Scattering properties of solitons in nonlinear disordered chains

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The scattering of a soliton from a disordered one-dimensional atomic lattice with nonlinear nearest-neighbor interactions of quartic type is studied numerically. The disorder is of the binary-alloy type with the concentration of the impurity masses m given by p. We numerically find that for large enough lengths L, the soliton transmission coefficient T decays as $1/\sqrt{L}$. This behavior has been obtained also by an analytical study of the transmission of a Gaussian wave packet in a linear disordered system. For short and intermediate lengths, T decays with a different power law for different nonlinear potentials. This behavior can be accounted by a simple independent scattering picture. Finally, the role of the boundary conditions in disordered nonlinear systems is discussed.

I. INTRODUCTION

Wave propagation in nonlinear disordered media has become an extensively studied subject recently.[1,2] The combined effects of disorder and nonlinearity introduce many novel and complex properties in the system under consideration. Disorder in a linear lattice will generally lead to localization,[3] a phenomenon first introduced by Landauer and Helland and by Anderson to describe the exponential decay of the electronic wave function in a disordered system. The Anderson localization can be easily generalized to other waves like phonon, acoustic, and electromagnetic waves.[4,5] The presence of the nonlinearity, on the other hand, drastically changes the behavior of the system and causes a lot of difficulties. Many techniques developed for the linear systems cannot be applied directly to the nonlinear systems because the superposition principle of waves does not hold in the nonlinear systems. However, nonlinear systems can have soliton solutions. Solitons are collective localized excitations and can propagate in pure systems without changing their shape or velocity. The effect of soliton interactions with each other can only introduce a phase shift.

For a disordered nonlinear system some interesting questions arise. First, how will the localization behavior be modified by the nonlinear effects or vice versa how will the nonlinearity be affected by the presence of disorder? Second, is it possible to define a transmission coefficient? What is the energy and length dependence of the transmission coefficient? These are difficult questions to answer either analytically or numerically. The analytical techniques applied to the linear problem seems to fail completely in the nonlinear one, while many numerical approaches are unstable. Some answers to these questions have been already given in Refs. 1 and 2. There the behavior of a plane wave is studied when it is transmitted through a medium satisfying a nonlinear Schrödinger equation. The main result[1,2] is that the transmission coefficient again tends to zero as the size of the system increases, but it decays not exponentially but with a power law. Here we take a different approach to study the effects of the disorder in nonlinear systems. In particular, we consider the transmission of a soliton in a disordered nonlinear system. In our approach to a time-dependent problem, the propagation of the soliton has to be numerically solved. There are no numerical instabilities in our approach. By integrating for a long time, we can easily identify the transmitted and reflected solitons. We view our method as complementary to that of Refs. 1 and 2. In two previous publications,[6,7] we studied the scattering of lattice solitons by either one or two impurities. The present paper is the extension of our previous work.[6] Our primary interest is the soliton transmission through a disordered binary lattice with cubic or quartic nearest-neighbor interactions.

We introduce our formalism in Sec. II, the numerical results and analysis are presented in Sec. III, and finally in Sec. IV we give our conclusions.

II. FORMALISM

A nonlinear lattice with nearest-neighbor interactions can be described by a Hamiltonian

$$H=\sum_n [\tfrac{1}{2} m_n \dot{y}_n^2 + V(y_n - y_{n-1})] , \tag{1}$$

where $y_n(t)$ is the displacement of the nth atom from its equilibrium position, $\dot{y}_n = dy_n(t)/dt$ and m_n are, respectively, the velocity and the mass of the nth particle. The interaction potential $V(r_n)$ is the bond strain, where $r_n = y_n - y_{n-1}$. We will consider only polynomial potentials of the following form:

$$V(r)=\frac{Gr^2}{2}+\frac{Ar^3}{3}+\frac{Br^4}{4} . \tag{2}$$

The equation of motion of the nth particle is

$$m_n \ddot{y}_n = -V'(y_n - y_{n-1}) + V'(y_{n+1} - y_n) , \tag{3}$$

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where $V'(r)=dV(r)/dr$. For homogeneous systems, Eq. (3) can be transformed to a generalized Boussinesq equation[6] in the continuum limit and the kink solution can be found. Stable narrow nontopological kink solitons can be also found numerically when the continuum approximation breaks down.[8]

We consider a disordered segment embedded between two ordered (homogeneous) seminfinite chains. The impurity mass $m=\gamma m_0$ is distributed randomly with probability p in the disordered part. The rest of the masses in the disordered part are taken to be equal to m_0. Our disorder is of a binary alloy type with two different masses m and m_0. A kink soliton is incident on the disordered segment from the left. The incident kink is decomposed into a reflected and a transmitted part. After passing through the disordered segment, the transmitted wave packet will reorganize itself and become one or more solitons, plus some small amplitude oscillations propagating in the right side of the ordered segment. In Fig. 1, we show qualitatively such a scattering process. Notice that for short times, the transmitted soliton has not yet adapted to a new single soliton. Only for longer times and well inside the homogeneous systems do we have a well-defined transmitted soliton. The reflected waves are decomposed into ripples, except if the reflection is dominant so that the reflected part can be reorganized to solitons.

We are interested in two quantities: (i) the total energy transmission coefficient $T=E_t/E_{\text{in}}$, which is the total transmitted energy E_t, including the energy of all the transmitted solitons plus the radiative part over the soliton incident energy E_{in} and (ii) the first soliton transmission coefficient $T_1=E_1/E_{\text{in}}$, which is the energy of the first transmitted soliton over the soliton incident energy E_{in}. We can define the reflection coefficient in a similar way. Such quantities can provide us with answers for the role of the transmission in disordered nonlinear media. It might also give answers to the practical problems such as the transfer of information in optical fibers with possible impurities.

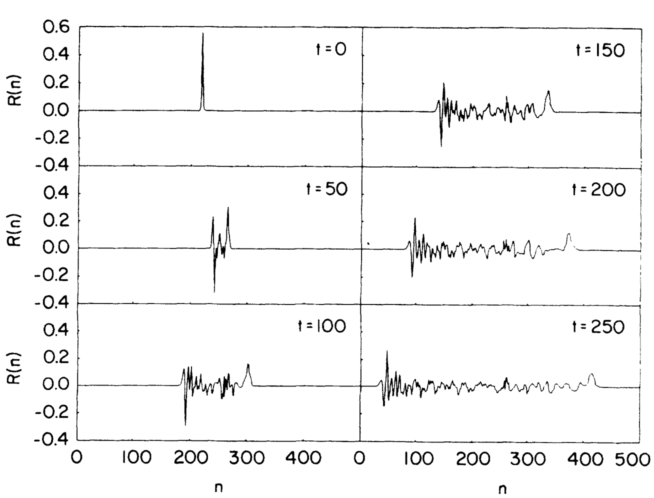

FIG. 1. Scattering process of a soliton in a quartic nonlinear disordered chain. This disordered segment starts at $n=250$ and ends at $n=450$ with 20% impurities of mass $m_i=5m_0$. The incident soliton energy is $E_i=0.66$. $R(n)\equiv y(n)-y(n-1)$ is the relative displacement at site n.

III. RESULTS AND DISCUSSION

We integrate the set of Eq. (3) using a fourth-order Runge-Kutta method. Our simulations begin with one of the analytical or numerical kink solutions of Eq. (3) as an initial state in the ordered part. We have chosen also the time step of the integration small enough as to keep the total energy error less than 0.01%. We numerically integrate Eq. (3) for different incident kink solitons and different disordered segment lengths L.

Then, we calculate the total energy transmission coefficient T as well as the first soliton transmission coefficient T_1 versus the length L of the disordered segment. In Fig. 2 we plot the dependence of T and T_1 on the length L of the disordered segment with concentration $p=0.20$ of impurity mass $m=5m_0$. The dimensionless energy of the initial kink soliton is $E_{\text{in}}=0.66$ and the dimensionless velocity $v=1.072$. The potential is quartic. Notice that Fig. 2 is a log-log plot of T vs L and therefore the transmission coefficients T and T_1 decrease only with a power law as the length increases. This has to be contracted with the exponential decrease of T with L in a linear disordered system for a plane-wave initial state. From Fig. 2 one obtains that T and T_1 behave as $1/\sqrt{L}$. In Fig. 3, we plot the dependence of T on the incident soliton energy for a disordered segment of length $L=250$, $m=5m_0$, and $p=0.20$.

To quantitatively understand our results, let us look at two limiting cases.

(1) *Independent scattering limit.* When the concentration p of the mass impurities is low, i.e., in the dilute limit, the average distance between two nearest-neighbor impurities is larger than the soliton size. In this limit, the scattering of a soliton by many impurities can be approximately treated independently. In other words, interference effects are not important. Therefore, in the dilute limit we have that $T_N\cong\prod_{i=1}^{N}t_i(E_i)$, where N is the number of impurities inside the disordered segment L, E_i is the incident wave-packet energy for the ith impurity, and t_i is the first soliton transmission coefficient of the ith impurity. The first transmitted soliton for the ith impurity is the

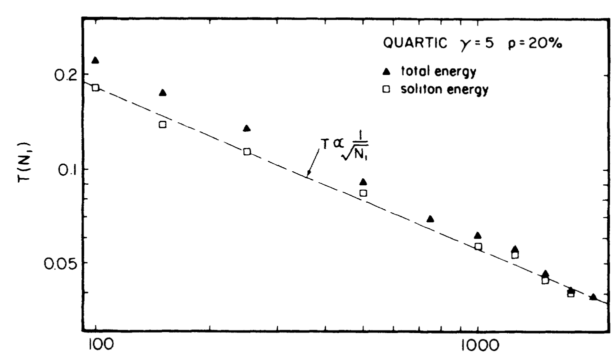

FIG. 2. Transmission coefficient of soliton with incident energy $E_i=0.66$ as a function of the disordered segment length N_i in a quartic nonlinear lattice.

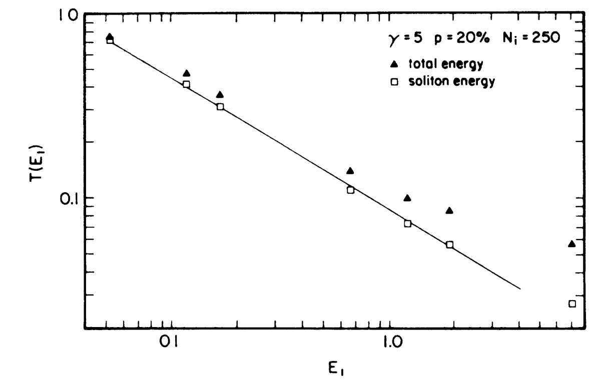

FIG. 3. Dependence of a transmission coefficient on incident soliton energy E_i for a disordered quartic nonlinear chain of length $N_i=250$. The solid line can be fitted as $T \propto E_i^{-0.70}$. Our independent scattering theory predicts $T \propto E_i^{-1}$.

incident soliton for the $(i+1)$th impurity. We can write that

$$E_{i+1}=E_i t_i(E_i)\,, \tag{4a}$$

and

$$\Delta E_{i+1}=E_{i+1}-E_i=-E_i(1-t_i(E_i))\,. \tag{4b}$$

Since on the average there are $(\Delta x)p$ impurities in the interval between x and $x+\Delta x$, from Eq. (4b) a differential equation can be established:

$$\frac{dE(x)}{dx}=-E(x)[1-t(E(x))]p\,, \tag{5}$$

where $t(E(x))$ is the kink soliton transmission coefficient scattered by an impurity with mass $m=\gamma m_0$, where $\gamma=5$ in this numerical simulation study. It has been found in Ref. 6 both numerically and within the linear theory that in the weak scattering limit

$$t(E)=1-a_0E^{\alpha}|\gamma-1|^2\,, \tag{6}$$

where $a_0=0.66$, $\alpha=2$ for quartic potential and $a_0=8.0$, $\alpha=\frac{2}{3}$ for cubic potential. Equation (5) can be easily solved and the solution is

$$T(x)=E(x)/E_{\rm in}=1/(1+c_0x)^{1/\alpha}\,, \tag{7}$$

where $c_0=p\alpha a_0E_{\rm in}^{\alpha}|\gamma-1|^2$. For length $x \gg 1/c_0$ Eq. (7) leads to

$$T(x)\cong x^{-1/\alpha}E_{\rm in}^{-1}(p|\gamma-1|^2)^{-1/\alpha}\,. \tag{8}$$

Therefore, the independent scattering limit gives for quartic potential $T(x)\propto x^{-1/2}$ in agreement with our numerical simulations and for cubic potential $T(x)\propto x^{-3/2}$. We see that $T(x)$ is implicitly dependent on the type of nonlinearity of the system. This is because within the linear scattering theory[6] the nonlinearity nevertheless enters the problem via the shape of the incident soliton.

The independent scattering approximation would not be a good approximation for very large lengths L, where the amplitude of the wave packet is small after so many collisions with mass impurities and its size is large compared to the average impurity distance. In this limit for the transmitted part of the soliton, the nonlinearity is not important anymore and therefore the linear approximation can be used.

(2) *Linear wave transmission approximation.* In the limit of $L\to\infty$, the quartic and cubic terms in Eq. (3) become smaller than the quadratic term. Therefore, we can totally adopt linear theory. The equation of motion is given by $m_n\ddot{y}_n=G(y_{n+1}+y_{n-1}-2y_n)$ and assuming a stationary state of the form $y_n(t)=y_ne^{i\omega t}$ we obtain for the equation of motion[3]

$$y_{n+1}+y_{n-1}+(m_n-\langle m_n\rangle)\frac{\omega^2}{G}y_n=\left[2-\langle m_n\rangle\frac{\omega^2}{G}\right]y_n\,. \tag{9}$$

This one-dimensional (1D) phonon problem can be easily mapped to the corresponding 1D electronic problem.[3] Hence, we identify the energy ε_n at site n with $(m_n-\langle m_n\rangle)(\omega^2/G)$ and the energy $E\equiv 2\cos k$ with $[2-\langle m_n\rangle(\omega^2/G)]$ where $\langle m_n\rangle$ is the average value of the masses m_n. It is easy to obtain the strength of the disorder

$$\begin{aligned} w^2&=\langle\varepsilon_n^2\rangle-\langle\varepsilon_n\rangle^2=(\langle m_n^2\rangle-\langle m_n\rangle^2)\frac{\omega^4}{G^2}\\ &=p(1-p)(\gamma-1)^2\omega^4/G^2\,,\end{aligned}$$

where G is the force constant for the harmonic approximation.

For the electronic problem, in the weak scattering limit, using second-order perturbation theory, the localization length[3] λ is given by $\lambda=2(4-E^2)/w^2$. Correspondingly, we have the following expression for the phonon localization length of frequency ω, and wave number k:

$$\lambda=\frac{8\langle m_n\rangle-2\langle m_n\rangle^2\omega^2/G}{(\langle m_n^2\rangle-\langle m_n\rangle^2)\omega^2/G}=\frac{2(1-p+\gamma p)^2}{p(1-p)(\gamma-1)^2}\cot^2\frac{k}{2}\,, \tag{10a}$$

and the transmission amplitude is

$$|t(k)|\cong e^{-L/\lambda}\,. \tag{10b}$$

These results are consistent with the results of Rubin[9] and Ishi[10] in the limit $\omega\to 0$. The phonon localization length diverges as $k\to 0$.

Now the total transmission coefficient[3] for linear wave packet (LWP) is given by

$$\begin{aligned} T_{\rm LWP}&=\frac{\int_{-\infty}^{\infty}|A(k)|^2|t(k)|^2dk}{\int_{-\infty}^{\infty}|A(k)|^2dk}\\ &\cong\frac{\int_{-\infty}^{\infty}|A(k)|^2e^{-2L/\lambda}dk}{\int_{-\infty}^{\infty}|A(k)|^2dk}\quad \text{as } L\to\infty\,, \end{aligned}\tag{11}$$

where $A(k)$ is the Fourier transform of the incident wave packet. Assuming that the incident wave packet is a Gaussian, then its Fourier transform $A(k)$ is also a Gaussian. Using Eqs. (9) and (11), we obtain in the $L\to\infty$ limit for the total transmission coefficient the expression

$$T_{\rm LWP}\sim\frac{1-p+\gamma p}{(\gamma-1)[p(1-p)]^{1/2}}\frac{1}{\sqrt{L}}\,. \tag{12}$$

We see that for $A(0)\neq 0$, we always obtain $T\propto 1/\sqrt{L}$. Ishii[9] has calculated the thermal conductivity of a disordered harmonic chain and found also that it behaves as $1/\sqrt{L}$. Although for a particular frequency exponential localization does take place, the total transmission coefficient of a wave packet decays much slower, in fact with a power law. This is a direct consequence of the divergence of localization length as $\omega\rightarrow 0$. Physically, when $\omega\rightarrow 0$, the wavelength is so large that the phonon does not see any inhomogeneity at all. On the other hand, the high-frequency parts are effectively filtered out completely since their localization length is relatively small. It is therefore the superposition of the different localization lengths that gives the power-law dependence of the transmission coefficient T of a wave packet traveling through a *linear* disordered segment. We therefore see that it is possible to obtain a much slower decay than exponential for T in a linear disordered system. It will be difficult somehow to separate the effects of the nonlinearity from the transmission of a wave packet in a linear disordered phonon segment.

Our analysis suggests that for short lengths and dilute systems, the independent scattering picture should give a good account for the soliton transmission T, which is given by Eq. (8) and its length decay depends on the type of the nonlinearity of the system. In particular, $T\sim 1/\sqrt{L}$ for a quartic potential and $T\sim L^{-3/2}$ for a cubic potential. However, for long enough lengths the nonlinear terms in the equation of motion can be neglected; then the total transmission must decay as $1/\sqrt{L}$, independent of the type of nonlinearity. This conclusion is weakly supported from some preliminary numerical results of a cubic potential which for long L, the transmission coefficient T behaves as $1/\sqrt{L}$, while for short length T is closer to a $L^{-3/2}$ dependence. The results shown in Fig. 2 are for a quartic potential where the short and long length dependence of T is of the form $1/\sqrt{L}$.

In Fig. 3, the energy dependence of T shows a simple power-law behavior $T(E)\propto E^{-0.70}$, where our independent scattering theory predicts $T(E)\propto E^{-1}$ for both cubic and quartic potentials. This discrepancy might be due to the large values of γ and p used in our numerical simulations. More detailed numerical work is needed to clarify this statement.

Finally, we would like to comment on the effect of the boundary conditions on the transmission of waves in a nonlinear disordered system. Due to the presence of the nonlinearity, bistability might exist, that is, for a fixed constant input amplitude, there might be more than one possible output amplitude. What value from the output amplitudes the system would take depends on experimental conditions. The asymptotic behavior of the transmission coefficient for a fixed input boundary condition is given by the behavior of the linear disordered system. This is because as $L\rightarrow\infty$, the amplitude of the transmitted wave decreases and the nonlinear term approaches zero, faster than the linear terms. Hence, in the electronic problem exponential localization will be present in the fixed input boundary condition. On the other hand, if the output rather than the input amplitude is fixed, as in the case of Ref. 1, the role of the nonlinearity of the system will increase with L. Then, the asymptotic behavior of such a system would not depend on the disorder at all. Therefore, the asymptotic behavior of the transmission coefficient for the second type of boundary conditions is determined by the ordered nonlinear system. Here again, a power-law behavior is obtained for the disordered nonlinear Schrödinger equation.[1,2] Our simulation is a time-dependent one and is closer to the first type of boundary condition; however, since we are looking at the transmission coefficient of a soliton, which is like a wave packet, we obtain a power law for the length dependence of the transmission coefficient.

IV. CONCLUSIONS

We have numerically studied the transmission of a soliton through a disordered nonlinear chain. We find that for intermediate lengths the transmission might be accounted for by a simple independent scattering picture from which a power-law length dependence results. The exponent in this power-law decay of the transmission coefficient seems to depend on the type of the nonlinear potential. More numerical results are needed to support this claim. However, when the size of the system is large enough, linear behavior is recovered. The transmission coefficient of the soliton decays then as $1/\sqrt{L}$ for all types of nonlinear potentials. Our numerical simulations agree with the prediction, which has been obtained also analytically, assuming an incident wave packet of Gaussian shape.

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