Magnetic field effects in Josephson networks

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The zero-temperature critical current of two-dimensional Josephson-junction arrays was studied in square and triangular geometry. The critical current for a low magnetic field is determined by a lattice-dependent pinning force, and is independent of the flux.

I. JOSEPHSON NETWORKS

Artificial devices have been constructed[1–4] that are two-dimensional arrays of small superconducting islands on a normal or insulating substrate. At temperatures somewhat below the transition temperatures of the individual islands, their phases become ordered due to the Josephson effect. Understanding the effects of magnetic field and temperature on these devices is a necessary first step in the study of granular and disordered superconductors.

It will be assumed in what follows that the fluctuations in the magnitude of the order parameter on an island can be ignored, so that the phase is the only relevant degree of freedom; and that the Hamiltonian for the system can be taken in the form

$$H=-E_J\sum_{i>j}\cos(\phi_i-\phi_j-K_{ij}) , \tag{1}$$

where the sum is over all neighboring sites i and j. Thus this is a site model for weakly coupled small chunks of superconductor, each having one degree of freedom. The magnetic field enters through the term K_{ij}, defined by

$$K_{ij}=\int_i^j \frac{2\pi}{\Phi_0}\mathbf{A}\cdot d\mathbf{r} , \tag{2}$$

where $\mathbf{A}$ is the vector potential and $\Phi_0=hc/2e$ is the flux quantum. In principle, the vector potential should contain the contributions from the fields set up by the Josephson currents, but these will be neglected on the assumption that they are small. Then there is no Meissner effect and the penetration length is infinite. The coupling energy E_J depends on temperature and magnetic field through the order parameter for the superconductor; it will be assumed that the temperature dependence can be incorporated by a rescaling of the explicit temperature,[5] and the magnetic effect will be ignored altogether by assuming that the applied field is less than H_{c1} for the islands. Then all physical properties are periodic functions of the flux Φ through an elemental plaquette of the lattice, with period Φ_0. This effect is special to the periodic Josephson array; it does not occur for a uniform superconducting sheet because there is no preferred length scale to relate field to flux, and for a random array the differing plaquette areas give incoherent contributions which suppress the periodic effect.[6,7] Associated with the Hamiltonian (1) are the Josephson currents

$$I_{ij}=J\sin(\phi_i-\phi_j-K_{ij}) , \tag{3}$$

where $J=2eE_J/\hbar$. Choosing the phases to minimize the Hamiltonian also forces the net current into each site to cancel.

Several discussions have been given to the effects of a magnetic field on a Josephson network. Alexander[8] noted that finding the equilibrium state for the linearized Landau-Ginzburg model is equivalent to determining the band edge for free electrons in a magnetic field, for which the spectrum is known.[9] This predicts a superconducting state at all fields at zero temperature, and a critical temperature curve which is a continuous function of flux. The effect of the magnetic field is a nearly smooth decrease in critical temperature, with maximum depression in the "half flux quantum" case.

The model Alexander studies is most appropriate to a lattice of thin wires, since it is assumed that the order parameter is everywhere small, and that it varies along a wire. Thus we may not directly transcribe this result to the site model; yet it is clearly relevant so that the behavior of the site model should be qualitatively similar. The theory gives the highest temperature at which the order parameter is nonzero anywhere, which is distinct from the resistive transition; it is a determination of $H_{c2}(T)$ for the network viewed as a type-II superconductor (though perhaps we should define a $T_{c2}(H)$, since at low temperatures the Josephson network is a superconductor at all fields, at least until the islands go normal or finite size of the junctions becomes relevant).

Teitel and Jayaprakash[10,11] have given a different theory for the critical temperature. They claim to determine the zero-temperature critical current density $I_{\rm TJ}(H)$ and from this determine a temperature $T_{\rm TJ}(H)$ that demarcates the lowest temperature at which flux-flow resistivity can occur. It is related to the flux as follows: If the flux per plaquette can be written as a rational fraction of the flux quantum

$$\Phi=(p/q)\Phi_0 , \tag{4}$$

where p and q are integers, then

$$I_{\rm TJ}\approx J/q \tag{5}$$

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and

$$kT_{\mathrm{TJ}} \approx E_J/q \ . \tag{6}$$

The theory is remarkable in that the dependence of critical current on field is highly discontinuous, and that the predicted critical current density is small for all but a few special values of Φ.

The present paper will show that the effects of a magnetic field on a Josephson network can be interpreted in the language of type-II superconductivity. The magnetic field gives rise to singularities in the order parameter field, which are the locus of vortices in the current distribution and would be regions of concentration of the magnetic field if the penetration depth were finite. The density of these vortices is B/Φ_c. The vortices move as if they were point particles subject to three forces: a pinning force due to the lattice structure, which tends to confine them to the interior of a plaquette; a Lorentz force whose direction and magnitude is given by $I\times B$; and a mutual repulsion. At zero temperature, an isolated vortex will move only if the Lorentz force exceeds the pinning force; this condition determines the critical current density. The Lorentz force and the pinning force are independent of the magnetic field, and thus for weak fields the critical current density is independent of the field, in disagreement with Eq. (5). The critical current density can be anomalously higher for special values of the applied field that give rise to a simple superlattice commensurate with the network itself, because the lattice moves piecewise and this is opposed by the mutual repulsion. This effect is most operative at $\Phi=\frac{1}{2}\Phi_0$ because this is the case where the vortex density is highest.

II. ZERO-TEMPERATURE SIMULATIONS

Rectangular sections of the square lattice having width W and length L were studied. The Landau gauge was chosen for the vector potential

$$\mathbf{A}=\mathbf{y}\times\mathbf{B} \ , \tag{7}$$

so that for the square lattice, $K_{ij}=0$ for links in the x direction, and

$$K_{ij}=axB/\Phi_0=2\pi mf$$

for links in the y direction, where the lattice sites are $x=ma$, $y=na$ for integers m,n. An additional term representing a "phase strain" was added to the Hamiltonian, and periodic boundary conditions were imposed. With the phase strain oriented along the x axis, Eq. (1) becomes

$$H=-E_J\sum_{m,n}\cos(\phi_{m,n}-\phi_{m+1,n}+\Delta) -E_J\sum_{m,n}\cos(\phi_{m,n}-\phi_{m,n+1}-2\pi mf) \ . \tag{8}$$

Here, $f=\Phi/\Phi_0$; periodicity requires that f be a multiple of $1/L$. The phase strain is represented by Δ; its effect can be most clearly seen in the expression for the current density in the x direction:

$$I=J\sum_{m,n}\frac{\sin(\phi_{m,n}-\phi_{m+1,n}+\Delta)}{LW} \tag{9}$$

In the absence of a magnetic field, where the phases can all be the same, we have a current density (current per horizontal bond) $I=J\sin(\Delta)$. In the absence of the phase strain no supercurrent would flow; the critical current density $I_c=J$.

The energy was minimized by simple relaxation. A typical starting configuration for $\Delta=0$ is shown in Fig. 1. The upper part is a representation of $\phi_{m,n}$ which resembles a twisted surface with cuts in it. The twist is the response of ϕ to the vector potential term. Well away from the ends of the cuts, the cuts are discontinuities of 2π; they are needed to allow the surface to twist while still matching (modulo 2π) at the opposite edges. The number of such cuts is LWf; this figure is for $f=1/L$, which is the smallest possible nonzero field.

The lower part of Fig. 1 is a representation of the current distribution. The ends of the cuts appear here as the location of the vortices; there is no physical meaning to the location of the rest of the cut.

The configuration shown in Fig. 1 is a local energy minimum but not an absolute minimum. The repulsion force favors a different arrangement with larger spacing, but the pinning force prevents motion.

The simulation proceeds by slowly increasing Δ, while continually relaxing the configuration, so that its adiabatic evolution can be observed. This induces a current (horizontally to the left in the representation of Fig. 1) which in turn gives rise to a vertical force on the vortices. The current, force, and energy increase until the pinning

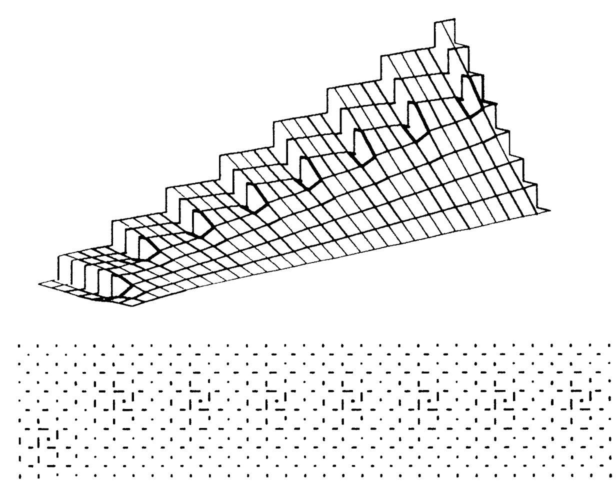

FIG. 1. The phase distribution, and the corresponding current pattern ($L=32$, $W=8$, $f=\frac{1}{32}$). Top panel: the negative of $\phi_{m,n}$ is represented as height in this projection view. Lower panel: A line has been drawn starting at the middle of each link in the direction of the current and with length proportional to the current. The scale is set by the maximum current J; the picture looks almost the same at the critical current because the average current per bond is still small.

force is overcome; then some of the vortices move, changing the length of the cut and thus relieving the phase strain. Moving one vortex the full width W of the strip effectively decreases the total phase strain $L\Delta$ by 2π, so moving one vortex one step is equivalent (on the average) to decreasing the phase strain by $\delta\Delta = 2\pi/LW$. Figure 2 shows how the energy and current density evolved as the phase strain increased from zero to 1 rad in steps of 0.001 rad, starting from the configuration shown in Fig. 1. In the initial stage (up to $\Delta \approx \frac{1}{4}$ rad), every other vortex in the row advances, which serves to spread them apart, eventually arriving at square superlattice of edge $\sqrt{32}$ (rotated by 45°). The subsequent motions of the vortices cause the superlattice origin to move vertically; the periodic drops in energy and current that appear in Fig. 2 occur when the whole superlattice moves one step downwards. Since this entails moving LWf vortices one step, the periodicity interval is $\delta\Delta = 2\pi f = 0.196$ rad for the $L = 32$ strip. The largest current density is $0.121J$ per horizontal bond. This value decreases slightly as the vortex density is lowered. For example, $W = 8$, $L = 64$, $f = 1/L$ gave $I_c = 0.11$, and $W = 10$, $L = 100$, $f = 1/L$ gave $I_c = 0.107$. The equilibrium configuration in these latter cases is a single row of vortices, of horizontal spacing L/W, implicitly forming an unrotated square lattice. Since the mutual repulsion of the vortices plays no role here, these values are a good estimate of the magnitude of the pinning force and the smallest possible critical current density.[12]

The largest simulations attempted were $L = 32$, $W = 32$, for $f = \frac{1}{32}$, which has 32 vortices present. One possible arrangement is the $\sqrt{32}$ square lattice discussed earlier; this was observed and the behavior was identical to that for the $L = 32$, $W = 8$ strip. Another possible arrangement is a 8×4 rectangular lattice; if the short edge is parallel to the direction of current flow, it transforms into the $\sqrt{32}$ lattice, but the alternate orientation is stable, with a critical current of 0.185.

This study also provides an estimate of the barrier height preventing free motion of the vortices. The maximum energy of a $W = 8$, $L = 32$, $f = \frac{1}{32}$ configuration (just before the eight vortices move) is $-1.8130E_J$ per site; thus the energy difference between a vortex just about to move over the barrier and the minimum energy is $0.36E_J$ per vortex.

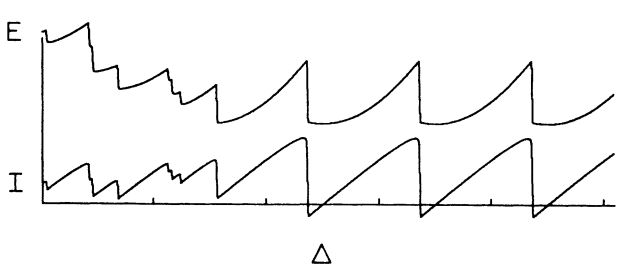

FIG. 2. Energy and current density with increasing phase strain ($L = 32$, $W = 8$, $f = \frac{1}{32}$). The top trace is the energy, and the lower is the current density. The initial configuration was not the ground state, and the first eight jumps occurred as vortices moved to better positions. Thereafter the whole vortex array moved one step every time the critical current was reached. The tic marks on the abscissa are drawn at intervals of $\pi/32$, which is the periodicity interval predicted by Eq. 10.

Other lattice sizes and field values were studied. Many of these supported the periodicity rule

$$\delta\Delta = 2\pi f \ , \tag{10}$$

although it was more common for the vortex array to move piecewise rather than all at once, giving a periodic succession of small drops in energy and current. These multistep patterns were especially common when the vortices were not spaced uniformly in the x direction. A few cases were observed in which the period was a multiple of Eq. (10), due to an alternate choice of ordered configurations.

Random arrangements were also constructed for $L = 32$, $W = 32$, $f = \frac{1}{32}$. These rapidly acquired and maintained a "liquid" arrangement, with rather uniform neighbor spacing but no orientational order. It is relevant to note that "sideways" (parallel to the current) motion of the vortices was only rarely observed; the mutual repulsion forces were generally balanced too well to overcome the pinning force. This effect, which is special to the geometry considered, obviously prevents the crystallization of the vortices. As Δ was slowly increased, only a few vortices reached criticality at a given time, so that the relaxation of the phase strain took place by many small steps. The current density stayed close to its critical value, which was $0.10 \pm 0.01J$. Eventually the sequence of configurations became periodic, with period $2\pi/32$.

Another study for $L = 32$, $W = 32$, $f = \frac{1}{32}$ imposed equal slowly increasing phase strains in both the x and y directions, so that vortices were forced to move in both directions. Now the $\sqrt{32}$ pattern was no longer stable; it decomposed into a liquid in which most of the vortices had six equally spaced neighbors. For small f the ground state is probably a large-scale periodic structure which closely approximates the (incommensurate) triangular superlattice, which is the ground state for the type-II superconductor.[13] The lowest energy observed for the liquid was $-1.8272E_J$ per site, which is less than the $\sqrt{32}$ state ($-1.8243E_J$). The horizontal and vertical components of the critical current density for this liquid were again $0.10 \pm 0.01J$.

For $f = \frac{1}{2}$, the ground state is a checkerboard pattern of vortices,[12] for which the critical current density is $0.414J$. Simulations on a 32×32 array showed that the relaxation process that occurs at the critical point is the advance of one column of vortices by one step (giving three vortices in a row), which quickly rearranges into two domain walls separating the original vortex lattice from a stripe domain which is again the checkerboard but with alternate registry. The domain walls are pairs of adjacent vortices. The simulations suggest that the "shear strength" of the domain wall is lower than that of the lattice, so that the wall moves whenever the current density exceeds $0.25J$. The walls repel each other, so that once formed, they do not anneal out. This suggests that the ground-state properties may not provide a complete description of the dynamical behavior.

As shown in Fig. 2, the advance of a vortex is a nonadiabatic irreversible process. The rapid change in phases occurring as a vortex advances is associated with a voltage field according to the second Josephson equation; the energy which the relaxation process removes from the system would be dissipated by normal processes. A more accurate simulation would include a resistive shunt to each Josephson junction, and find that the advance requires a time of order $\hbar/2eJR$. We may ignore this correction if we assume that the rate of increase of Δ is slow on this time scale. This is equivalent to requiring that the applied electrical field be less than RJ. Then we will observe a current whose temporal behavior is given by Fig. 2: a dc component whose magnitude is comparable to I_c independent of the applied field, and a strong ac component of frequency

$$2eV/L\hbar\,\delta\Delta = 2eV/L\hbar f \ .$$

The Teitel-Jayaprakash theory assumes the evolution in adiabatic at all stages, so that the configuration changes whenever a lower-energy state becomes available. This will occur at intervals $\delta\Delta = 2\pi/q$ [q is defined in Eq. (4)]; the current would then oscillate symmetrically about zero and never exceed $\pi J/q$. In fact the evolution is not adiabatic, because the phase configuration changes continuously, and the different vortex arrangements are then separated by energy barriers. Furthermore, the succession of configurations required to maintain the Teitel-Jayaprakash periodicity rule may require the vortex pattern to jump several squares at a time.

A similar study was undertaken on the triangular lattice. Since the 2π branch cut is shared by only three bonds (instead of four) at the end of the cut, the configurations are closer to criticality and the pinning force is lower. The phase strain was aligned to drive a current parallel to one of the bonds; in this geometry the critical current density in the absence of a field is $1.7602J$ per horizontal bond ($0.5867J$ per bond). For low-field configurations the critical current density was found to be $0.068J$ per horizontal bond.

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[1] J. Resnick, J. C. Garland, J. T. Boyd, S. Shoemaker, and R. S. Newrock, Phys. Rev. Lett. **47**, 1542 (1981).

[2] Richard F. Voss and Richard A. Webb, Phys. Rev. B **25**, 3446 (1982).

[3] B. Pannetier, J. Chaussy, and R. Rammal, J. Phys. (Paris), Lett. **44**, L-853 (1983).

[4] David W. Abraham, C. J. Lobb, M. Tinkham, and T. M. Klapwijk, Phys. Rev. B **26**, 5268 (1982).

[5] C. J. Lobb, D. W. Abraham, and M. Tinkham, Phys. Rev. B **27**, 105 (1983).

[6] J. P. Straley and P. B. Visscher, Phys. Rev. B **26**, 4922 (1982).

[7] C. Ebner and D. Stroud, Phys. Rev. B **31**, 165 (1985).

[8] S. Alexander, Phys. Rev. B **27**, 1541 (1983).

[9] D. R. Hofstadter, Phys. Rev. B **14**, 2239 (1976).

[10] S. Teitel and C. Jayaprakash, Phys. Rev. Lett. **51**, 1999 (1983).

[11] S. Teitel and C. Jayaprakash, J. Phys. (Paris), Lett. **46**, L-33 (1985).

[12] A similar estimate for I_c has been given by T. C. Halsey, Phys. Rev. Lett. **55**, 1018 (1985).

[13] W. H. Kleiner, L. M. Roth, and S. H. Autler, Phys. Rev. **133**, A1226 (1966).