Diffusion in a sparsely connected space: A model for glassy relaxation

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A model for diffusion in configuration space is proposed which combines the features of infinite dimensionality and low connectivity thought to be important for glassy relaxation. Specifically, a random walk amongst a set of N points, with each of the $N(N-1)/2$ pairs connected independently with probability p/N (and the mean connectivity p finite for $N\to\infty$), is considered. The model can be solved exactly by the replica method, but the behavior in the long-time regime is difficult to extract. From, instead, intuitive arguments based on the dominance for $t\to\infty$ of a particular type of statistical fluctuation in the network connectivity, the mean probability $f(t)$ of return to the origin after time t is predicted to approach its infinite-time limit according to a "stretched-exponential" law, $f(t)-f(\infty)\sim\exp[-(t/\tau)^{1/3}]$ for all finite p, with $\tau\sim|p-1|^{-3}$ near the percolation threshold $p_c=1$.

I. INTRODUCTION

Nonexponential relaxation is ubiquitous in glassy systems,[1,2] but so far has not really received a satisfactory explanation. Experimental results are often well described over many decades of time by "Kohlrausch" or "stretched exponential" decays[2,3] of the form $\exp[-(t/\tau)^{\beta}]$, with $0<\beta<1$. Models devised thus far to explain such behavior usually involve putting in a hierarchically organized set of (free) energy barriers.[4-7] While nonexponential relaxation can certainly be obtained by this approach, the assumptions are somewhat *ad hoc* and in a sense feed in the desired result at the outset.

The present paper was motivated by the recent work of Campbell and coworkers, who have proposed a general picture for the onset of glassy relaxation and ergodicity breaking in terms of the geometrical properties of the configuration space.[8-10] The general idea is that under cooling the configuration space gradually breaks up into mutually inaccessible pockets. The ergodicity breaking transition is identified as a kind of "percolation transition" in the configuration space. Nonexponential relaxation sets in at higher temperatures, however, as the different parts of the configuration space, and of the "infinite cluster" in particular, become more and more tenuously connected. Glassy relaxation can then be understood in terms of diffusion in an infinite-dimensional, sparsely connected space.[8-10]

As a concrete model exhibiting the above features, Campbell *et al.* have studied numerically random walks on site-diluted high-dimensional hypercubes, in an attempt to model Ising spin glasses.[9,10] (The 2^N vertices of an N-dimensional hypercube can represent the states of an Ising model with N spins.) The results have been interpreted in terms of Kohlrausch relaxation with a stretched exponent β which depends continuously on the level of dilution, varying from $\beta=1$ for no dilution to $\beta=\frac{1}{3}$ at the percolation threshold. The authors note that this is in good agreement with Ogielski's spin-glass data,[11] for which the spin autocorrelation function can also be fitted to a stretched exponential form, with an exponent β which depends continuously on temperature, varying from $\beta=1$ at the "Griffiths temperature" (i.e., the transition temperature of the unfrustrated system) to $\beta=\frac{1}{3}$ at the spin-glass transition temperature.

In the present paper we will not pursue this analogy in detail. Rather we concentrate on the general notion[8] that diffusion in a sparsely connected space of infinite dimensionality can be a useful, general way to model glassy dynamics, and we introduce and solve a simple model. Indeed, we argue that it is the simplest such model: The geometrical aspects are described by the mean-field theory of percolation, but the dynamics are nontrivial. The model has a geometrical interpretation in terms of a random walk on a bond-diluted hypertetrahedron. The physical quantity of interest is the probability $f(t)$, averaged over initial positions, for the particle to be back at its initial position at time t. The model can be solved exactly, in principle, by means of replicas (Sec. III). Unfortunately, however, we have so far been unable to extract the asymptotic behavior of $f(t)$ from the replica solution, except in the limit that the mean coordination number p is large. Instead, therefore, we appeal to intuitive arguments based on the dominance for $t\to\infty$ of a certain type of statistical fluctuation in the network connectivity. These arguments, which are expounded in Sec. IV, have the character of Lifshitz-like arguments for the density of states in the tail of a random matrix,[12] or the arguments used to extract the effects of Griffiths singularities[13] in the properties of random spin systems.[14] They lead to the prediction

$$R(t)\equiv[f(t)-f(\infty)]/[1-f(\infty)]\sim\exp[-(t/\tau)^{1/3}]\ , \tag{1}$$

for the "relaxation function" $R(t)$, for all finite p. [Here $f(\infty)$ is nonzero because of the possibility that the initial site belongs to a finite cluster. The normalization in Eq. (1) is such that $R(0)=1$.] Hence the exponent β is pre-

dicted to be $\frac{1}{3}$ for all finite p. The asymptotic timescale $\tau(p)$, however, is small far from the percolation threshold $p_c=1$, and (1) only holds for very large times where $R(t)$ is very small. Close to $p=1$, on the other hand, τ is large, $\tau \propto |p-1|^{-3}$, and the form (1) should be observable over a wide range of times.

The relevance of these results to the simulations of Campbell *et al.*[9] is discussed in Sec. V, where it is suggested that the observation of a p-dependent Kohlrausch exponent β is possibly due to crossover effects associated with the finiteness of the systems studied.

II. THE MODEL

We consider a system consisting of N sites (or vertices) $\{i\}$ connected pairwise by $N(N-1)/2$ bonds (or edges) $\{J_{ij}\}$. The sites can be regarded as the vertices of an $(N-1)$-dimensional hypertetrahedron. The model is now diluted by removing at random all but a fraction p/N of the bonds. Thus the elements J_{ij} of the connectivity matrix **J** are independent random variables (subject to $J_{ij}=J_{ji}$) with distribution

$$P(J)=\frac{p}{N}\delta\left[J-\frac{1}{p}\right]+\left[1-\frac{p}{N}\right]\delta(J) . \tag{2}$$

The p dependence chosen for the nonzero elements $J_{ij}=1/p$ ensures that the dynamics (see below) has a sensible limit for $p\to\infty$.

Equation (2) and related models with an Ising spin placed at each vertex and the $\{J_{ij}\}$ regarded as exchange interactions have received much attention recently as suitable nontrivial mean-field models of dilute spin systems[15] and in the context of combinatorial optimisation theory. We will be primarily interested in the case of intensive average connectivity, where p (the average coordination number) is fixed as $N\to\infty$. The other possibility, extensive average connectivity, where $p\propto N$, though much simpler to solve, has rather uninteresting dynamics characterized by simple exponential relaxation. We will, however, consider this case also, both for completeness, and because it may have some relevance to understanding the diffusion problem on hypercubes in a space of high (but finite) dimension.[9]

To describe a random walk on this network, we let $c_i(t)$ be the probability for the walker to occupy site i at time t, and J_{ij} be the transition rate from site i to site j. Then the probabilities $\{c_i\}$ obey the master equation,

$$\frac{dc_i}{dt}=-\sum_j M_{ij}c_j , \tag{3}$$

where

$$M_{ij}=\delta_{ij}\sum_k J_{ij}-J_{ij} , \tag{4}$$

and we have exploited the symmetry $J_{ij}=J_{ji}$. Equation (3) is readily solved in terms of the eigenvalues $\{\mu\}$ and normalized eigenvectors $\{|\mu\rangle\}$ of the real symmetric matrix M:

$$c_i(t)=\sum_{\mu,j}\langle i|\mu\rangle\langle\mu|j\rangle c_j(0)\exp(-\mu t) . \tag{5}$$

If the walker starts at a particular site α, such that $c_j(0)=\delta_{j,\alpha}$, then Eq. (5) gives the site-averaged probability of being again at site α time t, $f(t)=N^{-1}\sum_\alpha c_\alpha(t)$, as

$$f(t)=N^{-1}\sum_\mu \exp(-\mu t) \tag{6}$$

$$=\int d\mu\,\rho(\mu)\exp(-\mu t) , \tag{7}$$

where $\rho(\mu)$ is the eigenvalue density.

Thus computing the function $f(t)$ is equivalent to determining the eigenvalue spectrum of the matrix **M**: $\rho(\mu)$ is just the distribution of relaxation rates. The related problem of determining the eigenvalue spectrum of the matrix **J** was recently addressed by us using the replica method,[16] and the replica approach adopted here (Sec. III) follows closely our earlier work.

In contrast to **J**, **M** is a non-negative definite matrix. This is obvious on physical grounds, and easily proved by constructing the quadratic form

$$Q(\{x_i\})\equiv\sum_{i,j}M_{ij}x_ix_j=\tfrac{1}{2}\sum_{i,j}J_{ij}(x_i-x_j)^2\geq 0 .$$

This construction also shows that **M** has a null eigenvector, with amplitude independent of i. If the system consists of a number of disconnected clusters, there is a null eigenvector localized on each cluster, so that the number of null eigenvectors is equal to the number of clusters $N_{\rm cl}$. For intensive average connectivity, the model described by Eq. (2) is equivalent to the mean-field theory of percolation, and the cluster number is extensive, yielding a nonzero number of clusters per site, $n_{\rm cl}=N_{\rm cl}/N$. Equation (6) then gives the nonzero large-time limit $f(\infty)=n_{\rm cl}$. In this paper we will be concerned with how $f(t)$ approaches this limit. For extensive average connectivity there is only one cluster, and $f(\infty)$ vanishes in the thermodynamic limit.

From Eq. (7) it is clear that the behavior of $\rho(\mu)$ for $\mu\to 0$ determines the large t form of $f(t)$. In particular, the stretched exponential form $f(t)-f(\infty)\sim\exp[-(t/\tau)^\beta]$ requires an essential singularity in the eigenvalue density of the form $\rho(\mu)\sim\exp[-(A/\mu)^{\beta/(1-\beta)}]$.

In the following section, an exact formal solution for $\rho(\mu)$ will be derived using the method of replicas. Readers preferring a more direct, intuitive approach may skip this section and proceed directly to Sec. IV.

III. FORMAL SOLUTION WITH REPLICAS

The replica approach follows closely Ref. 16, and we refer the reader to that paper for details of the method. The eigenvalue density $\rho(\mu)$, averaged over realizations of the disorder, is obtained from the generating function

$$Z(\mu)=\int\prod_i(d\phi_i)\exp\left[(i/2)\left[\mu\sum_i\phi_i^2-\sum_{i,j}M_{ij}\phi_i\phi_j\right]\right] \tag{8}$$

via

$$\rho(\mu)=(2/N\pi)\mathrm{Im}\partial[\ln Z]/\partial\mu , \tag{9}$$

where $[\cdots]$ indicates a disorder average, and μ in Eq. (8) contains implicity an infinitesimal positive imaginary part. To compute $[\ln Z]$, we use

$$[\ln Z]=\lim_{n\to 0} n^{-1}\ln[Z^n] .$$

For integer n, Z^n can be written as a multiple integral over n sets of integration variables $\{\phi_{i\alpha}\}$, $(\alpha=1,\ldots,n)$. After averaging over the disorder, the result is analytically continued to real n, and the $n\to 0$ limit taken. The disorder average yields, using (2),

$$[Z^n]=\int D\phi \exp\left[(i/2)\mu\sum_{i,\alpha}\phi_{i\alpha}^2\right]\prod_{(i,j)}\left[1-\frac{p}{N}+\frac{p}{N}\exp\left[-(i/2p)\sum_{\alpha}(\phi_{i\alpha}-\phi_{j\alpha})^2\right]\right] , \tag{10}$$

where $D\phi\equiv\prod_{i\alpha}d\phi_{i\alpha}$, and (i,j) indicates that each pair is to be included only once.

Consider first the case of *extensive* average connectivity, $p=O(N)$. The final exponential can be expanded in powers of its argument, and only the zeroth- and first-order terms need be retained in the thermodynamic limit. This gives

$$[Z^n]=\int D\phi \exp\left[(i/2)\mu\sum_{i,\alpha}\phi_{i\alpha}^2-(i/4N)\sum_{i,j,\alpha}(\phi_{i\alpha}-\phi_{j\alpha})^2\right] , \tag{11}$$

a result which is identical to that which would have been obtained by simply replacing each bond J_{ij} by its averaged value $1/N$. For extensive average connectivity, therefore, we can replace the matrix elements M_{ij} by their mean values $[M_{ij}]=\delta_{ij}-1/N$. The latter matrix has one zero eigenvalue. The remaining $(N-1)$ eigenvalues are equal to unity. Inserting these values into Eq. (6) gives, for $N\to\infty$,

$$f(t)=\exp(-t) , \tag{12}$$

a simple exponential decay. We will find that this result is recovered for *intensive* average connectivity in the limit $p\to\infty$ at fixed t. (For $t\to\infty$ at fixed p, however, the relaxation is asymptotically nonexponential.)

For intensive average connectivity (p remains finite for $N\to\infty$), Eq. (10) yields

$$[Z^n]=\int D\phi \exp\left\{(i\mu/2)\sum_{i,\alpha}\phi_{i\alpha}^2+(p/2N)\sum_{i,j}\left[\exp\left[-(i/2p)\sum_{\alpha}(\phi_{i\alpha}-\phi_{j\alpha})^2\right]-1\right]\right\} . \tag{13}$$

The next step is to reduce the problem to a self-consistent single-site problem by the introduction of auxiliary integration variables in the standard way. The result can be expressed in terms of an order function $g(\phi)$, with ϕ an n-component vector, given by

$$g(\phi)=\langle\exp[-(i/2p)(\phi-\psi)^2]\rangle_\psi , \tag{14}$$

where for any function $A(\psi)$

$$\langle A(\psi)\rangle_\psi=\int d\psi\, A(\psi)\exp[(i\mu/2)\psi^2+pg(\psi)]/\int d\psi\exp[(i\mu/2)\psi^2+pg(\psi)] . \tag{15}$$

Following Ref. 16, we investigate the solution which preserves rotational invariance in the replica space, such that $g(\phi)=g(|\phi|)$. Then the angular integral can be evaluated, and the $n\to 0$ limit explicitly taken to give, with $x=|\phi|$,

$$g(x)=e^{-(i/2p)x^2}\left[1-(x/p)e^{-p}\int_0^\infty dy\exp[(i/2)(\mu-1/p)y^2+pg(y)]J_1(xy/p)\right] , \tag{16}$$

where J_1 is the Bessel function of order one. The density of states is given by

$$\rho(\mu)=(1/n\pi)\mathrm{Re}\langle\psi^2\rangle_\psi=(1/\pi)\mathrm{Re}e^{-p}\int_0^\infty y\,dy\exp[(i\mu/2)y^2+pg(y)] . \tag{17}$$

Alternatively, $\rho(\mu)$ can be obtained from the term of order x^2 in the power series expansion of $g(x)$. Using $J_1(z)=z/2+O(z^3)$ in Eq. (16) yields

$$g(x)=1+(i/2)\alpha(\mu)x^2+O(x^4) ,$$

where

$$\alpha(\mu)=-1+(i/p^2)e^{-p}\int_0^\infty y\,dy\exp[(i/2)(\mu-1/p)y^2+pg(y)] .$$

Comparison with Eq. (17) yields

$$\rho(\mu)=(p^2/\pi)\mathrm{Im}\alpha(\mu+1/p) . \tag{18}$$

Equations (16)–(18) are the central results of this section. They determine $\rho(\mu)$ exactly, in principle, via the solution of the nonlinear integral equation (16). To determine the behavior of $f(t)$ at large times, we require the solution for small μ. Unfortunately, we have been unable to solve Eq. (16) in this limit for arbitrary p. We can, however, obtain the solution for arbitrary μ in the limit $p\to\infty$. Although this is not the most interesting regime, the solution does reveal

some general qualitative features of $\rho(\mu)$.

To proceed we introduce a function $h(x)$ via

$$g(x)=[1+h(x)]\exp(-ix^2/2p)$$

and rescale the variables as follows:

$$x=p^{1/4}X,\quad y=p^{1/4}Y,\quad h(p^{1/4}X)=p^{-1}H(X)\ .$$

Making these substitutions in (16), and retaining only terms of order p, $\sqrt{p}$ and unity in the equation for $H(X)$ gives

$$H(X)=-(X^2/2)\int_0^\infty Y\,dY\exp[(i\sqrt{p}/2)(\mu-1)Y^2-Y^4/8+H(Y)]\ . \tag{19}$$

In the large-p limit, therefore, $\rho(\mu)$ depends on μ only through the combination $\sqrt{p}(\mu-1)$. It follows that $\rho(\mu)$ is peaked around $\mu=1$, with a width of order $1/\sqrt{p}$. To reveal the structure of the peak we set

$$\mu=1+\epsilon/\sqrt{p}\ ,$$

and note from (19) that we can write

$$H(X)=(i/2)\gamma(\epsilon)X^2$$

with

$$\begin{aligned}\gamma&=i\int_0^\infty Y\,dY\exp[(i/2)(\epsilon+\gamma)Y^2-Y^4/8]\\&=i\int_0^\infty Y\,dY\int_{-\infty}^\infty[dz/(2\pi)^{1/2}]\exp[-z^2/2+(i/2)(\epsilon+\gamma+z)Y^2]\\&=-\int_{-\infty}^\infty[dz/(2\pi)^{1/2}]\exp(-z^2/2)(\epsilon+\gamma+z)^{-1}\ ,\end{aligned} \tag{20}$$

where the (implicit) positive imaginary part of ϵ ensures the convergence of the Y integral. Equation (20) has been solved numerically for the real and imaginary parts of $\gamma(\epsilon)$. The eigenvalue density is extracted from the imaginary part:

$$\rho(\mu)=(\sqrt{p}/\pi)\mathrm{Im}\gamma(\sqrt{p}(\mu-1))\ . \tag{21}$$

The form of $\rho(\mu)$ given by Eq. (21) is displayed in Fig. 1 for $p=25$. We would expect this result to be accurate to within a few percent, since corrections to (21) appear as a power series in $1/p$. Also shown in Fig. 1 is the eigenvalue density for a Cayley tree with coordination number p (or branching ratio $p-1$). (See Sec. IV C for details.) The comparison with a Cayley tree is relevant because, as we shall see in the following section, the network topology in the case of intensive average connectivity is essentially that of a set of randomly branching trees, a randomly chosen site having a coordination number given by a Poisson distribution with mean p. For a regular Cayley tree, the eigenvalue spectrum has a *gap* between the isolated eigenvalue at $\mu=0$ and the continuum, which begins (see Sec. IV C) at $\mu_g=1-(2/p)\sqrt{(p-1)}$. [The gap vanishes for a linear chain, $p=2$, a fact that has important consequences for the behavior of $\rho(\mu)$ for $\mu\to0$ in the random system—see Sec. IV.] The introduction of disorder of the type considered here has the effect of filling in the gap. In fact, the leading large-p result, Eq. (21), overdoes this, yielding negative eigenvalues when none are possible. Equation (21) is readily solved in the "tails" of the distribution, $|\mu-1|\sqrt{p}\gg1$, to give $\rho(\mu)\sim(p/2\pi)^{1/2}\exp[-p(\mu-1)^2/2]$. Since, however, the predicted value of $\rho(0)$ is exponentially small for large p, $\rho(0)\sim(p/2\pi)^{1/2}\exp(-p/2)$, there is no contradiction since the predicted value of $\rho(0)$ [and of $\int_{-\infty}^0 d\mu\,\rho(\mu)$] is nonperturbative in $1/p$.

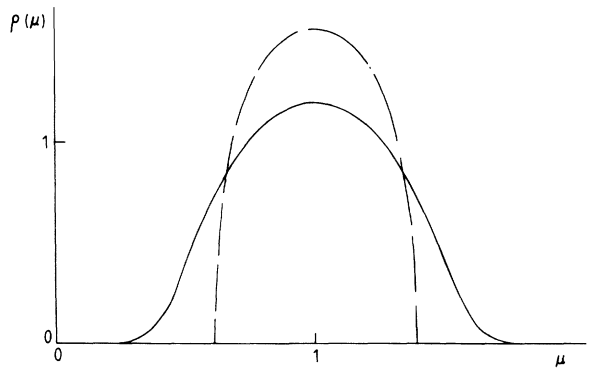

FIG. 1. Eigenvalue density $\rho(\mu)$ of the transition matrix **M** for $p=25$. The solid curve is the large-p result of the replica method, Eq. (21), for the random connectivity model described by Eq. (2). The dashed-dotted curve is the corresponding result for the Bethe lattice (infinite Cayley tree) with fixed coordination number $p=25$.

We conclude that a power series expansion in $1/p$ is unlikely to yield useful information on the limit $\mu\to0$. Since we have been unable to investigate this limit directly for arbitrary p, we resort (in Sec. IV) to heuristic arguments based on the dominance for $\mu\to0$ of certain topological features, namely long chainlike structures (recall that a one-dimensional system has no gap in its eigenvalue spectrum).

The calculations of the present section yield the following qualitative picture as a function of p. For $p\to\infty$, $\rho(\mu)\to\delta(\mu-1)$, corresponding to a single relaxation rate, i.e., one recovers the result obtained for extensive average

connectivity, $f(t)=\exp(-t)$. For p large but finite, the δ function broadens, with a nonzero width of order $1/\sqrt{p}$ (and a finite height of order $\sqrt{p}$). The eigenvalue density has weight right down to $\mu=0$, i.e., there is *no gap*. The *asymptotic* behavior of $f(t)$ is determined by the behavior of $\rho(\mu)$ for $\mu\to 0$. For large p, however, nearly all the *weight* in $\rho(\mu)$ is concentrated around $\mu=1$. For large p, therefore, $f(t)$ will be essentially indistinguishable from $\exp(-t)$, except for extremely large t. In the following section we will argue that ultimately $\rho(\mu)\sim\exp[-A(p)/\sqrt{\mu}]$, for $\mu\to 0$, with $A(p)$ increasing with p for large p, implying $f(t)\sim\exp[-(t/\tau)^{1/3}]$ for $t\to\infty$, for all finite p. In view of the above, however, it is easy to see how a fit, over a finite range of t, to a numerically generated $f(t)$ could be interpreted in terms of a p-dependent stretched exponent.

IV. THE ROLE OF GRIFFITHS SINGULARITIES FOR $\mu\to 0$

We start by considering the general topological features of the network generated by Eq. (2). Although some of the results for these geometrical aspects of the problem have been derived before,[17] we rederive them here since the derivations are simple and we will be making extensive use of the results in our discussion of the dynamics.

The first point to notice is that the system breaks up into disconnected finite and infinite clusters of sites. Here an infinite cluster is one which contains, in the thermodynamic limit, a nonvanishing fraction of the sites. The existence of an infinite cluster requires that p exceed the percolation threshold p_c.

After presenting a simple argument for the percolative order parameter—the fraction P of sites in the infinite cluster—we derive the cluster-size distribution for the finite clusters. These results are then used to determine the probability of forming a quasilinear section of network of a given length, and to estimate its contribution to the eigenvalue density. Since the dominant quasilinear sections will turn out to be longer than those occurring typically, we can make an analogy with the role played by Griffiths singularities[13] in the dynamics of random magnets.[14]

A. The infinite cluster

Consider a set of N sites, randomly connected as described by Eq. (2), and suppose that a fraction P of them, i.e., NP sites, belong to the infinite cluster. Now add a further site, and let its connections with the existing points be also described by the distribution (2). The added site will become part of the infinite cluster with probability

$$P=1-(1-p/N)^{NP}\to 1-\exp(-pP)\ , \tag{22}$$

where the second term in the intermediate expression is the probability that all connections to the existing infinite cluster are absent, and the final expression follows on taking the limit $N\to\infty$. For $p<1$, the trivial solution $P=0$ is the only physically acceptable one (since one cannot have $P<0$), whereas for $p>1$ a nontrivial solution with $P>0$ appears. Hence $p=1$ is the percolation threshold in this model. Equation (22) is identical to the usual mean-field result (obtained, for example, from the q-state Potts model with infinite-range interactions in the limit $q\to 1$, or from the bond-diluted Bethe lattice in the limit of large branching ratio[18]).

B. The cluster-size distribution

Let $w_n(p)$ be the probability that a randomly chosen site belongs to finite cluster of n sites. For example,

$$w_1=(1-p/N)^{N-1}\to\exp(-p)$$

is the fraction of sites which are isolated. Similarly

$$w_2=(N-1)(p/N)(1-p/N)^{2(N-2)}\to p\exp(-2p)$$

is the fraction of sites which belong to isolated pairs, and so on. The smallest few clusters are shown in Fig. 2(a), together with their weights w. In computing the weights, one can ignore constants of order unity relative to N in powers of $(1-p/N)$ and in polynomial prefactors. Note that two types of $n=3$ clusters are possible, corresponding to the chosen site (always shown at the bottom) being an end site or the middle site. The two possibilities have weights $p^2\exp(-3p)$ and $\frac{1}{2}p^2\exp(-3p)$, respectively. The factor $\frac{1}{2}$ in the latter expression derives from the $\frac{1}{2}(N-1)(N-2)$ ways of choosing a pair of sites to couple to the chosen site. It is a consequence of the two equivalent "branches" in the cluster, as seen from the chosen site.

There is, of course, one other type of $n=3$ cluster, not shown in Fig. 2(a), namely a triangular cluster in which

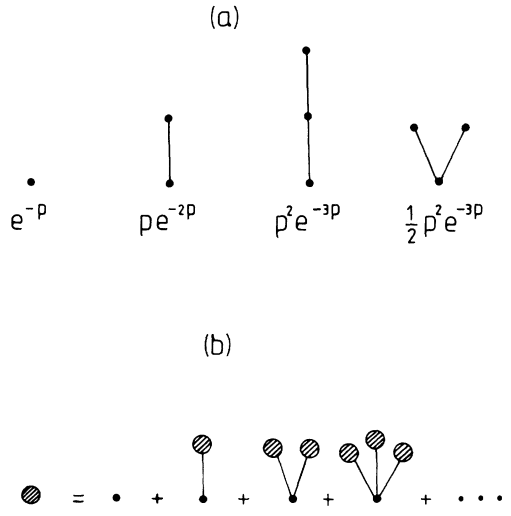

FIG. 2. (a) The first few finite clusters, rooted to a given site, and the corresponding probability weights: a factor $\exp(-p)$ is associated with each site, a factor p with each bond, and a factor $1/n!$ with a set of n identical branches. (b) Diagrammatic equation for the generating function $G(z)$, represented by a shaded blob. The graph rules are as in (a), but with an extra factor of z per site.

all three pairs of sites are linked. The weight for this cluster, however, vanishes in the thermodynamic limit: $w_\Delta \to (1/2N)p^3\exp(-3p)$. The only clusters which have nonvanishing weight for $N\to\infty$ are those in which the factor $1/N$ associated with an additional bond is compensated by a free summation over the position of an additional site. Thus only treelike, or branched clusters, having no closed loops, have nonzero weight. These have the character of randomly branched Cayley trees. (Since there are N sites, there will typically be a finite number of clusters with one closed loop: However, these will not contribute to the extensive part of the eigenvalue spectrum.)

These observations lead to the following rules for computing the weight of an arbitrary branched cluster. (i) Associate a factor $\exp(-p)$ with each site, and a factor p with each bond. (ii) Divide by a symmetry factor $1/n_1!n_2!\ldots$, where $n_1, n_2, \ldots$ are the numbers of equivalent branches of type 1, type 2, etc.

To proceed further, we introduce the generating function

$$G(z)=\sum_{n=1}^{\infty} w_n z^n \ . \tag{23}$$

A diagrammatic expansion in powers of z can be constructed by using the above rules, with an additional factor of z per site. The expansion can be resummed as in Fig. 2(b), where a shaded blob represents $G(z)$, to give

$$\begin{aligned} G &= z\exp(-p)\sum_{n=0}^{\infty}(pG)^n/n! \\ &= z\exp[p(G-1)] \ . \end{aligned} \tag{24}$$

The total weight to belong to a finite cluster is given by $G(1)$, so the fraction of sites in the infinite cluster is given by $P=1-G(1)$, yielding $P=1-\exp(-pP)$, in agreement with the simple argument in Sec. IV A.

We can use Eq. (24) to compute the mean number of finite clusters per site, $n_{\rm cl}=\sum_{n=1}^{\infty} w_n/n$. [Recall that this quantity also equals $f(\infty)$.] From (23), $n_{\rm cl}=\int_0^1 (dz/z)G(z)$. Differentiating (24) yields $G/z=(1-pG)G'$, and hence

$$\begin{aligned} f(\infty)=n_{\rm cl} &= G(1)-(p/2)\{G(1)\}^2 \\ &= 1-P-(p/2)(1-P)^2 \ . \end{aligned}$$

The cluster weight w_n can also be obtained from Eq. (24), via the contour integral

$$w_n=(2\pi i)^{-1}\oint_C (dz/z^{n+1})G(z) \ , \tag{25}$$

where C is the unit circle. Consideration of Eq. (24) reveals that $G(z)$ is analytic in the complex z plane, apart from a branch cut along the positive real axis beginning at $z_b=p^{-1}\exp(p-1)$. Deforming the contour C into the circle at infinity plus integrals along both sides of the cut yields

$$w_n=\pi^{-1}\int_{z_b}^{\infty}(dz/z^{n+1})\,{\rm Im}G(z) \ , \tag{26}$$

since the contribution from the circle at infinity vanishes. Of particular interest is the limit $n\to\infty$. For $n\gg 1$, the integral in (26) is dominated by the vicinity of $z=z_b$, where, from (24), $G\simeq p^{-1}\{1+[-2(z/z_b-1)]^{1/2}\}$. Inserting this into (26) yields

$$w_n\simeq(\sqrt{2}/\pi p)z_b^{-n}\int_1^{\infty}(dx/x^{n+1})(x-1)^{1/2} \ ,$$

giving

$$w_n\simeq(2\pi)^{-1/2}p^{-1}n^{-3/2}\exp[-n(p-1-\ln p)] \ , \qquad n\gg 1 \ . \tag{27}$$

Thus w_n decreases exponentially with n, except at the percolation threshold ($p=1$) where an algebraic dependence is obtained. Not unexpectedly, in view of the mean-field character of the model, these geometrical properties are identical to those derived by Fisher and Essam[18] for a bond-diluted Cayley tree, if in the latter model one takes the limit of large branching ratio and small-bond-occupation probability, at fixed mean coordination number p.

C. Eigenvalue spectrum for a Bethe lattice

In order to understand what type of statistical fluctuations in the connectivity are responsible for small eigenvalues (i.e., small relaxation rates) in a randomly branching Cayley tree, it is important to first examine the properties of an infinite, regular Cayley tree (i.e., a Bethe lattice) with fixed coordination number p. Since the result is well known,[19] we shall simply quote it here. Taking $J_{ij}=1/p$ as usual gives, for the eigenvalue spectrum of the *connectivity matrix* $\mathbf{J}$,

$$\rho_J(\mu)=(p/2\pi)(1-\mu^2)^{-1}[4(p-1)/p^2-\mu^2]^{1/2} \ ,$$

for $\mu^2<4(p-1)/p^2$, and $\rho_J(\mu)=0$ otherwise. The matrix $\mathbf{M}$ which controls relaxation becomes $\mathbf{M}=\mathbf{I}-\mathbf{J}$ (where $\mathbf{I}$ is the unit matrix), with eigenvalue spectrum $\rho(\mu)=\rho_J(1-\mu)$. The result for $p=25$ is shown in Fig. 1. Apart from the isolated eigenvalue at $\mu=0$,[20] the eigenvalues are continuously distributed in the interval

$$1-(2/p)\sqrt{(p-1)}<\mu<1+(2/p)\sqrt{(p-1)} \ .$$

Thus there is a *gap* in the eigenvalue spectrum, *except* for $p=2$ which corresponds to a one-dimensional system. This result is in stark contrast to that for a *compact* lattice, where no gap is expected in general. It suggests that the dominant contributions to $\rho(\mu)$ for $\mu\to 0$ come from quasi-one-dimensional (in a sense to be clarified) sections of the network. In the following subsection we attempt to estimate such contributions semiquantitatively.

D. The eigenvalue density for $\mu\to 0$

To fix our ideas we consider initially a chain of bonds, attached to the infinite cluster at both ends. The results can be generalized to chains which have one or both ends free, and/or are part of a finite cluster. As a first step (the argument will be made more precise later) we consider a simple chain, containing l interior sites each of which is connected only to its two neighbors on the chain. The

probability per site to be part of such a chain is of order $(pe^{-p})^l$, where the factors p and e^{-p} are associated with the chain bonds and sites, respectively. An open chain with free or periodic boundary conditions would have an eigenvalue spectrum $\mu_k \propto (1/p)k^2$, for small k, where k is the usual wave-vector index. Since the chain is connected to the infinite cluster at both ends, the $k=0$ contribution is absent. (There is a single zero eigenvalue from the whole infiinte cluster.) The boundary conditions, however, have little effect on the smallest nontrivial eigenvalue, $\mu_{\min} \sim 1/pl^2$, which corresponds for an open chain to $k \sim 1/l$. Summing over all l with the appropriate weight gives

$$\rho(\mu) \sim \sum_l p^l \exp(-lp)\delta(\mu - 1/pl^2)$$
$$\sim \exp[-(p - \ln p)/\sqrt{(p\mu)}] \ . \tag{28}$$

This expression is already of the form $\rho(\mu) \sim \exp[-A(p)/\sqrt{\mu}]$, which leads to the time dependence $f(t) \sim \exp[-(t/\tau)^{1/3}]$, and contains the essential physics. In the remainder of this subsection we refine these ideas to improve our estimate for the amplitude $A(p)$. In particular we are interested in how $A(p)$ vanishes (or equivalently how τ diverges) for $p \to 1$.

The crucial point is that it is not necessary to have a simple chain of the type described above to generate small eigenvalues—arbitrary finite clusters can be appended to the interior sites of the chain without changing the essential physics. A typical such quasi-one-dimensional section is sketched in Fig. 3(a). Allowing such appendages will clearly significantly increase the weight for forming such a structure.

It is useful to introduce the concepts of nodes, chain sites, and dead-end sites. On the infinite cluster, a node is a site from which there are at least three distinct initial steps which lead to infinity; from a chain site there are exactly two such steps (in the two directions along the chain), while from a dead-end site there is a unique direction for the initial step (i.e., towards the nearest chain site). Clearly the chain terminates in nodes at either end. The important chains are those whose length is large compared with the typical distance between nodes, distances being measured in terms of steps on the network. For such a chain we will show that the smallest eigenvalue is determined by the *sizes* (and not the *shapes*) of the finite clusters attached to the chain. In fact only the average cluster size matters.

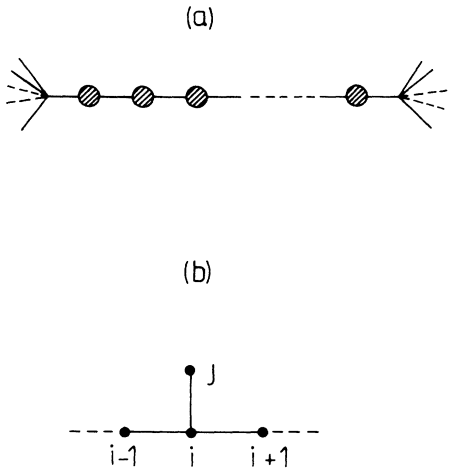

FIG. 3. (a) Diffusion on this type of chain structure dominates the asymptotic dynamics. The shaded blob represents any of the finite clusters that can be generated by iterating the diagrammatic equation of Fig. 2(b). (b) The simplest nontrivial cluster. Eliminating the dead-end site j from the eigenvalue equation (29) yields a one-dimensional problem with coefficients which can be obtained in a simple way for small μ (see text).

To show this we use the eigenvalue equation,

$$(q_i/p - \mu)u_i = \sum_j J_{ij}u_j \ , \tag{29}$$

where $q_i \equiv p\sum_k J_{ij}$ is the coordination number of site i, and u_i is the eigenvector amplitude, to successively eliminate dead-end sites, starting with the singly coordinated sites furthest from the chain sites. As an example, consider the simplest case of a single dead-end site j attached to the chain site i, as illustrated in Fig. 3(b). Use of (29) at sites j (with $q_j=1$) and i (with $q_i=3$), respectively, yields

$$\begin{aligned}(1-p\mu)u_j &= u_i \ , \\ (3-p\mu)u_i &= u_j + u_{i-1} + u_{i+1} \ .\end{aligned} \tag{30}$$

Solving the first equation for u_j gives

$$u_j = u_i/(1-p\mu) = u_i(1+p\mu) \ ,$$

to leading order in μ. Substituting the result in the second of Eqs. (30) gives

$$(2-2p\mu)u_i = u_{i-1} + u_{i+1} \ .$$

This equation is the same as for a one-dimensional system, but with μ replaced by 2μ in the equation for u_i. The factor of 2 has a simple interpretation: It is simply the total mass of the chain site and the attached dead end. A little thought soon convinces one that this result has a simple generalization. For a chain with arbitrary attachements one finds, after eliminating all dead-end sites,

$$(2 - M_i p\mu)u_i = u_{i-1} + u_{i+1} \ , \tag{31}$$

where M_i is the total mass (i.e., total number of sites) associated with the chain site i, including the attached dead-end sites.

At this point an analogy with a one-dimensional phonon problem is useful. With the identifications $\mu \equiv \omega^2$ and $p \equiv 1/K$, Eq. (31) gives the normal frequencies ω for a chain of masses $\{M_i\}$ connected by harmonic springs of stiffness K, where $\{u_i\}$ are the displacements from equilibrium. In the long-wavelength limit (i.e., for small μ), the chain can be treated as an elastic continuum, and the velocity of sound depends only on the mean density, i.e, on the average mass $\langle M \rangle$, yielding the dispersion

$$\mu \sim k^2/p\langle M \rangle, \quad k \to 0 \ .$$

For a finite chain of length l, therefore, $k_{\min} \sim 1/l$ yields for the smallest eigenvalue

$$\mu_{\min} \sim (p\langle M \rangle l^2)^{-1} . \tag{32}$$

This result is independent of whether the chain is part of the infinite cluster or of a finite cluster. In the latter case the chain, including the attached dead-end sites, constitutes the entire cluster, and Eq. (32) gives an estimate of the smallest nonzero eigenvalue associated with the cluster. (There is, of course, one zero eigenvalue per cluster.)

Next we have to compute the probability per site to form a chain of length l. This is given by

$$\begin{aligned} &\text{prob}(l) \sim p^l (1-P)^l = \exp[-a(p)l] , \\ &a(p) = -\ln[p(1-P)] , \end{aligned} \tag{33}$$

with P given by Eq. (22). The factor p^l is associated with the bonds linking the chain sites, the factor $(1-P)$ per chain site is the total probability for attaching a *finite* cluster of arbitrary size, i.e., the probability that the site is *not* connected to the infinite cluster other than through its connections to the (two) neighboring chain sites.

It remains to determine $\langle M \rangle$ in terms of p. Using the generating function $G(z)$ [Eq. (23)] gives

$$\begin{aligned} \langle M \rangle &= \sum_n n w_n \Big/ \sum_n \omega_n = G'(1)/G(1) \\ &= [1-p(1-P)]^{-1} , \end{aligned} \tag{34}$$

where $G'(z)$ means dG/dz and we have used Eq. (24) to express $G'(1)$ in terms of $G(1) = 1-P$.

Combining (32) and (33) yields

$$\begin{aligned} \rho(\mu) &\sim \sum_l \exp[-a(p)l]\delta(\mu - 1/p\langle M \rangle l^2) \\ &\sim \exp[-a(p)/(p\langle M \rangle \mu)^{1/2}] \\ &= \exp[-A(p)/\sqrt{\mu}] . \end{aligned} \tag{35}$$

Hence we recover the form for $\rho(\mu)$ given in (28), but with the correct p dependence for the amplitude $A(p)$. Note that including higher eigenvalues associated with a given chain would not change (35), since their contributions are negligible for $\mu \to 0$.

The function $f(t)$ is the Laplace transform Eq. (7) of $\rho(\mu)$. Evaluating the μ integral by steepest descents for large t, using (35) for $\rho(\mu)$, yields a saddle point $\mu^* \sim (A/t)^{2/3}$ and

$$f(t) - f(\infty) \sim \exp[-(t/\tau)^{1/3}] , \tag{36}$$

$$\begin{aligned} \tau \sim A^{-2} &= p\langle M \rangle / [a(p)]^2 \\ &= p[1-p(1-P)]^{-1} \{\ln[p(1-P)]\}^{-2} . \end{aligned} \tag{37}$$

This expression for τ is correct up to numerical factors associated with Eq. (32). Three limiting regimes are of interest: $p \to \infty$, $p \to 0$, and $p \to p_c = 1$.

(i) For $p \to \infty$, $1-P \to \exp(-p)$ and $\tau \sim 1/p$. Hence the asymptotic decay for large p has the form $\sim \exp[-(pt)^{1/3}]$. Comparison with the form $\exp(-t)$ obtained from the replica method in the large-p limit shows that the true asymptotic behavior discussed here only sets in for $t \sim \sqrt{p}$, when $R(t)$ [see Eq. (1)] is already very small, of order $\exp(-\text{const}\sqrt{p})$, and will therefore be difficult to observe.

(ii) For $p \to 0$, $\tau \sim p/(\ln p)^2$. For most of its decay, however, $f(t)$ is dominated for small p by contributions from isolated sites and pairs of sites: a power series expansion in p gives

$$f(t) = 1 - p/2 + (p/2)\exp(-2t/p) + O(p^2) .$$

The asymptotic behavior discussed above sets in only when $(t/\tau)^{1/3} \sim t/p$, i.e., when $t \sim p|\ln p|$. At this timescale, $R(t)$ is already small, of order $\exp[-\text{const}|\ln p|]$, so the asymptotic regime will again be difficult to observe.

(iii) The limit $p \to 1$ is the most interacting regime, because τ diverges, making the asymptotic stretched exponential behavior more readily accessible. In addition we can compare our results with the predictions of scaling theories of the percolative critical point.[21] Therefore we devote a separate subsection to this limit.

E. The limit $p \to 1$

For p near 1, Eq. (22) gives $P \simeq 2(p-1)\Theta(p-1)$, where $\Theta(x)$ is the usual step function. In the expression for τ, Eq. (37), therefore, $1-p(1-P) \to |p-1|$ and $\ln p(1-P) \to -|p-1|$. Thus

$$\tau \sim |p-1|^{-3}, \quad p \to 1 . \tag{38}$$

This result can be compared with the predictions of scaling theory. For the critical infinite cluster obtained by dilution of a regular d-dimensional lattice, the probability of finding a particle which started at the origin at $t=0$ at position $\mathbf{r}$ at time t should have a scaling form[21]

$$f(\mathbf{r},t) = \lambda^{-D} g(r/\lambda) , \tag{39}$$

where D is the Haussdorf dimension of the cluster, $\lambda(t)$ is the diffusion length, and $g(x)$ is a scaling function. The diffusion index θ is defined by

$$\lambda(t) \propto t^{1/(2+\theta)} .$$

Inserting this into (39), and making the natural generalization to nonzero (but small) $p - p_c$, yields

$$f(\mathbf{r},t) = t^{-D/(2+\theta)} g(rt^{-1/(2+\theta)}, r|p-p_c|^{\nu}) , \tag{40}$$

where the second argument of the scaling function is r/ξ, where $\xi \propto |p-p_c|^{-\nu}$ is the correlation length. For $r \ll \xi$ and $r \ll t^{1/(2+\theta)}$, $f(\mathbf{r},t)$ becomes independent of r and equal to the usual function $f(t)$ which gives the probability of return to the origin. Scaling out the r dependence from Eq. (40) yields

$$f(t) = t^{-D/(2+\theta)} h(t|p-p_c|^{\nu(2+\theta)}) , \tag{41}$$

where $h(x)$ is another scaling function.

To make contact with Eqs. (36) and (37) we note that our model is equivalent to the *mean-field theory* of percolation,[17] and the exponents therefore take the mean-field values[21] $\nu = \frac{1}{2}$, $D = \theta = 4$ so that (41) becomes (with $p_c = 1$)

$$f(t) = t^{-2/3} h(t|p-1|^3) . \tag{42}$$

Thus the intuitive arguments leading to (36) and (37) reproduce the scaling form (apart from the algebraic prefactor), and also give explicitly the large argument form of the scaling function, $h(x)\sim\exp(-\text{const}x^{1/3})$.

The scaling approach suggests that the asymptotic form (36) might be restricted to $t\gg\tau$ (i.e., large values of the scaling variable) for $p\to 1$. Our intuitive arguments confirm this. Specifically, the replacement of the masses $\{M_i\}$ by the mean mass $\langle M\rangle$ in Eq. (31), which leads to (32), is valid only if the relative fluctuations in the total chain mass $M_T=\sum_{i=1}^{l}M_i$ are small. But

$$\langle M_T\rangle = l\langle M\rangle = l[1-p(1-P)]^{-1} ,$$

from Eq. (34), and

$$\langle M_T^2\rangle-\langle M_T\rangle^2 = l(\langle M^2\rangle-\langle M\rangle^2) = lp(1-P)/[1-p(1-P)]^3 ,$$

where the final expression follows from

$$\langle M^2\rangle=\sum_n n^2 w_n \Big/ \sum_n w_n = [zG'(z)]'|_{z=1}/G(1)$$

and use of Eq. (24). The validity of our approach requires

$$\langle M_T^2\rangle-\langle M_T\rangle^2 \ll \langle M_T\rangle^2$$

which implies

$$l\gg p(1-P)/[1-p(1-P)] . \tag{43}$$

Far from the percolation threshold, this condition is always satisfied since the right-hand side is small. For $p\to 1$ however the inequality becomes $l\gg 1/|p-1|$. This has a simple physical interpretation: $1/|p-1|$ is the typical distance (measured along the bonds) between nodes (corresponding to a Euclidean distance $\xi\propto|p-1|^{-1/2}$ between nodes (or to the cluster size for finite clusters) if the cluster is imbedded in a high-dimensional hypercubic lattice with all bonds mutually perpendicular). The behavior of $f(t)$ for large values of the scaling variable is therefore dominated by chains which are longer than typical. Equations (36) and (37) imply that $f(t)$ can be written as $f(t)-f(\infty)\sim\exp[-a(p)l]$, with l the chain length which dominates at time t. Since $a(p)\propto|p-1|$, the condition $l\gg 1/|p-1|$ is equivalent to $a(p)l\gg 1$, i.e., to $t/\tau\gg 1$. Even for $p\to 1$, therefore, our derivation of stretched exponential behavior requires $R(t)\ll 1$. This restriction is, however, much less severe than when p is large or small compared to 1, when $R(t)\ll\exp(-\text{const}\sqrt{p})$ and $R(t)\ll\exp(-\text{const}|\ln p|)$ are required, respectively.

V. DISCUSSION

We have studied the simplest model for diffusion in a sparsely connected space. The geometrical aspects of the problem are those of the mean-field theory of percolation: the percolation clusters consist of randomly branching trees, with mean coordination number p. The dynamics, however, are nontrivial. Heuristic arguments lead to asymptotic stretched-exponential decay of the relaxation function,

$$R(t)=[f(t)-f(\infty)]/[1-f(\infty)] ,$$

according to $R(t)\sim\exp[-(t/\tau)^{1/3}]$.

It is important to appreciate that the use of a mean-field-like, infinite-dimensional model (i.e., a model for which mean-field results become exact) is *not*, as is commonly the case with mean-field approaches, a first step towards a more realistic treatment. On the contrary, the mean-field aspects of the model are *of the essence* here; only for an infinite-dimensional space does the relaxation become exponential in the limit of high connectivity and stretched exponential for low connectivity. Consider, for example, diffusion on a randomly diluted regular lattice. At the percolation threshold, one obtains (see Sec. IV E) anomalous diffusion with $f(t)\sim t^{-D/(2+\theta)}$. For all p different from p_c however, the conventional behavior $f(t)\sim t^{-d/2}$ is recovered at sufficiently long times. This is because at large enough length scales (large compared to the correlation length) the network is no longer ramified but becomes highly connected. For the model considered here, this is not true: The model remains sparsely connected away from $p=1$, provided p remains finite. This sparse connectedness, together with the infinite dimensionality of the space, are the crucial features leading to stretched exponential relaxation. Other models which retain these features should yield similar results.

A question of great interest concerns the range of validity of the asymptotic expressions derived. The answer depends on the value of p. Near the percolation threshold ($p_c=1$), $\tau\propto|p-1|^{-3}$ and the asymptotic form holds when $R(t)$ is small compared to unity, i.e., for large values of the scaling variable $t|p-1|^3$. Far from p_c the asymptotic expression holds only when $R(t)$ is small compared with $\exp(-\text{const}\sqrt{p})$ (for large p) or with $\exp(-\text{const}|\ln p|)$ (for small p), and will therefore be difficult to observe.

It is interesting to compare our predictions with the simulations by Campbell *et al.* of random walks on high-dimensional site-diluted hypercubes.[9] For an N-dimensional hypercube the concentration of occupied sites at the percolation threshold (which is strictly defined only for $N\to\infty$), is $1/N$.[8,9] For this model too, therefore, one can define a regime of intensive mean coordination number: since the coordination number of before dilution is N, the choice $c=p/N$ for the site occupation probability c yields mean coordination number p. In this regime we would expect this model to be qualitatively similar to our model, in view of the low density of closed loops, and therefore to be described by Kohlrausch behavior with $\beta=\frac{1}{3}$ for all finite p. For $c=O(1)$, on the other hand, the model is highly connected, and we would expect pure exponential relaxation. Problems arise, however, for finite N, when the distinction between $c=O(1/N)$ and $c=O(1)$ becomes blurred. In their simulations, Campbell *et al.* are limited to $N\le 17$. We believe that the c-dependent stretched exponent observed in these studies might well be associated with a crossover between two regimes which are not well separated for finite N. Even for large N we have seen that the Kohlrausch regime will be difficult to observe for p far

from p_c and this could again appear as an effective stretched exponent in simulations. For the above reasons, the vicinity of p_c seems the best place for numerical studies, and it is reassuring that $\beta=\frac{1}{3}$ *is* recovered in this regime.[9] In addition, the timescale τ gets large very rapidly for $p\to p_c$, in qualitative agreement with the $|p-p_c|^{-3}$ prediction. As a caveat to the above, however, we note that in a recent report Flesselles and Botet[22] have argued that for the hypercube problem the Kohlrausch law with $\beta=\frac{1}{3}$ holds right at p_c, whereas we obtain a power-law decay for our model at p_c [see Eq.(42)]. If the former result is correct (and an approximation, whose consequences are unclear, was used in its derivation), the two models are inequivalent even for finite average connectivity. Further work is needed to clarify any common features of, and differences between, the two models. In addition, the relevance (if any) of these models to random spin systems,[9] for which direct consideration of the role of Griffiths singularities yield quite different asymptotic dynamics,[14] also needs clarification.

In summary, we have introduced a simple model with stretched-exponential relaxation. The results were obtained without the necessity of introducing a wide distribution of energy barriers. The barriers are, in fact, purely entropic in character, being a consequence of the low connectivity of the configuration space. A distribution of energy barriers can be included in the model, if desired, by a simple generalization in which the term $\delta(J-1/p)$ in Eq. (2) is replaced by a general function of J. This generates a distribution of barriers E via the relaxation $J\sim\exp(-E/T)$. The asymptotic relaxation might then pend on the form of the J distribution for $J\to 0$.

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