factor of order $|g(T)|^2$. Similar comments apply to the polar and d-wave states, except that the largest components of the viscosity are $\eta_{zz,zz}$, $\eta_{xx,xx}$, $\eta_{xx,zz}$, $\eta_{xx,yy}$, and $\eta_{xy,xy}$. The corresponding ratios of the reduced viscosities are given for arbitrary phase shifts by $H_{xx,xx} = \frac{13}{16} H_{zz,zz}$, $H_{xy,xy} = \frac{3}{4} H_{zz,zz}$, $H_{xx,zz} = H_{zz,zz}$, and $H_{xx,yy} = \frac{11}{16} H_{zz,zz}$. For $\delta_N \ll \pi/2$, $H_{zz,zz} = \frac{5}{4}$ for $T/\Delta \to 0$. The low-temperature limits of these components of the viscosity are *exactly* the same for the d-wave state as for the polar state. This is a consequence of the fact that the low-temperature properties of both states are dominated by quasiparticles close to the equatorial nodal line on the Fermi surface.

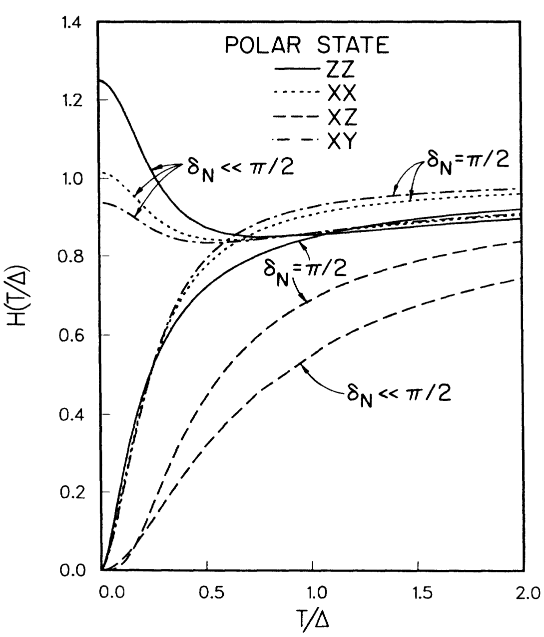

FIG. 9. Same as Fig. 8 but for the polar state.

To predict the temperature dependence of the ultrasonic attenuation, we use the expression for the temperature dependence of the gap already used to predict the temperature dependence of the thermal conductivity. In Figs. 11 and 12, we show the ultrasonic attenuation coefficients in the superconducting state normalized to their values at T_c as functions of T/T_c for the axial p-wave state for very small phase shifts $\delta_N \ll \pi/2$ and for $\delta_N = \pi/2$. In Figs. 13 and 14, the same quantities are plotted for the polar p-wave state, and in Figs. 15 and 16 for the d-wave state. For $\delta_N = \pi/2$, in the case of the axial state, one sees that $\alpha_{zz,zz}/\alpha_N > 1$ for $T \gtrsim 0.75 T_c$. This feature can be understood as being due to the fact that in

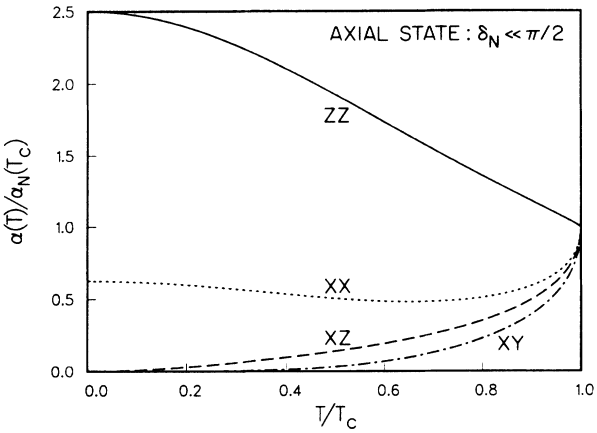

FIG. 11. Components of the ultrasonic attenuation relative to their normal state values at T_c for the axial state for the case of a very small phase shift ($\delta_N \ll \pi/2$).

the axial state the relaxation time at energies of the order of the maximum gap or less is significantly greater than in the normal state. For small values of Δ, this increase in the relaxation time more than compensates for the reduction in the quasiparticle velocity in the expression for $\alpha_{zz,zz}$, and consequently $\alpha_{zz,zz}$ exceeds its normal-state value. The other components of the viscosity that we show all fall as T decreases below T_c. This is due to the fact that, while the largest contributions to $\alpha_{zz,zz}$ come from quasiparticles moving in directions close to the polar axis, which are least affected by the gap in the quasiparticle spectrum, the maximum contribution to the other components come from quasiparticles moving at an angle to the polar axis, and these are more strongly affected by the gap. For other superconducting states, the relaxation time is never much larger than the normal-state value, and as a result the components of the attenuation decrease with increasing Δ. The results in the case of the polar and d-wave states are very similar to those obtained allowing for finite width of the quasiparticle states.[9–12]

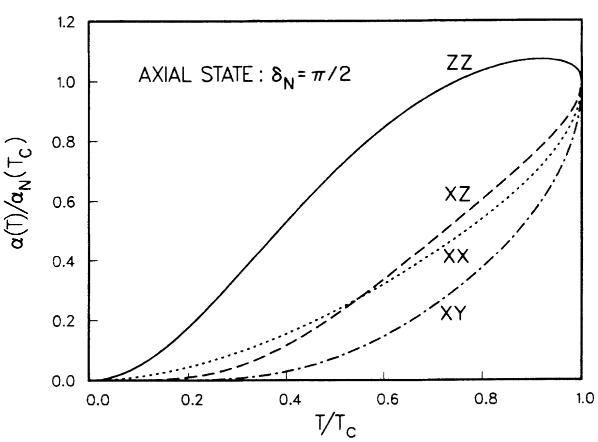

FIG. 12. Same as Fig. 11 but for resonant scattering ($\delta_N = \pi/2$).

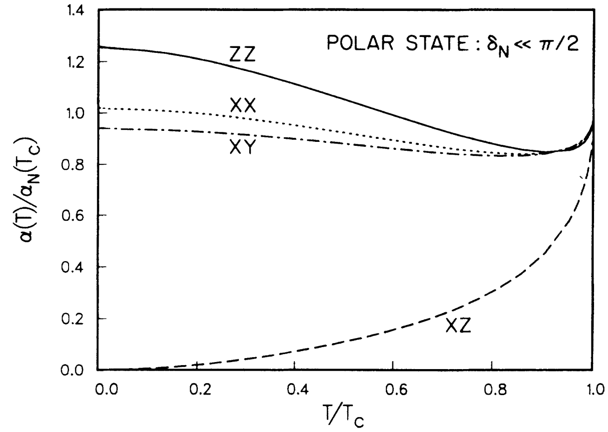

FIG. 13. Same as Fig. 11 but for the polar state.

We now comment briefly on the experimental results. Shivaram *et al.*[2] measured the attenuation of transverse sound in UPt_3 with the propagation vector $\mathbf{q}$ along the b axis in the basal plane of the hexagonal structure. They found that when the polarization vector $\hat{\epsilon}$ is along the a axis the attenuation varies roughly linearly with temperature between $T \simeq 35$ and 400 mK, and when $\hat{\epsilon}$ is along the c axis the temperature dependence of the attenuation is approximately quadratic. Taking the z direction as the c axis, we see that our results, in the case of both polar and d-wave states, for $\alpha_{xy,xy}$ and $\alpha_{xz,xz}$ agree qualitatively with the experimental ones for most of the temperature region between 0 and T_c. For the longitudinal sound attenuation the experimental results obtained by different groups[4,5] do not agree, and consequently one cannot make a sensible comparison between theoretical and experimental results. In the case of longitudinal sound, order parameter fluctuations may play an important role.

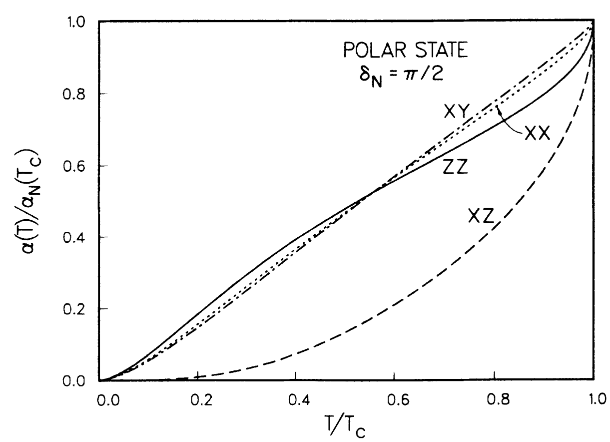

FIG. 14. Same as Fig. 12 but for the polar state.

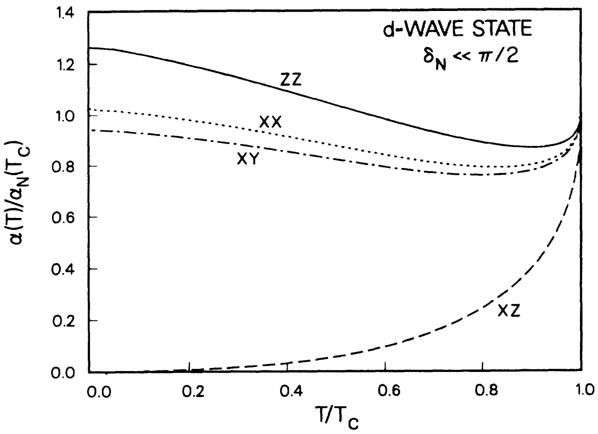

FIG. 15. Same as Fig. 11 but for the *d*-wave state.

V. CONCLUSION

Our calculations show clearly the very different behaviors for the relaxation time and transport coefficients expected for the small phase shifts in the normal state, and for resonant scattering. The results confirm in detail the earlier conclusions of Ref. 8.

In our work the scattering amplitude was calculated neglecting the width of intermediate states in the scattering process. Comparison of our calculations with field-theoretic ones shows that this is a good approximation for $T/T_c \gtrsim 0.1$ for a pair-breaking parameter $\hbar/[\tau_N \Delta(T=0)] \lesssim 10^{-2}$, the range of values for which detailed calculations have been performed.[9–12]

We have presented detailed calculations of the anisotropy of the thermal conductivity, and for comparison with these results it would be valuable to have measurements of the anisotropy for one particular specimen, at least for specimens taken from a single ingot.

The experimentally observed behaviors of the thermal conductivity and the ultrasonic attenuation cannot be understood on the basis of the calculations for small normal-state phase shifts. However, for $\delta_N \simeq \pi/2$, the theoretical results exhibit a fall-off with decreasing temperature, as is found experimentally. The measured ultrasonic attenuation of transverse waves in UPt_3 (Ref. 2) is in qualitative agreement with the calculations for states with a nodal line around the equator of the Fermi surface, a result previously pointed out by Schmitt-Rink *et al.*[11]

In this paper we shall not make detailed comparisons between theory and experiment, but shall defer the discussion until we consider the calculations for general phase shifts. These calculations reveal a number of unexpected features of scattering processes for intermediate phase shifts which we shall discuss in a future paper.

One general point is that the theoretical results for the transport coefficients for the polar and *d*-wave states are qualitatively very similar, except that for the *d*-wave state the characteristic temperature that enters is twice the maximum gap, while for the polar state it is just the maximum gap.

In the calculations we have assumed the Fermi surface to be spherical. Also in the calculations of the ultrasonic attenuation, we have assumed the simplest possible form for the deformation potential. The formal transport theory is not crucially dependent on these assumptions, and calculations for more realistic Fermi surfaces and deformation potentials may be carried out straightforwardly, at the expense only of more numerical work.

In the calculation of the ultrasonic attenuation we did not estimate the contribution due to order-parameter relaxation[27] and possible collective modes of the gap.[31–33] Detailed calculations of this need to be carried out, allowing for the different roles elastic and inelastic scattering processes play in relaxation the order parameter.

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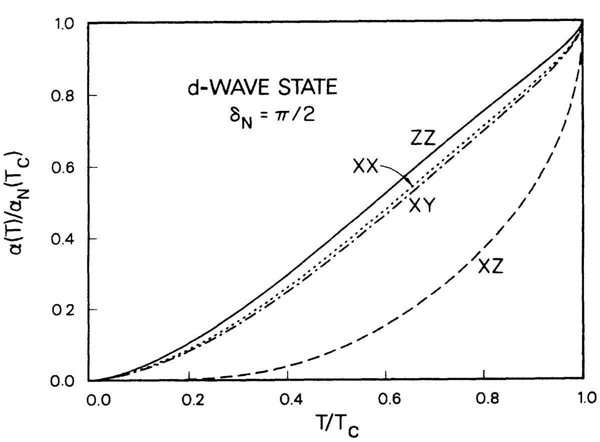

FIG. 16. Same as Fig. 12 but for the *d*-wave state.

APPENDIX

Here we give results for the angular integrals required in the calculation of the transport coefficients. These are $\langle 1 \rangle$, $g(E)$, $\langle \bar{v}^2 \rangle$, $\langle \bar{v}^2 \mu_i^2 \rangle$, $\langle \lambda_{ij} \bar{v}^2 \rangle$, and $\langle \lambda_{ij}^2 \bar{v}^2 \rangle$. The integrals involving μ_i^2 and λ_{ij} may be expressed in terms of the basic integrals

$$I_{2n} = \langle \bar{v}^2 \mu^{2n} \rangle \quad (n=0,\ 1,\ \text{or}\ 2)\ , \tag{A1}$$

where μ is shorthand notation for μ_2. The integrals in the thermal conductivity are

$$\langle \bar{v}^2 \mu_x^2 \rangle = \langle \bar{v}^2 \mu_u^2 \rangle = \tfrac{1}{2}(I_0 - I_2)\ , \tag{A2}$$

and

$$\langle \tilde{v}^2 \mu_z^2 \rangle = I_2 \; . \tag{A3}$$

For the components of the viscosity that we evaluate, we need only the integrals $\langle \tilde{v}^2 \lambda_{ij} \rangle$ and $\langle \tilde{v}^2 \lambda_{ij}^2 \rangle$, which may be expressed as

$$\langle \tilde{v}^2 \lambda_{ij} \rangle = b_{ij}^{(0)} I_0 + b_{ij}^{(2)} I_2 \tag{A4}$$

and

$$\langle \tilde{v}^2 \lambda_{ij}^2 \rangle = c_{ij}^{(0)} I_2 + c_{ij}^{(2)} I_2 + c_{ij}^{(4)} I_4 \; , \tag{A5}$$

when the coefficients b_{ij} and c_{ij} are shown Table I.

The integrals $\langle 1 \rangle$, $|g|^2$ [Eq. (28)], and I_{2n} have the following forms.

Axial state:

$$|g(x)|^2 = \left[\frac{\pi x}{2} \right]^2 + \left[\tfrac{1}{2} x \ln \left[\frac{1+x}{1-x} \right] \right]^2 , \quad x<1 \tag{A6}$$

$$|g(x)|^2 = \left[\tfrac{1}{2} x \ln \left| \frac{1+x}{1-x} \right| \right]^2 , \quad x>1 \tag{A7}$$

$$\langle 1 \rangle = \frac{x}{2} \ln \left| \frac{1+x}{1-x} \right| , \tag{A8}$$

$$\langle \tilde{v}^2 \rangle = \tfrac{1}{2} + \frac{1}{4x}(x^2-1) \ln \left| \frac{1+x}{1-x} \right| , \tag{A9}$$

$$\langle \tilde{v}^2 \mu^2 \rangle = \tfrac{1}{4} + \tfrac{1}{8}(x^2-1) - \frac{1}{16x}(x^2-1)^2 \ln \left| \frac{1+x}{1-x} \right| , \tag{A10}$$

$$\langle \tilde{v}^2 \mu^4 \rangle = \tfrac{1}{6} + \tfrac{1}{16}(x^2-1)(\tfrac{5}{3}-x^2) + \frac{1}{32x}(x^2-1)^3 \ln \left| \frac{1+x}{1-x} \right| . \tag{A11}$$

Polar state.

For $x<1$:

$$|g(x)|^2 = \left[\frac{\pi x}{2} \right]^2 + \left[x \ln \left[\frac{1+(1-x^2)^{1/2}}{x} \right] \right]^2 , \tag{A12}$$

TABLE I. Coefficients b_{ij} and c_{ij}.

ij	$b^{(0)}$	$b^{(2)}$	$c^{(0)}$	$c^{(2)}$	$c^{(4)}$
zz	$-\frac{1}{3}$	1	$\frac{1}{9}$	$-\frac{2}{3}$	1
xx	$-\frac{1}{6}$	$-\frac{1}{2}$	$\frac{11}{72}$	$-\frac{5}{12}$	$\frac{3}{8}$
xz	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
xy	0	0	$\frac{1}{8}$	$-\frac{1}{4}$	$\frac{1}{8}$

$$\langle 1 \rangle = \frac{\pi x}{2} , \tag{A13}$$

$$\langle \tilde{v}^2 \rangle = \frac{\pi x}{4} , \tag{A14}$$

$$\langle \tilde{v}^2 \mu^2 \rangle = \frac{\pi x^3}{16} , \tag{A15}$$

$$\langle \tilde{v}^2 \mu^4 \rangle = \frac{\pi x^5}{32} . \tag{A16}$$

For $x>1$:

$$|g(x)|^2 = \left[x \arcsin \frac{1}{x} \right]^2 , \tag{A17}$$

$$\langle 1 \rangle = x \arcsin \frac{1}{x} , \tag{A18}$$

$$\langle \tilde{v}^2 \rangle = \tfrac{1}{2}(1-\tfrac{1}{2})^{1/2} + \frac{x}{2} \arcsin \frac{1}{x} , \tag{A19}$$

$$\langle \tilde{v}^2 \mu^2 \rangle = \tfrac{1}{4} \left[-\frac{x^2}{2} + 1 \right] \left[1 - \frac{x}{2} \right]^{1/2} + \frac{x^3}{8} \arcsin \frac{1}{x} , \tag{A20}$$

$$\langle \tilde{v}^2 \mu^4 \rangle = \left[-\frac{x^4}{16} - \frac{x^2}{24} + \frac{1}{6} \right] \left[1 - \frac{1}{x^2} \right]^{1/2} + \frac{x^5}{16} \arcsin \frac{1}{x} . \tag{A21}$$

d-wave state.

For $x<1$:

$$|g(x)|^2 = \left[\frac{x}{2} \right]^2 \left[\left[\int_0^{\mu_1} + \int_{\mu_2}^1 \frac{d\mu}{(\mu^4-\mu^2+x^2/4)^{1/2}} \right]^2 + \left[\int_{\mu_1}^{\mu_2} \frac{d\mu}{(\mu^2-\mu^4-x^2/4)^{1/2}} \right]^2 \right] , \tag{A22}$$

$$\langle 1 \rangle = \frac{x}{2} \left[\int_0^{\mu_1} + \int_{\mu_2}^1 \right] \frac{d\mu}{(\mu^4-\mu^2+x^2/4)^{1/2}} , \tag{A23}$$

and

$$I_{2n} = \langle \tilde{v}^2 \mu^{2n} \rangle = \frac{2}{x} \left[\int_0^{\mu_1} + \int_{\mu_2}^1 \right] d\mu \, \mu^{2n} (\mu^4-\mu^2+x^2/4)^{1/2} , \tag{A24}$$

where

$$\mu_1=\frac{1}{\sqrt{2}}[1-(1-x^2)^{1/2}]^{1/2}$$

and

$$\mu_2=\frac{1}{\sqrt{2}}[1+(1-x^2)^{1/2}]^{1/2} .$$

For $x>1$:

$$|g(x)|^2=\left[\frac{x}{2}\right]^2\left[\int_0^1\frac{d\mu}{(\mu^4-\mu^2+x^2/4)^{1/2}}\right]^2 , \tag{A25}$$

$$\langle 1\rangle=\frac{x}{2}\int_0^1\frac{d\mu}{(\mu^4-\mu^2+x^2/4)^{1/2}} , \tag{A26}$$

and

$$I_{2n}=\langle \tilde{v}^2\mu^{2n}\rangle=\frac{2}{x}\int_0^1 d\mu\,\mu^{2n}(\mu^4-\mu^2+x^2/4)^{1/2} . \tag{A27}$$

Analytic results can be obtained in a number of limiting cases. For $x\to\infty$, $g(x)$ tends to unity, and I_{2n} tends to $1/(2n+1)$. For $x=1$, $I_0=(2\sqrt{2}-1)/6\simeq 0.304$, $I_2=(1+\sqrt{2})/15\approx 0.160$, and $I_4=(6+\sqrt{2})/70\simeq 0.105$.

Img vanishes for $x>1$, and for $x=1-\epsilon$ where ϵ is infinitesimal, $\mathrm{Im}g=-\pi/2^{3/2}\simeq -1.110$. For $x\ll 1$, the leading terms are the same as for the polar state, but with the expressions evaluated for an argument $y=2x$. Thus $\langle 1\rangle\simeq\pi/4$, $I_0\simeq\pi x/8$, $I_2=\pi x^3/128$, and $I_4=\pi x^5/1024$.

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