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Negative resistance fluctuations at resistance minima in narrow quantum Hall conductors

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Negative resistances are a signature of four-terminal resistance measurements. We construct a simple model of a localized impurity state in a four-probe conductor. A recently proposed multiprobe resistance formula is combined with the multichannel Breit-Wigner formalism to describe tunneling between edge states via the localized state. Deviations from the quantized Hall resistance are discussed and the symmetry with regard to magnetic field reversal is investigated. The model permits negative longitudinal resistance fluctuations which have been observed in recent experiments.

Hall resistances and longitudinal resistances are measured in a four-probe setup with two contacts used as carrier source and carrier sink and two contacts used as voltage probes. Four-probe resistances are not simply related to the diagonal and off-diagonal elements of the conductance tensor. As a consequence these resistances do not exhibit a simple symmetry with respect to magnetic field reversal[1,2] and need not be positive.[3] Here we are concerned with negative resistances in small conductors in very high fields. Ideally, the "longitudinal" resistance is zero when the Hall voltage is quantized. In narrow samples, isolated impurities can bring about the transfer of carriers from one sample edge to the other,[4,5] and under certain circumstances this leads to fluctuations in the Hall resistance (deviations from exact quantization). As observed by Chang *et al.*,[6] at a resistance minimum both positive and negative resistance fluctuations are possible.

We use a Landauer approach[7] which emphasizes that voltage differences are determined by the equilibrium chemical potentials of Fermi baths connected to the conductor.[1,3] Figure 1(a) shows a conductor with four leads connected to electron reservoirs at chemical potentials μ_i, $i=1,2,3,4$. Inelastic events occur only in the reservoirs. Scattering in the conductor is elastic and described by a scattering matrix s of dimension $(M_1+M_2+M_3+M_4)$.[2] M_i is the number of quantum channels in reservoir i. For a carrier incident in channel n in lead i and reflected into channel m in lead i the reflection amplitude is denoted by $r_{ii,mn}$. A total reflection probability for carriers incident in lead i is introduced, $R_{ii}=\sum_{m,n}|r_{ii,mn}|^2=\mathrm{Tr}(r_{ii}^{\dagger}r_{ii})$. Here Tr denotes the trace and $\dagger$ stands for the Hermitian conjugate. The transmission of carriers incident in channel n in lead j to channel m in lead i is described by the amplitude $t_{ij,mn}$. The total transmission probability is $T_{ij}=\mathrm{Tr}(t_{ij}^{\dagger}t_{ij})$. The current I_i incident from the reservoir at contact i is

$$I_i=\frac{e}{h}\left[(M_i-R_{ii})\mu_i-\sum_{j\neq i}T_{ij}\mu_j\right]. \qquad (1)$$

Equation (1) is strictly applicable only to the case where carriers move from one reservoir to another without experiencing inelastic events. However, even in the presence of inelastic events Eq. (1) remains correct if we replace the probabilities for coherent transmission and reflection by more general expressions which invoke both a coherent and an incoherent contribution.[3,8] In a configuration where reservoirs m and n are used as a source and sink and contacts k and l are voltage probes, the resistance is $\mathcal{R}_{mn,kl}=(\mu_k-\mu_l)/eI$, where $I=I_m=-I_n$ is the current impressed on the sample. At the voltage contacts, there is

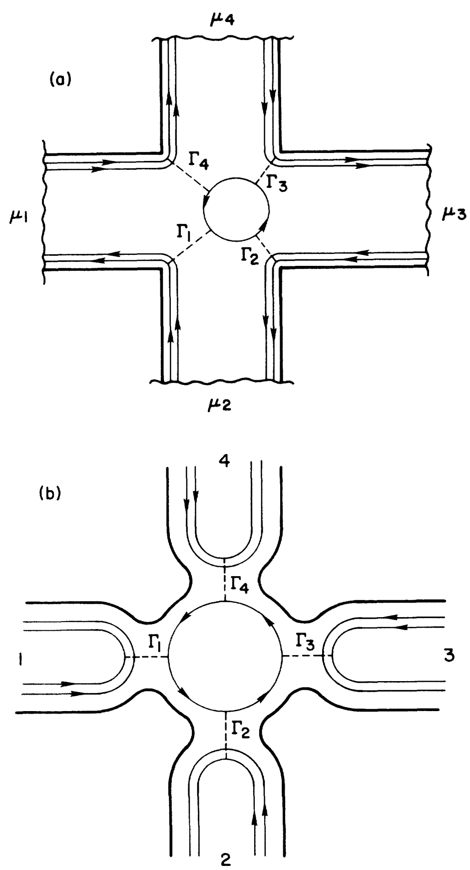

FIG. 1. (a) Four-probe Hall conductor with localized impurity state. Γ_i is the total decay width of the localized state due to coupling to lead i. (b) Quantum dot coupled to four leads with the same states at the Fermi energy as in (a).

zero net current flow, $I_k = I_l = 0$. These conditions on the currents determine the resistance

$$\mathcal{R}_{mn,kl} = (h/e^2)(T_{km}T_{ln} - T_{kn}T_{lm})/D . \tag{2}$$

Here D is a subdeterminant of rank three of the matrix formed by the coefficients in Eq. (1), which multiply the chemical potentials. All subdeterminants of rank three of this matrix are equal and independent of the indices m, n, k, and l. Microreversibility, the symmetry of the s matrix, implies

$$T_{ij}(H) = T_{ji}(-H), \quad R_{ii}(H) = R_{ii}(-H) .$$

Using this in Eq. (2) gives rise to the reciprocity of four-terminal resistances, $\mathcal{R}_{kl,mn}(H) = \mathcal{R}_{mn,kl}(-H)$. The reciprocity of the resistances is related to the Onsager-Casimir relations for the (global) conductances of a four-probe conductor.[1,3]

Let us next consider how the quantum Hall effect is established. A discussion of the quantum Hall effect based on Eqs. (1) and (2) has been proposed independently by Beenakker and van Houten,[9] Peeters,[10] and by the author.[11] In a clean conductor, in a high magnetic field, carrier flux occurs via edge states,[12] the quantum-mechanical equivalent of skipping orbits. The resulting carrier paths are indicated in Fig. 1(a) with thin solid lines. Let us first disregard the motion along the broken path via a localized impurity state which we study later on. The carriers that transmit through the contact from the reservoir to the lead propagate without reflection to the next-nearest contact in a clockwise fashion. For simplicity let us assume that each carrier which reaches a reservoir leaves the conductor (no internal reflection[11]). Each edge state provides a path along which carriers can traverse the conductor without backscattering. If each of the incident edge states carries a unit current (is full), all outgoing edge states carry also a unit current (are full). Given N edge states, the total transmission probabilities in the conductor of Fig. 1(a) are $T_{41} = N$, $T_{34} = N$, $T_{23} = N$, and $T_{12} = N$. All other T_{ij} are zero. The total reflection probabilities in the absence of internal reflection are $R_{ii} = M_i - N$. The Hall resistance $\mathcal{R}_{13,42}$ is determined by $T_{41}T_{23} - T_{43}T_{21}$, which is equal to N^2. Evaluation of the subdeterminant yields $D = N^3$. All Hall resistances of the conductor of Fig. 1(a) are quantized and yield $\pm h/e^2N$. The "longitudinal" resistances are zero. The important feature of this discussion of the quantum Hall effect is that it invokes only the states at the Fermi energy. It is remarkable that a single and simple expression for the resistance, Eq. (2), is applicable both at low fields[1-3,8] and at high quantizing fields.

Even if the conductor is disordered, the situation described above prevails as long as scattering from one set of edge states near one side of the sample to another set of edge states near another side of the sample does not occur.[11] The notion[4,5,9-11] that the resistance is determined by carrier flow along a few edge states has received some experimental support. Washburn *et al.* and Haug *et al.*[13] observed quantized four-terminal resistances in gated narrow constrictions which are closely related to the quantization of point-contact resistances observed by van Wees *et al.* and Wharam *et al.*[14]

In the absence of a magnetic field, a long-lived state leads to transmission and reflection probabilities of the Breit-Wigner[15,16] form. We extend this approach to high fields and to the multiprobe conductors of Fig. 1(a). Consider first the conductor in Fig. 1(b). A small disk (quantum dot), recently investigated by Sivan and Imry,[17] is weakly coupled to four leads. The conductor of Fig. 1(b) is equivalent to the conductor shown in Fig. 1(a) in the following sense: The topology of the current-carrying states is the same. The edge state, which in the conductor of Fig. 1(a) describes transmission into lead 2 of carriers incident in lead 1, reappears in the conductor of Fig. 1(b) as a state describing carriers incident in lead 1 and reflected back into lead 1. The Breit-Wigner formalism is applicable since a unitary transformation U_i of the current amplitudes in lead i leaves the incident and the outgoing current invariant. Equations (1) and (2) are invariant under the transformations $U = U_1 \times U_2 \times U_3 \times U_4$, since $\mathrm{Tr}[(U_i^{-1}r_{ii}U_i)^\dagger(U_i^{-1}r_{ii}U_i)]$ and $\mathrm{Tr}[(U_i^{-1}t_{ij}U_j)^\dagger \times (U_j^{-1}t_{ij}U_j)]$ are invariant. Consider, as a starting point, the case where coupling between the leads and the localized state in Fig. 1(b) is switched off. In this scattering matrix only the reflection amplitudes are nonzero. There exists a unitary transformation U which makes the scattering matrix diagonal. The scattering matrix can be diagonalized since $s = s_1 + is_2$ leads to Hermitian matrices $s_1 = (s+s^\dagger)/2$, $s_2 = -i(s-s^\dagger)/2$, which commute $[s_1,s_2] = 0$ because $s^\dagger s = 1$. Hence, there exists a set of channels for which the reflection amplitudes are of the form $\exp(2i\phi_{i,n})$. Here, $2\phi_{i,n}$ is the phase a carrier incident in lead i and channel n accumulates during reflection. The interaction of these channels with the localized state is taken into account with the ansatz,[15,16]

$$s_{ij,mn} = \left(\delta_{ij,mn} - i\frac{\Gamma Q_{ij,mn}}{E - E_r + i\Gamma/2} \right) e^{i(\phi_{i,m}+\phi_{j,n})} . \tag{3}$$

Here, E_r is the energy and Γ is the total width of the localized state. Q is a matrix which remains to be determined. Since s is unitary, differing rows of the s matrix must be orthogonal. This yields a set of equations which can only be satisfied identically for all energies if $Q^*_{ij,mn} = Q_{ji,nm}$. Using this, these equations imply that Q is equal to its own square. Since Q is Hermitian, it follows that the eigenvalues of Q are either 1 or 0. The remaining part of the derivation is as in Refs. 15 and 16 and yields $|Q_{ij,mn}|^2 = \Gamma_{i,m}\Gamma_{j,n}/\Gamma$. Here $\Gamma_{i,n}$ is the partial width due to decay into channel n in lead i. The total width is the sum of all the partial widths, $\Gamma = \sum_i \Gamma_i = \sum_{i,n}\Gamma_{i,n}$. The total transmission and reflection probabilities are

$$S_{ij} = \mathrm{Tr}(s_{ij}^\dagger s_{ij}) = \Gamma_i\Gamma_j/\Delta , \tag{4}$$

$$S_{ii} = \mathrm{Tr}(s_{ii}^\dagger s_{ii}) = |s_{ij}|^2 = N - \Gamma_i(\Gamma - \Gamma_i)/\Delta . \tag{5}$$

Here we have introduced the resonant denominator $\Delta = (E - E_r)^2 + \Gamma^2/4$. Equations (4) and (5) are correct if all four leads are characterized by N edge states. The discussion given above can be generalized to treat leads with a differing number of edge states, and can be generalized to describe the decay into reservoir states instead of a small number of edge states. Note, in the Breit-Wigner limit, the resulting matrix of transmission and reflection probabilities is symmetric $S_{ij}=S_{ji}$. Using Eqs. (4) and (5) in Eq. (2) and taking $M_i=N$ yields a vanishing Hall effect. This result is a consequence of the asymptotic validity of the Breit-Wigner formulas. Suppose the decay widths Γ_i are of order ε. Terms of order ε^3 in the nominator (and denominator) of Eq. (4) are neglected. An "exact" analysis shows indeed that the asymmetry $S_{ij}(H)-S_{ji}(H)$ is of order ε^3. These subtleties are unimportant for the problem [the conductor of Fig. 1(a)] which is of interest here.

The point of our digression to the conductor of Fig. 1(b) is the following: The transmission and reflection matrix of the conductor Fig. 1(a) is obtained by a permutation of the transmission and reflection matrix of the conductor of Fig. 1(b). The role of the reflection probabilities in the conductor of Fig. 1(b) is taken by the transmission probabilities of the conductor in Fig. 1(a). Hence the Breit-Wigner formulas corresponding to Fig. 1(a) are

$$T_{ij}=S_{i,j+3},\quad R_{ii}=S_{i,i+3}\,. \tag{6}$$

The second index of S is taken to be mod4. The four-terminal resistances are determined by Eq. (2) with the help of Eq. (6). A little algebra yields for D,

$$\Delta^3 D=(N\Delta)^3-\sum_{i<j}\Gamma_i\Gamma_j(N\Delta)^2+\left(\sum_{i=1}^{i=4}\Gamma_i^{-1}\right)\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma(N\Delta)+\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma^2\,. \tag{7}$$

Let us first consider the Hall effect. We consider the symmetric component $\mathcal{J}=(\mathcal{R}_{13,42}+\mathcal{R}_{42,13})/2$ and the antisymmetric component, $\mathcal{A}=(\mathcal{R}_{13,42}-\mathcal{R}_{42,13})/2$. Using Eqs. (2) and (6) gives

$$\mathcal{J}=(h/e^2)[(\Gamma_1\Gamma_3-\Gamma_2\Gamma_4)N\Delta+\Gamma(\Gamma_1\Gamma_4(\Gamma_2-\Gamma_3)+\Gamma_2\Gamma_3(\Gamma_4-\Gamma_1))]/D\Delta^2\,, \tag{8}$$

$$\mathcal{A}=(h/e^2)[N^2\Delta^2+D_2N\Delta+(D_1/2)]/D\Delta^2\,. \tag{9}$$

Here D_2 and D_1 are the coefficients in Eq. (7) multiplying $(N\Delta)^2$ and $N\Delta$. To elucidate the content of these equations we discuss below some special cases.

For $\Gamma_3=\Gamma_4=0$, the edge states in lead 2 of the conductor of Fig. 1(a) are connected by impurity scattering. $\mathcal{J}$ stays at zero and $\mathcal{A}$ is quantized and given by h/e^2N. As shown in Ref. 11, two nonadjoining leads can exhibit impurity-induced transitions between their edge states without effect on the quantum Hall resistance. For $\Gamma_2=\Gamma_4=0$, there exists a "diagonal path" across the Hall bar junction. The Hall voltage acquires a symmetric part,

$$\mathcal{J}=(h/e^2)\Gamma_1\Gamma_3/N(N\Delta-\Gamma_1\Gamma_3)\,.$$

The antisymmetric part remains quantized, $\mathcal{A}=(h/e^2)\times(1/N)$. The symmetric part can be quite large. For $N=1$, and maximum resonant transmission through the impurity state, i.e., for $\Gamma_1=\Gamma_3$, the symmetric part diverges when $E=E_r$. If all the decay widths are equal the symmetric part is zero, but the antisymmetric part falls below the quantized value.

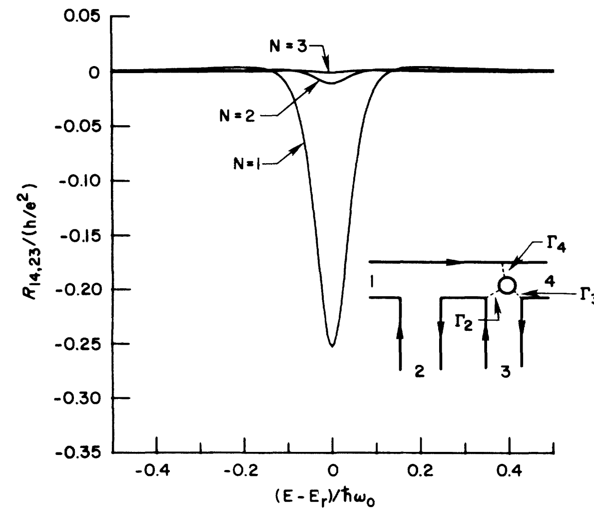

FIG. 2. Negative resistance peak as a function of Fermi energy with arbitrary energy unit $\hbar\omega_0$ for the four-probe conductor shown in the inset.

Consider the "longitudinal" resistance $\mathcal{R}_{14,23}$ proportional to $T_{21}T_{34}-T_{24}T_{31}$. The net transport is from contact 1 to contact 4 and contacts 2 and 3 are voltage probes. The inset of Fig. 2 shows a conductor which is equivalent to the conductor of Fig. 1(a) with $\Gamma_1=0$. For this resistance to be negative $T_{24}T_{31}$ must exceed $T_{21}T_{34}$. This cannot be achieved simply by redirecting noninteracting edge states; the path from lead 1 to lead 3 and the path from lead 4 to lead 2 necessarily cross. In our example, the intersection of these paths occurs at the localized state. Equations (6) and (2) give

$$\mathcal{R}_{14,23}=(h/e^2)(\Gamma_2\Gamma_4)(N\Delta-\Gamma\Gamma_3)/D\Delta^2\,. \tag{10}$$

Using the definition of Δ, we find that Eq. (10) describes a positive resistance peak if $4\Gamma_3<N\Gamma$ and a negative resistance peak if $4\Gamma_3>N\Gamma$. Our model allows for negative resistances if $N<4$. In samples where the spin degeneracy is not lifted by the magnetic field, our model allows for negative resistance fluctuations at the minima associated with a Hall resistance $h/6e^2$ and the higher plateaus. In the experiment of Chang *et al.*[6] the most pronounced negative resistance fluctuations occur at the minimum associated with the $h/4e^2$ Hall "plateaus." Figure 2 shows an example for $\Gamma_1=0$, $\Gamma_2=\Gamma_4=0.02\hbar\omega_0$, and $\Gamma_3=0.14\hbar\omega_0$. For $N=1$ the negative peak height is of order h/e^2. As N increases the peak height decreases sharply, for our example it is approximately 260 Ω for $N=2$ and 50 Ω for $N=3$. These fluctuations provide yet another example of huge conductance fluctuations[8] far exceeding h/e^2. Since $\Delta G\simeq-\Delta R/R^2$, where ΔG and ΔR denote the mean-square deviations from the average values G and R, and since $R\cong 0$ at a minimum ΔG can be expected to be very large indeed. Real conductors are likely to be more complex then the simple model studied here. Our model, however, gives insight on how negative resistance fluctuations appear in these conductors.

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