Classical Hall effect in two-dimensional composites: A characterization of the set of realizable effective conductivity tensors

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After Dykhne, the classical conductivity problem in any two-dimensional inhomogeneous medium is shown to be isomorphic to the conductivity problem in a family of associated inhomogeneous media. The transformation generating this isomorphism generalizes Keller's duality transformation. For conduction in two-component media in a magnetic field, there is a transformation to a conductivity problem with the magnetic field absent. This implies a linear relation between the determinant and skew part of the effective conductivity tensor which, in the low-field limit, reduces to Shklovskii's formula. A complete characterization is given of the set of all effective conductivity tensors that are obtained as the composite geometry is varied over all possible configurations.

I. INTRODUCTION

In a material composed of insulating and conducting phases the macroscopic conductivity is intimately correlated with the composite geometry; it clearly depends on whether the conducting phase is connected or not. Most effective parameters are similarly structure dependent. Of particular interest, however, are structure-independent equalities or inequalities (bounds) between effective parameters.

One well-known and illustrative example is Levin's exact relation[1] between the effective bulk modulus and effective thermal expansion coefficient of an isotropic composite of two and only two isotropic phases. More recently Hashin[2] and Schulgasser[3] have found a similar type of relation for various types of polycrystalline aggregates. Such relations are due to an underlying isomorphism between the different problems of hydrostatic compression and thermal expansion.

Many exact microstructure independent relations are known for conduction in two-dimensional composites or equivalently for conduction in thin films. Keller[4] obtained an expression for the conductivity of a two-phase composite in terms of the conductivity when the phases are interchanged. Dykhne[5] independently derived an exact microstructure-independent expression for the conductivity of an isotropic polycrystalline film formed from a single anisotropic material. Schulgasser[6] had the clever idea of replacing each crystallite in the polycrystal by a laminate of equal portions of two appropriate isotropic phases and obtained the same result as Dykhne using Keller's phase-interchange equality. Mendelsohn[7] obtained a more general relation which unified these results. This relation was based on an isomorphism between any two-dimensional conductivity problem and its dual. Other proofs of Mendelsohn's relation in a rigorous mathematical setting were given by Kohler and Papanicolaou,[8] Nevard and Keller,[9] and Tartar.[10]

Separate progress was made by Dykhne[11] and Shklovskii[12] who obtained an expression for the low-field Hall constant of a two-phase composite in terms of the effective conductivity in the absence of a magnetic field and in terms of the Hall constants and conductivities of the two isotropic phases. Later Stroud and Bergman[13] noted that a uniform antisymmetric part of the conductivity tensor field could be incorporated trivially into the general theory of conduction in an inhomogeneous medium.

By enlarging upon the work of Dykhne,[11] we will establish an isomorphism between any given two-dimensional conductivity problem and a whole family of associated conductivity problems. These associated problems are each generated from the original problem by applying a special fractional linear matrix transformation to the local conductivity tensor. Within a single framework the isomorphism accounts for all known exact microstructure independent relations for conductivity in two-dimensional composites. In particular since the family of associated conductivity problems includes the dual problem as a member, the isomorphism accounts for the results of Keller,[4] Dykhne,[5] and Mendelsohn.[7] The transformation is based on the simple observation that any two-dimensional divergence-free field when rotated locally at each point by 90° produces a curl-free field and vice versa.

For composites of two anisotropic phases we use the isomorphism to transform a problem in which the conductivity tensors of the two phases are not symmetric into a problem with symmetric conductivity tensors. This generalizes what Stroud and Bergman[13] accomplished for the special case where the two phases have equal skew parts of their conductivity tensors. Since an asymmetric conductivity tensor physically corresponds to a material in an applied magnetic field (the Hall effect generates the skew part of the tensor) we effectively transform to a problem in which the magnetic field is absent. In the absence of the magnetic field the effective conductivity tensor must be symmetric. This applies a linear, experimentally testable, microstructure independent relation between the determinant of the effective conductivity tensor (with the magnetic-field present) and the skew part of that tensor. In the low-magnetic-field limit the relation reduces to Shklovskii's formula[12] for the effective Hall constant.

The transformation leads to another important result, namely a complete characterization of the set of all effective conductivity tensors that are generated as the microstructure of the two-dimensional, two-component composite is varied over all possible configurations while keeping the (possibly asymmetric) conductivity tensors of the components fixed. The characterization follows directly from the characterization obtained by Lurie and Cherkaev[14,15] and Francfort and Murat[10] for the case where the conductivity tensors of the components are symmetric. Such characterizations are needed for the solution of general optimal design problems.[15–19]

Throughout the paper we neglect quantum effects. The conduction in each phase is modeled by the classical continuum equations [see (2.7) and (2.8)] with continuity of potential and flux at the interfaces between phases. For this treatment to be justified the mean free path of the electrons must be much smaller than the typical inhomogeneities in the composite.

II. A FAMILY OF SIMILAR CONDUCTIVITY PROBLEMS

Two different conductivity problems, for conduction in a medium with tensor $\sigma(x)$ and for conduction in a medium with tensor $\sigma'(x)$ can be said to be "similar" if the electric and current fields that solve one problem can be used to obtain a solution for the fields in the other problem and vice versa.

Here we establish that conduction in a two-dimensional medium with a possibly asymmetric tensor $\sigma(\mathbf{x})$, is similar to conduction in a medium with tensor

$$\sigma'(\mathbf{x})=[a\sigma(\mathbf{x})+b\mathbf{R}_\perp][c\mathbf{I}+d\mathbf{R}_\perp\sigma(\mathbf{x})]^{-1}\ , \tag{2.1}$$

for any choice of constants a, b, c, and d, where

$$\mathbf{R}_\perp=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \tag{2.2}$$

is the matrix for a 90° rotation. Furthermore, assuming that the media are macroscopically homogeneous, we will prove their effective conductivity tensors are related via

$$\sigma'_*=(a\sigma_*+b\mathbf{R}_\perp)(c\mathbf{I}+d\mathbf{R}_\perp\sigma_*)^{-1}\ . \tag{2.3}$$

These results for the case where $\sigma(\mathbf{x})$ and σ_* are isotropic are implicit in the work of Dykhne[11] and the proof given here follows Dykhne's approach: see also the remarks at the end of this section.

In general transformation (2.1) will map a symmetric tensor field $\sigma(\mathbf{x})$ into a tensor field $\sigma'(\mathbf{x})$ which is not symmetric. However, if we take $a=0$, $b=1$, $c=0$, and $d=1$ the symmetry of $\sigma(\mathbf{x})$ will be preserved. Then a problem with conductivity tensor $\sigma(\mathbf{x})$ and effective conductivity tensor σ_* gets mapped to a problem with conductivity

$$\sigma'(\mathbf{x})=[\mathbf{R}_\perp\sigma(\mathbf{x})\mathbf{R}_\perp^T]^{-1}\ , \tag{2.4}$$

and its effective conductivity is

$$\sigma'_*=(\mathbf{R}_\perp\sigma_*\mathbf{R}_\perp^T)^{-1}\ . \tag{2.5}$$

This *duality* transformation is a well-known result due to Keller,[4] Dykhne,[5] and Mendelsohn.[7] The same ideas which underlie their proof of (2.5) are helpful in establishing (2.3).

To simplify the subsequent analysis, we restrict our attention to periodic two-dimensional composite media, i.e., to materials with conductivity tensors $\sigma(\mathbf{x})$ satisfying

$$\sigma(\mathbf{x})=\sigma(\mathbf{x}+\mathbf{v}_1)=\sigma(\mathbf{x}+\mathbf{v}_2),\quad \forall \mathbf{x}\in\mathbb{R}^2\ , \tag{2.6}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are primitive lattice vectors. The arguments we give can be reformulated for wider classes of composite materials such as statistical ensembles of materials[8] or for limits of sequences of materials with successively finer microstructure[17,19] without altering the conclusions. And although composites produced in a laboratory are seldom periodic, the effective conductivity tensor is perturbed only slightly if we take a sufficiently large square sample (larger than the relevant correlation lengths in the material) and extend it periodically throughout space.

For periodic composites the effective conductivity tensor σ_* is calculated from solutions to the equation

$$\mathbf{j}(\mathbf{x})=\sigma(\mathbf{x})\mathbf{e}(\mathbf{x}) \tag{2.7}$$

over the set of periodic current fields $\mathbf{j}(\mathbf{x})$ and electric fields $\mathbf{e}(\mathbf{x})$ satisfying

$$\nabla\times\mathbf{e}(\mathbf{x})=0,\quad \nabla\cdot\mathbf{j}(\mathbf{x})=0\ . \tag{2.8}$$

By computing the averages of these fields,

$$\mathbf{e}_*=\int_\Omega \mathbf{e}(\mathbf{x})d\mathbf{x},\quad \mathbf{j}_*=\int_\Omega \mathbf{j}(\mathbf{x})d\mathbf{x} \tag{2.9}$$

for at least two independent solutions, where the integral is taken over the unit cell Ω of the composite, we obtain the effective tensor σ_* via its defining relation

$$\mathbf{j}_*=\sigma_*\mathbf{e}_*\ . \tag{2.10}$$

When the symmetric part of the conductivity tensor is bounded,

$$k^-\mathbf{I}\le[\sigma(\mathbf{x})+\sigma^T(\mathbf{x})]/2\le k^+\mathbf{I} \tag{2.11}$$

for some constants $k^+\ge k^->0$, the uniqueness and existence[20] of the solutions $\mathbf{j}(\mathbf{x})$ and $\mathbf{e}(\mathbf{x})$ for a given applied average field $\mathbf{e}_*$ or $\mathbf{j}_*$, and the linearity of Eqs. (2.7)–(2.9) guarantee that the tensor σ_* is well defined and unique.

The key to proving (2.3) is a simple observation[4,5,7] that the fields

$$\mathbf{e}_\perp(\mathbf{x})\equiv\mathbf{R}_\perp\mathbf{j}(\mathbf{x}),\quad \mathbf{j}_\perp(\mathbf{x})\equiv\mathbf{R}_\perp\mathbf{e}(\mathbf{x}) \tag{2.12}$$

are periodic and satisfy

$$\nabla\times\mathbf{e}_\perp(\mathbf{x})=0,\quad \nabla\cdot\mathbf{j}_\perp(\mathbf{x})=0\ . \tag{2.13}$$

It immediately follows that electric and current fields

$$\begin{aligned}\mathbf{e}'(\mathbf{x})&\equiv c\,\mathbf{e}(\mathbf{x})+d\,\mathbf{e}_\perp(\mathbf{x})=[c\mathbf{I}+d\mathbf{R}_\perp\sigma(\mathbf{x})]\mathbf{e}(\mathbf{x})\ ,\\ \mathbf{j}'(\mathbf{x})&\equiv a\,\mathbf{j}(\mathbf{x})+b\,\mathbf{j}_\perp(\mathbf{x})=[a\sigma(\mathbf{x})+b\mathbf{R}_\perp]\mathbf{e}(\mathbf{x})\end{aligned} \tag{2.14}$$

are periodic and satisfy

$$\nabla\times\mathbf{e}'(\mathbf{x})=0,\quad \nabla\cdot\mathbf{j}'(\mathbf{x})=0\ , \tag{2.15}$$

and are related via

$$\mathbf{j}'(\mathbf{x})=\boldsymbol{\sigma}'(\mathbf{x})\mathbf{e}'(\mathbf{x}) , \tag{2.16}$$

where $\boldsymbol{\sigma}'(\mathbf{x})$ is the tensor (2.1). Similarly the average fields

$$\begin{aligned}\mathbf{e}'_*&=\int_\Omega \mathbf{e}'(\mathbf{x})d\mathbf{x}=c\mathbf{e}^*+d\mathbf{R}_\perp\mathbf{j}^*=(c\mathbf{I}+d\mathbf{R}_\perp\boldsymbol{\sigma}_*)\mathbf{e}_* ,\\ \mathbf{j}'_*&=\int_\Omega \mathbf{j}'(\mathbf{x})d\mathbf{x}=a\mathbf{j}_*+b\mathbf{R}_\perp\mathbf{e}_*=(a\boldsymbol{\sigma}_*+b\mathbf{R}_\perp)\mathbf{e}_*\end{aligned} \tag{2.17}$$

satisfy

$$\mathbf{j}'_*=\boldsymbol{\sigma}'_*\mathbf{e}'_* , \tag{2.18}$$

where $\boldsymbol{\sigma}'_*$ is the tensor (2.3).

In summary from the electric and current fields that solve the conduction problem with tensor $\boldsymbol{\sigma}(\mathbf{x})$ we generate via (2.14) a new set of fields that solve the conduction problem with tensor $\boldsymbol{\sigma}'(\mathbf{x})$ given by (2.1). The converse is also true: the relations (2.14) can be inverted to give $\mathbf{e}(\mathbf{x})$ and $\mathbf{j}(\mathbf{x})$ in terms of $\mathbf{e}'(\mathbf{x})$ and $\mathbf{j}'(\mathbf{x})$. Specifically we find

$$\mathbf{e}(\mathbf{x})=c'\mathbf{e}'(\mathbf{x})+d'\mathbf{e}'_\perp(\mathbf{x}),\quad \mathbf{j}(\mathbf{x})=a'\mathbf{j}'(\mathbf{x})+b'\mathbf{j}'_\perp(\mathbf{x}) , \tag{2.19}$$

where

$$\mathbf{e}'_\perp(\mathbf{x})=\mathbf{R}_\perp\mathbf{e}'(\mathbf{x}),\quad \mathbf{j}'_\perp(\mathbf{x})=\mathbf{R}_\perp\mathbf{j}'(\mathbf{x}) , \tag{2.20}$$

and

$$\begin{aligned}&c'=c/(c^2+d^2),\quad d'=-d/(c^2+d^2) ,\\ &a'=a/(a^2+b^2),\quad b'=-b/(a^2+b^2) .\end{aligned} \tag{2.21}$$

Hence, the two conductivity problems are similar and their effective conductivity tensors are related via (2.3).

This result could alternatively be proved by noting the similarity of conductivity problems (i) with $\boldsymbol{\sigma}'(\mathbf{x})=k\boldsymbol{\sigma}(\mathbf{x})$ (dilation), (ii) with $\boldsymbol{\sigma}'(\mathbf{x})=\boldsymbol{\sigma}(\mathbf{x})+s\mathbf{R}_\perp$ (translation in the direction of $\mathbf{R}_\perp$), (iii) with $(\boldsymbol{\sigma}'(\mathbf{x}))^{-1}=(\boldsymbol{\sigma}(\mathbf{x}))^{-1}+t\mathbf{R}_\perp$ (translation of the inverse tensor in the direction of $\mathbf{R}_\perp$).

The similarity property (i) is obvious and follows directly from the linearity of the field equations, (ii) was noted by Stroud and Bergman[13] and follows from the observation that $\nabla\cdot\mathbf{j}_\perp=0$, and (iii) can similarly be proved once it is recognized that $\nabla\times\mathbf{e}_\perp=0$. Now (ii) implies conduction with conductivity $\boldsymbol{\sigma}(\mathbf{x})$ is similar to conduction with conductivity $\boldsymbol{\sigma}(\mathbf{x})+(b/a)\mathbf{R}_\perp$ and from (i) this is similar to conduction with conductivity $[a\boldsymbol{\sigma}(\mathbf{x})+b\mathbf{R}_\perp]/(c+db/a)$, which from (iii) is equivalent to conduction with conductivity $\boldsymbol{\sigma}'(\mathbf{x})$ given by

$$\begin{aligned}(\boldsymbol{\sigma}'(\mathbf{x}))^{-1}&=(c+db/a)[a\boldsymbol{\sigma}(\mathbf{x})+b\mathbf{R}_\perp]^{-1}+(d/a)\mathbf{R}_\perp\\ &=[c\mathbf{I}+d\mathbf{R}_\perp\boldsymbol{\sigma}(\mathbf{x})][a\boldsymbol{\sigma}(\mathbf{x})+b\mathbf{R}_\perp]^{-1} .\end{aligned} \tag{2.22}$$

Thus the transformation (2.1) can be regarded as a composition of a translation and a dilation followed by a translation of the inverse conductivity tensor. Subsequent translations or dilations do not change the form of the overall transformation (2.1).

While I was studying this general transformation, using the idea of successive dilations and translations of the conductivity and inverse conductivity tensor, Tartar told me he had independently derived this same result with Murat. I liked their approach and have followed it in the treatment given in (2.7)–(2.12): later I became aware of Dykhne's earlier discovery of this transformation for isotropic composites of isotropic conductors.

III. THE TRANSFORMATION TO A PROBLEM WITH A SYMMETRIC CONDUCTIVITY-TENSOR FIELD

Consider a two-component two-dimensional composite (or thin film) in the presence of a uniform magnetic field of magnitude H (applied perpendicular to the film). We assume the magnetic field generated by currents in the composite is small in comparison with H: otherwise it is unrealistic to assume the magnetic field is uniform. The conductivity tensor $\boldsymbol{\sigma}(\mathbf{x})$ is thus taken to have the form

$$\boldsymbol{\sigma}(\mathbf{x})=\mathbf{R}(\mathbf{x})[\chi_1(\mathbf{x})\boldsymbol{\sigma}_1+\chi_2(\mathbf{x})\boldsymbol{\sigma}_2]\mathbf{R}^T(\mathbf{x}) , \tag{3.1}$$

where $\mathbf{R}(\mathbf{x})$ is a field of rotation matrices giving the orientation of the material at any point, χ_1 and χ_2 are the characteristic functions

$$\chi_1(\mathbf{x})=1-\chi_2(\mathbf{x})=\begin{cases}1, & \text{if } \mathbf{x} \text{ is in component 1}\\ 0, & \text{if } \mathbf{x} \text{ is in component 2} ,\end{cases} \tag{3.2}$$

and $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are constant tensors of the form

$$\boldsymbol{\sigma}_1=\begin{bmatrix}\alpha_1 & -\lambda_1\\ \lambda_1 & \beta_1\end{bmatrix},\quad \boldsymbol{\sigma}_2=\begin{bmatrix}\alpha_2 & -\lambda_2\\ \lambda_2 & \beta_2\end{bmatrix} . \tag{3.3}$$

By Onsager's theorem[21] α_1, α_2, β_1, and β_2 are even functions of H, while the constants λ_1 and λ_2 representing the antisymmetric part of the conductivity tensor of each component are odd functions of H. We will assume $\lambda_1\neq\lambda_2$. The special case where $\lambda_1=\lambda_2$ is easily treated: as shown by Stroud and Bergmann[13] a uniform antisymmetric part of the conductivity tensor field can be incorporated trivially into the conductivity problem. Since problems with conductivity tensor fields that differ only by a proportionality constant are obviously similar it suffices to consider transformations of the form (2.1) with $a=d=1$. Our aim is to adjust the two remaining constants b and c so that the transformed tensor field $\boldsymbol{\sigma}'(\mathbf{x})$ becomes symmetric. This is why it is crucial to assume only two components are present: for composites with more than two components it is usually impossible to find a transformation of the form (2.1) that makes $\boldsymbol{\sigma}'(\mathbf{x})$ symmetric. Recalling that $\mathbf{R}_\perp$ and $\mathbf{R}(\mathbf{x})$ are two-dimensional rotations, which therefore commute, the transformed tensor $\boldsymbol{\sigma}'(\mathbf{x})$ takes the form

$$\boldsymbol{\sigma}'(\mathbf{x})=\mathbf{R}(\mathbf{x})[\chi_1(\mathbf{x})\boldsymbol{\sigma}'_1+\chi_2(\mathbf{x})\boldsymbol{\sigma}'_2]\mathbf{R}^T(\mathbf{x}) , \tag{3.4}$$

where for $i=1$ and 2,

$$\sigma_i'=\begin{bmatrix}\alpha_i' & -\lambda_i'\\ \lambda_i' & \beta_i'\end{bmatrix}=(\sigma_i+b\mathbf{R}_\perp)(c\mathbf{I}+\mathbf{R}_\perp\sigma_i)^{-1}$$
$$=\frac{1}{(c-\lambda_i)^2+\alpha_i\beta_i}\begin{bmatrix}\alpha_i(c+b) & \alpha_i\beta_i-(b+\lambda_i)(c-\lambda_i)\\ (b+\lambda_i)(c-\lambda_i)-\alpha_i\beta_i & \beta_i(c+b)\end{bmatrix}. \tag{3.5}$$

If b and c are to be chosen so σ_1', σ_2', and hence $\sigma'(\mathbf{x})$ are symmetric then clearly

$$\begin{aligned}\alpha_1\beta_1&=(b+\lambda_1)(c-\lambda_1)=bc+\lambda_1(c-b)-\lambda_1^2\,,\\ \alpha_2\beta_2&=(b+\lambda_2)(c-\lambda_2)=bc+\lambda_2(c-b)-\lambda_2^2\,.\end{aligned} \tag{3.6}$$

This implies

$$bc=(\lambda_2\Delta_1-\lambda_1\Delta_2)/(\lambda_2-\lambda_1),\quad c-b=(\Delta_2-\Delta_1)/(\lambda_2-\lambda_1)\,, \tag{3.7}$$

where Δ_1 and Δ_2 are the determinants of σ_1 and σ_2, i.e.,

$$\Delta_1=\alpha_1\beta_1+\lambda_1^2,\quad \Delta_2=\alpha_2\beta_2+\lambda_2^2\,. \tag{3.8}$$

Hence c must be a root of the equation

$$c^2-c[(\Delta_2-\Delta_1)/(\lambda_2-\lambda_1)]-[(\lambda_2\Delta_1-\lambda_1\Delta_2)/(\lambda_2-\lambda_1)]=0\,, \tag{3.9}$$

which has two solutions

$$c=\frac{(\Delta_2-\Delta_1)\pm[(\Delta_2-\Delta_1)^2+4(\lambda_2-\lambda_1)(\lambda_2\Delta_1-\lambda_1\Delta_2)]^{1/2}}{2(\lambda_2-\lambda_1)}\,, \tag{3.10}$$

and the associated solutions for b are, respectively,

$$b=\frac{(\Delta_1-\Delta_2)\pm[(\Delta_2-\Delta_1)^2+4(\lambda_2-\lambda_1)(\lambda_2\Delta_1-\lambda_1\Delta_2)]^{1/2}}{2(\lambda_2-\lambda_1)}\,. \tag{3.11}$$

We need to check these solution for c and b are real. To do this note that

$$(y-x)(y\Delta_1-x\Delta_2)\geq -x^2(\Delta_2-\Delta_1)^2/4\Delta_1\,, \tag{3.12}$$

for any $x,y,\in\mathbb{R}$ and Δ_1, $\Delta_2>0$. (The inequality is easily established by taking the minimum of the left-hand side over y.) This implies, with $x=\lambda_1$ and $y=\lambda_2$, that

$$(\Delta_2-\Delta_1)^2+4(\lambda_2-\lambda_1)(\lambda_2\Delta_1-\lambda_1\Delta_2)\geq(\Delta_2-\Delta_1)^2(\Delta_1-\lambda_1^2)/\Delta_1\geq(\Delta_2-\Delta_1)^2\alpha_1\beta_1/\Delta_1\,, \tag{3.13}$$

which is positive provided α_1 and β_1 are both positive.

Physical considerations imply that α_i, β_i, and hence Δ_i must be non-negative for $i=1$ and 2 since otherwise the components will generate energy rather than dissipating it. Hence there exit real constants c and b given by (3.10) and (3.11) such that $\sigma'(\mathbf{x})$ is symmetric, i.e., that transform to a problem in which the magnetic field is effectively absent.

When $\sigma'(\mathbf{x})$ is symmetric, the relations (3.6) imply

$$(c-\lambda_i)^2+\alpha_i\beta_i=(c-\lambda_i)^2+(b+\lambda_i)(c-\lambda_i)=(b+c)(c-\lambda_i)\,, \tag{3.14}$$

and hence the tensors σ_1' and σ_2' given by (3.5) simplify to

$$\sigma_1'=\frac{1}{c-\lambda_1}\begin{bmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{bmatrix},\quad \sigma_2'=\frac{1}{c-\lambda_2}\begin{bmatrix}\alpha_2 & 0\\ 0 & \beta_2\end{bmatrix}. \tag{3.15}$$

It does not matter which sign of the square root we take in the expression (3.10) for c. As could be expected the two possible transformed problems are duals of each other. To prove this we need to show there exists a scaling constant k such that

$$\tilde{\sigma}'(\mathbf{x})=k[\mathbf{R}_\perp\sigma'(\mathbf{x})\mathbf{R}_\perp{}^T]^{-1}\,, \tag{3.16}$$

where $\tilde{\sigma}'(\mathbf{x})$ is the transformed problem obtained by choosing the other root $\tilde{c}$ of the quadratic (3.10). Equivalently from (3.15) we need to show that for $i=1$ and 2.

$$\frac{1}{\tilde{c}-\lambda_i}\begin{bmatrix}\alpha_i & 0\\ 0 & \beta_i\end{bmatrix}=k(c-\lambda_i)\begin{bmatrix}1/\beta_i & 0\\ 0 & 1/\alpha_i\end{bmatrix}, \tag{3.17}$$

which clearly holds if and only if

$$\frac{\alpha_i\beta_i}{c\tilde{c}-\lambda_i(c+\tilde{c})+\lambda_i^2}=k . \tag{3.18}$$

From the formulas for the sum and products of the roots of the quadratic (3.9) we have

$$c\tilde{c}-\lambda_i(c+\tilde{c})=[(\lambda_1\Delta_2-\lambda_2\Delta_1)-\lambda_i(\Delta_2-\Delta_1)]/(\lambda_2-\lambda_1)=-\Delta_i=-\alpha_i\beta_i-\lambda_i^2 . \tag{3.19}$$

Hence, (3.18) is satisfied with $k=-1$. Thus, the two possible transformed problems are duals.

The two prefactors $1/(c-\lambda_1)$ and $1/(c-\lambda_2)$ in (3.15) necessarily have the same sign. To see this note that

$$c-\lambda_i=\frac{\Delta_2-\Delta_1+2\lambda_i(\lambda_1-\lambda_2)\pm[(\Delta_2-\Delta_1)^2+4(\lambda_2-\lambda_1)(\lambda_2\Delta_1-\lambda_1\Delta_2)]^{1/2}}{2(\lambda_2-\lambda_1)} \tag{3.20}$$

and assuming that α_i,β_i are non-negative we have

$$(\Delta_2-\Delta_1)^2+4(\lambda_2-\lambda_1)(\lambda_2\Delta_1-\lambda_1\Delta_2)-[\Delta_2-\Delta_1+2\lambda_i(\lambda_1-\lambda_2)]^2=4(\lambda_2-\lambda_1)^2(\Delta_i-\lambda_i^2)=4(\lambda_2-\lambda_1)^2\alpha_i\beta_i\geq 0 . \tag{3.21}$$

Hence, in the above expression for $c-\lambda_i$ the term with the square root will be larger in magnitude than the preceding term. Therefore, $c-\lambda_1$ and $c-\lambda_2$ will both be positive or both negative depending on whether $\lambda_2-\lambda_1$ is positive or negative and on whether we take the positive of negative square root. By if necessary multiplying both σ_1' and σ_2' by a negative scale factor, say $k=-1$, we obtain a conductivity problem in which the components have positive semidefinite conductivity tensors. Thus the transformed problem is physically realistic whenever the original conductivity problem is physically realistic. In practice this means we could design experiments to measure the effective tensor σ_*' of the transformed problem and then use σ_*' to compute σ_*.

We will now prove that σ^* satisfies a microstructure independent identity. Let us represent σ_* in the form

$$\sigma_*=\mathbf{R}_*\begin{bmatrix}\alpha_* & -\lambda_*\\ \lambda_* & \beta_*\end{bmatrix}\mathbf{R}_*^T , \tag{3.22}$$

where $\mathbf{R}_*$ is a rotation matrix determined by the orientation of σ_*. Since the effective tensor

$$\sigma_*'=(\sigma_*+b\mathbf{R}_\perp)(c\mathbf{I}+\mathbf{R}_\perp\sigma^*)^{-1} \tag{3.23}$$

must be symmetric because σ_1' and σ_2' are both symmetric, it follows by analogy with (3.6) that

$$\alpha_*\beta_*=(b+\lambda_*)(c-\lambda_*)=bc+\lambda_*(c-b)-\lambda_*^2 . \tag{3.24}$$

From (3.7) this relation has the equivalent form

$$\Delta_*(\lambda_2-\lambda_1)=\Delta_1(\lambda_2-\lambda_*)-\Delta_2(\lambda_1-\lambda_*) , \tag{3.25}$$

where

$$\Delta_*=\alpha_*\beta_*+\lambda_*^2 \tag{3.26}$$

is the determinant of σ_*. Presumably it should be possible to experimentally test (3.25) over a range of microstructures and applied magnetic fields. As the volume fractions of the components are varied continuously from pure component 1 to pure component 2 at fixed magnetic field (Δ_*,λ_*) should *trace out a straight line in the* (Δ,λ) *plane joining* (Δ_1,λ_1) *with* (Δ_2,λ_2). When only one component, say component 1, is present in the composite then clearly $\Delta^*=\Delta_1$, and $\lambda^*=\lambda_1$ irrespective of the microstructure of the polycrystal.

By analogy with (3.15) the effective tensor σ_*' is given by

$$\sigma_*'=\mathbf{R}_*\begin{bmatrix}\alpha_*/(c-\lambda_*) & 0\\ 0 & \beta_*/(c-\lambda_*)\end{bmatrix}\mathbf{R}_*^T . \tag{3.27}$$

Let us suppose σ_*' is known from experiment or numerical calculations and that we want to determine σ_*. The eigenvectors of σ_*' obviously determine $\mathbf{R}_*$. Now let

$$\alpha_*'=\alpha_*/(c-\lambda_*) , \tag{3.28}$$

$$\beta_*'=\beta_*/(c-\lambda_*) \tag{3.29}$$

denote the eigenvalues of σ_*'. From (3.24) and (3.29) we have

$$\alpha_*=(b+\lambda_*)/\beta_*' , \tag{3.30}$$

which when substituted in (3.28) implies

$$\lambda_*=(\alpha_*'\beta_*'c-b)/(1+\alpha_*'\beta_*') . \tag{3.31}$$

Substituting this back in (3.28) and (3.29) gives

$$\alpha_*=\alpha_*'(b+c)/(1+\alpha_*'\beta_*') , \tag{3.32}$$

$$\beta_*=\beta_*'(b+c)/(1+\alpha_*'\beta_*') . \tag{3.33}$$

These three equations (3.31)–(3.33) together with a knowledge of $\mathbf{R}_*$ completely determine σ_* once σ_*' is given.

IV. A COMPLETE CHARACTERIZATION OF THE SET OF POSSIBLE EFFECTIVE TENSORS

Our goal is to determine the set of all effective tensors σ_* that can be obtained as the microstructure of the two-dimensional, two-component composite is varied over all possible configurations while keeping the tensors σ_1 and σ_2 fixed. Actually we have nearly achieved this objective. The results of Sec. III imply σ_* is a realizable effective tensor amongst composites of σ_1 and σ_2 if and only if σ'_* is a realizable effective tensor amongst composites of σ'_1 and σ'_2. Hence, from the characterization, obtained by Lurie and Cherkaev[14,15] and Francfort and Murat,[10] of the set of all possible tensors σ'_* of composites built from two phases that have *symmetric* tensors σ'_1 and σ'_2 we can easily obtain the set of all possible tensors σ_*.

To describe the set of possible effective tensors σ'_* let us assume, without loss of generality, that

$$\alpha'_1 \geq \beta'_1 > 0, \quad \alpha'_2 \geq \beta'_2 > 0, \quad \alpha'_* \geq \beta'_* > 0 \; , \tag{4.1}$$

$$\alpha'_2\beta'_2 \geq \alpha'_1\beta'_1 \; , \tag{4.2}$$

where the α'_i and β'_i for $i=1$, 2, and $*$ are the eigenvalues of the σ'_i. (This may necessitate relabeling of the eigenvalues and of the components.) With these assumptions σ'_* is an effective tensor of some microgeometry if and only if

$$\alpha'_1\beta'_1 \leq \alpha'_*\beta'_* \leq \alpha'_2\beta'_2 \; , \tag{4.3}$$

and at least one of the inequalities

$$\alpha'_*\beta'_*(\beta'_2-\beta'_1) \leq \alpha'_1\beta'_1(\beta'_2-\beta'_*) - \alpha'_2\beta'_2(\beta'_1-\beta'_*) \; , \tag{4.4}$$

$$\alpha'_*\beta'_*(\alpha'_2-\alpha'_1) \leq \alpha'_1\beta'_1(\alpha'_2-\alpha'_*) - \alpha'_2\beta'_2(\alpha'_1-\alpha'_*) \tag{4.5}$$

is satisfied. In the well-ordered case,

$$(\alpha'_1-\alpha'_2)(\beta'_1-\beta'_2) \geq 0 \; , \tag{4.6}$$

it suffices to check if (4.3) and (4.4) hold since (4.4) is weaker than (4.5), while in the badly-ordered case

$$(\alpha'_1-\alpha'_2)(\beta'_1-\beta'_2) < 0 \; , \tag{4.7}$$

it suffices to check if (4.3) and (4.5) hold since (4.5) is weaker than (4.4).

For any fixed $\alpha'_*\beta'_*$ in the range (4.3) the lower bound (4.4) on β'_* is realized when the two components are layered together. The crystal eigenvectors that have smallest eigenvalue, i.e., β'_1 or β'_2, are aligned normal to the layers and the volume fractions of the components are adjusted according to the value of the product $\alpha'_*\beta'_*$. Similarly the bound (4.5) is realized when the crystal eigenvectors that have largest eigenvalue, i.e., α'_1 or α'_2, are aligned normal to the layers. All other tensors σ'_* compatible with (4.3) and (4.4) or (4.5) can be realized by forming a polycrystal from this laminate that attains the bound (4.4) or (4.5) and that has the same product $\alpha'_*\beta'_* = \det\sigma'_*$: see Lurie and Cherkaev[14,15] and Francfort and Murat.[10]

From (3.15) and (3.27) the eigenvalues of σ'_i for $i=1,2$, and $*$ are related to the invariants of σ_i via the equations

$$\alpha'_i = \alpha_i/(c-\lambda_i), \quad \beta'_i = \beta_i/(c-\lambda_i) \; , \tag{4.8}$$

where c is given by (3.10). Since we are free to choose either root of the quadratic (3.9) as a solution for c let us for concreteness take c as the largest root; this ensures that both $c-\lambda_1$ and $c-\lambda_2$ are positive as established in the paragraph following (3.21). We suppose the eigenvalue of the symmetric part of σ'_i for $i=1$, 2, and $*$ have been labeled so that

$$\alpha_1 \geq \beta_1 > 0, \quad \alpha_2 \geq \beta_2 > 0, \quad \alpha_* \geq \beta_* > 0 \; , \tag{4.9}$$

where we have assumed on the basis of physical considerations that these eigenvalues are positive (or at least non-negative). Further let us suppose that components have been labeled so that

$$\alpha_2\beta_2/(c-\lambda_2)^2 \geq \alpha_1\beta_1/(c-\lambda_1)^2 \; . \tag{4.10}$$

Under these assumptions the eigenvalues of the transformed tensors σ'_1, σ'_2, and σ'_* satisfy (4.1) and (4.2).

Hence, by substituting the relations (4.8) in (4.3)–(4.5) we deduce that σ_* is an effective tensor of some microgeometry if and only if

$$\Delta_*(\lambda_2-\lambda_1) = \Delta_1(\lambda_2-\lambda_*) - \Delta_2(\lambda_1-\lambda_*) \; , \tag{4.11}$$

$$\alpha_1\beta_1/(c-\lambda_1)^2 \leq \alpha_*\beta_*/(c-\lambda_*)^2 \leq \alpha_2\beta_2/(c-\lambda_2)^2 \; , \tag{4.12}$$

and at least is one of the inequalities

$$\begin{aligned} \alpha_*\beta_*[\beta_2(c-\lambda_1)-\beta_1(c-\lambda_2)]/(c-\lambda_*) &\leq \alpha_1\beta_1[\beta_2(c-\lambda_*)-\beta_*(c-\lambda_2)]/(c-\lambda_1) \\ &\quad - \alpha_2\beta_2[\beta_1(c-\lambda_*)-\beta_*(c-\lambda_1)]/(c-\lambda_2) \; , \\ \alpha_*\beta_*[\alpha_2(c-\lambda_1)-\alpha_1(c-\lambda_2)]/(c-\lambda_*) &\geq \alpha_1\beta_1[\alpha_2(c-\lambda_*)-\alpha_*(c-\lambda_2)]/(c-\lambda_1) \\ &\quad - \alpha_2\beta_2[\alpha_1(c-\lambda_*)-\alpha_*(c-\lambda_1)]/(c-\lambda_2) \; , \end{aligned} \tag{4.13}$$

is satisfied where c given by (3.10) is the largest root of (3.9). Of course the same laminate microgeometries that realize the bounds (4.3)–(4.5) also attain these transformed bounds (4.11)–(4.13).

When the volume fractions f_1 and $f_2 = 1-f_1$ of the components are prescribed the set of possible symmetric effective tensors σ'_* is reduced. The resulting set has not been characterized except when both components are isotropic, i.e., when $\beta'_1 = \alpha'_1$ and $\beta'_2 = \alpha'_2$. In this case Lurie and Cherkaev[22] and Tartar and Murat[23] have established that σ'_* is an effective tensor of some microgeometry if and only if α'_* and β'_* satisfy both the lower bound

$$f_2[(\alpha'_*-\alpha'_1)^{-1}+(\beta'_*-\alpha'_1)^{-1}] \leq 2(\alpha'_2-\alpha'_1)^{-1}+f_1/\alpha'_1 \; , \tag{4.14}$$

and the upper bound

$$f_1[(\alpha_2'-\alpha_*')^{-1}+(\alpha_2'-\beta_*')^{-1}] \le 2(\alpha_2'-\alpha_1')^{-1}-f_1/\alpha_2' , \quad (4.15)$$

where we have assumed that $\alpha_2' \ge \alpha_1'$. The bounds are attained when the composite is an assemblage of coated ellipses when confocal inner and outer surfaces: the coated ellipses are aligned and identical to one another up to a scale factor and must range to the infinitesimally small to fit all space. The lower (upper) bound is attained when we take ellipses with a core of component 2 (component 1) surrounded up a coating of the other component. By varying their eccentricity we obtain composites of various degrees of anisotropy. For isotropic composites with $\alpha_*'=\beta_*'$ the bounds (4.14) and (4.15) correspond to the Hashin-Shtrikman bounds and the microgeometries that achieve them are the Hashin-Shtrikman coated circle assemblages.[24] Lurie and Cherkaev[22] and Braidy and Pouilloux[23] found that simple second rank laminate geometries also suffice to attain the bounds.

Hence, σ_* is an effective tensor of some composite with prescribed proportions f_1 and $f_2=1-f_1$ of the two isotropic components if and only if the identity (4.11) holds and the inequalities (4.14) and (4.15) hold where α_*', β_*', α_1', and α_2' are given by (4.8) and c is the largest root of the quadratic (3.9).

V. SOME RESULTS FOR ISOTROPIC COMPOSITES OF TWO ISOTROPIC CONDUCTORS

When the components and the composite are isotropic, i.e., $\beta_1=\alpha_1$, $\beta_2=\alpha_2$, and $\beta_*=\alpha_*$, the tensors σ_1, σ_2, and σ_* can be represented by points

$$\sigma_1=\alpha_1+i\lambda_1, \quad \sigma_2=\alpha_2+i\lambda_2, \quad \sigma_*=\alpha_*+i\lambda_* \quad (5.1)$$

in the complex plane. The idea of this representation is due to Tartar and Murat (private communication) who noted it is particularly appropriate because $\mathbf{R}_\perp$ has the same algebraic properties as i, satisfying $\mathbf{R}_\perp^2=-1$. The transformation (2.1) from the isotropic σ tensors to the isotropic σ' tensors is thus equivalent to the fractional linear transformation

$$\sigma'=(a\sigma+ib)/(c+id\sigma) \quad (5.2)$$

in the complex plane, where a, b, c, and d are real. These fractional linear transformations clearly map the imaginary axis onto itself. [Conversely any fractional linear transformation mapping the imaginary axis onto itself can be expressed in the form (5.2) with real constants a, b, c, and d].

Any pair of points that are symmetrically placed about the imaginary axis remain symmetrically placed after the transformation. Hence, the preimage of the real axis is a circle centered on the imaginary axis, intersecting it at the two points $\sigma=-ib/a$ and $\sigma=ic/d$. (These points get mapped, respectively, to $\sigma'=0$ and $\sigma'=\infty$.) When $a=d=1$ the equation of this circle is clearly

$$\alpha^2+(\lambda+b)(\lambda-c)=0 . \quad (5.3)$$

From this viewpoint the determination of the constants b and c is equivalent to finding a circle centered on the imaginary axis that passes through σ_1 and σ_2. Once the circle is found any possible isotropic effective tensor σ_* must lie on it because its image point σ_*' necessarily lies on the real axis between $\sigma_1'=\alpha_1'$ and $\sigma_2'=\alpha_2'$. In the (Δ,λ) plane the circle corresponds to the straight line (3.25) which has the equivalent form

$$\frac{\lambda_2-\lambda_*}{\lambda_2-\lambda_1}=\frac{\Delta_2-\Delta_*}{\Delta_2-\Delta_1}=\frac{\alpha_2^2-\alpha_*^2+\lambda_2^2-\lambda_*^2}{\alpha_2^2-\alpha_1^2+\lambda_2^2-\lambda_1^2} . \quad (5.4)$$

Experimental measurements are usually in terms of the elements of the resistivity tensors $\rho=\sigma^{-1}$ rather than the conductivity tensor σ. The resistivity tensors ρ_1, ρ_2, and ρ_* of the components and composite can likewise be represented by points

$$\rho_1=\gamma_1+i\tau_1H, \quad \rho_2=\gamma_2+i\tau_2H, \quad \rho_*=\gamma_*+i\tau_*H \quad (5.5)$$

in the complex plane, in which $\gamma_1(H)$, $\gamma_2(H)$, and $\gamma_3(H)$ are the *transverse magnetoresistances* and $\tau_1(H)$, $\tau_2(H)$, and $\tau_*(H)$ are the *Hall constants*.[21,25] These measured parameters are related to the elements of the conductivity tensors via

$$\gamma_i=\alpha_i/(\alpha_i^2+\lambda_i^2), \quad \tau_i=-\lambda_i/H(\alpha_i^2+\lambda_i^2) \quad (5.6)$$

for $i=1$, 2, and $*$.

For isotropic composites the duality transformation (2.4) reduces to the reciprocal transformation $\sigma'=\sigma^{-1}$. Hence, any general result which holds for conductivity tensors must also apply to resistivity tenors. So by analogy with (5.4) we have

$$\frac{\tau_2(H)-\tau_*(H)}{\tau_2(H)-\tau_1(H)}=\frac{\gamma_2(H)^2-\gamma_*(H)^2+H^2[\tau_2(H)^2-\tau_*(H)^2]}{\gamma_2(H)^2-\gamma_1(H)^2+H^2[\tau_2(H)^2-\tau_1(H)^2]} \quad (5.7)$$

and this relation holds irrespective of the strength of the magnetic field H. In the low-field limit, $H\to 0$, the Hall constants and magnetoresistances approach constant values, and (5.7) implies

$$\frac{\tau_2(0)-\tau_*(0)}{\tau_2(0)-\tau_1(0)}=\frac{\gamma_2(0)^2-\gamma_*(0)^2}{\gamma_2(0)^2-\gamma_1(0)^2} . \quad (5.8)$$

This is precisely the same result which Shklovskii[12] derived following the work of Dykhne.[11] Later Bergman[26] gave an alternative and simpler derivation. Thus, the formula (5.7), which needs to be experimentally verified, generalizes Shklovskii's result to large magnetic fields.

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