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Perturbation calculation around the two-dimensional Ising model

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The dispersion expansions for the two-point electric correlation functions in the eight-vertex model are calculated to first order in the four-spin coupling in the scaling limit. This is a systematic study of perturbation around the Ising system in the massive regime; features having general relevance are emphasized.

The two-dimensional Ising model is the most studied solvable model which can model some real physical systems. This modeling often involves neglecting some weaker interactions; therefore, it will be interesting to study perturbations around the Ising model. Furthermore, many interesting models such as the axial next-nearest-neighbor Ising (ANNNI) model,[1] which exhibits a fascinating phase diagram, or the Ising model with next-neighbor interactions,[2] are in the category of Ising models with extra interactions which can be studied by the exact technique presented here.

Quite a few nontrivial exactly solved models exist in statistical mechanics; especially notable is the class of models solved by the method of commuting transfer matrices.[3] However, most of them have been solved only for the thermodynamics: the free energy, the critical exponents, and the bulk expectation values. Despite the importance of the thermodynamics and the spectacular results which have been found, these models are far from fully understood, because not much information on their correlation functions are available. In the last few years, an important development[4] on the relation between conformal algebra and two-dimensional (2D) statistical systems has given much insight into the universality classes of many solvable models and has provided their correlation functions at the critical point in terms of some linear differential equations; however, when there is a mass gap, correlation functions are not attainable by the method. The two-dimensional (2D) Ising model is an exception in this respect. The extent of the available information is demonstrated in the remarkable work by Bariev,[5] who computed the dispersion expansion for any correlation function formed with an arbitrary number of basis operators in the Kadanoff algebra.[6]

In addition to the general importance of studying perturbation calculation around the Ising model, it has an immediate significance in understanding the connection between solvable models in statistical mechanics and soliton theory. For the Ising model, it has been established[7] that the two-point spin-correlation function satisfies Painlevé equations, which is an important equation in soliton theory.[8] For other solvable models in statistical mechanics, some have been widely believed[9] to be "equivalent" to 1D quantum field theories, which have been extensively studied in the framework of classical and quantum inverse scattering.[10] Some examples are the six-vertex model and the nonlinear Schrödinger equation, the sine-Gordon model or the Thirring model and the eight-vertex model. Understanding their "equivalence" at the level of correlation functions will be most fruitful. Since the Ising model is contained in these statistical models as a special case, exact results for perturbing around the Ising model gives valuable information toward this effort to relate these fields.

A first attempt to do a straightforward perturbation calculation around the Ising model has been the exact computation of the two-particle contribution to the two-point energy-density correlation function in the Ising model in a magnetic field.[11] There the lowest-order term in an expansion in the external magnetic field at $T \rightarrow T_c^+$ has been found to be a Bessel function, confirming the scaling theory.[12] A second calculation[13] in the same spirit produces new results for the Baxter eight-vertex model, which can be formulated as two Ising sublattices coupled by a four-spin interaction.[14] There the exact results attained are the two-particle contributions to the two-point electric correlation functions to the first order in the four-spin coupling at $T \rightarrow T_c^-$ and $T \rightarrow T_c^+$. In both works,[11,13] the correlation functions are differentiated with respect to the small parameters and evaluated at the Ising points. This gives an expression in terms of an integral of correlation functions in the Ising model for each case as follows:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\mathbf{r}_1 d\mathbf{r}_2 \langle \delta\varepsilon_0 \delta\varepsilon_r \sigma_{\mathbf{r}_1}\sigma_{\mathbf{r}_2}\rangle_I \ , \qquad (1)$$

$$\int_{-\infty}^{\infty} d\mathbf{r}_1 \langle \sigma_0 \sigma_r \delta\varepsilon_{\mathbf{r}_1}\rangle I \ , \qquad (2)$$

where $\langle \ldots \rangle_I$ is the Ising correlation for spin σ , and net energy density $\delta\varepsilon$. An apparent reason that such calculations have not been attempted more may be seen by examining the calculation in Ref. 11. There the number of integrals in the four-point Ising correlation which contribute to the leading exponential order for small magnetic field is 25; it reduces to six integrals after taking into account the geometric symmetries. That the sum of those integrals is a Bessel function is by no means obvious. But if the apparent complexity has inhibited calculations, the simplicity of this result is an indication of hidden structures yet to be exploited. In this paper, we shall present a generalization of the work in Ref. 13 from two-particle contributions to an arbitrary number of particles, and indicate the key features which are shared by other perturbation calculation around the Ising model.

The eight-vertex model can be formulated as two Ising systems coupled by a four-spin coupling: The lattice consists of two Ising sublattices (1) and (2) with nearest-neighbor Ising interaction K within each sublattice; in addition, there is a four-spin interaction K_4 which couples the two Ising lattices. For the scaling limit calculation, the total energy E can be written as

$$-\frac{E}{kT}=2\sum_{\mathbf{R}}[K(\varepsilon_{\mathbf{R}}^{(1)}+\varepsilon_{\mathbf{R}}^{(2)})+K_4\varepsilon_{\mathbf{R}}^{(1)}\varepsilon_{\mathbf{R}}^{(2)}], \tag{3}$$

where the summation is over the sublattice index $\mathbf{R}$. The sublattice energy-density operator $\varepsilon_{\mathbf{R}}^{(j)}$ is the product of two nearest-neighbor spins in an Ising system. We have not distinguished between the horizontal and vertical energy density in the four-spin term, which gives the difference between eight-vertex and Ashkin-Teller model,[15] because we consider the scaling limit $T\to T_c$, $\xi\to\infty$, and $r=R/\xi$ fixed. The $O(K_4)$ term of the electric correlation function can be expressed in the form of Eq. (2), which was first given by Kadanoff and Wegner.[16] The results to all exponential orders in the dispersion expansions for low- and high-temperature regimes are given below. The low-temperature part has been reported in Ref. 17.

Let

$$F_\pm(r)\equiv\lim_{T\to T^\pm}P^{-2}\partial_{K_4}\langle\sigma_0^{(1)}\sigma_0^{(2)}\sigma_R^{(1)}\sigma_R^{(2)}\rangle_{K_4=0}/\langle\sigma_0\sigma_r\rangle^2_{I,T_c^\pm}$$
$$=\left(2\int_{-\infty}^{\infty}d\mathbf{r}_1\langle\sigma_0\sigma_r\delta\varepsilon_{\mathbf{r}_1}\rangle_I\right)_{\text{finite}}, \tag{4}$$

where P is the spontaneous polarization;[18] it has a K_4 dependence which is reflected in the divergence in formula (2) which can be unambiguously extracted in the present approach.[19] Define

$$Q_i\equiv(1+q_i^2)^{1/2},\quad df_i=(dq_i)e^{-rQ_i}/(2\pi Q_i),\quad \Gamma(1,\ldots,2l)\equiv\left(\prod_{i=1}^{2l}X_{i\,i+1}\right)_{\text{mod}2l},$$
$$\tilde q_i\equiv\ln\left|\frac{Q_i-q_i}{Q_i+q_i}\right|,\quad X_{ij}\equiv\frac{Q_i-Q_j}{q_j-q_i},\quad Y_{ij}\equiv\frac{Q_i+Q_j}{q_i-q_j}, \tag{5}$$
$$Z_1(i,j)=\tilde q_iX_{ij},\quad Z_2(i,j)=\tilde q_iY_{ij},\quad Z_3(i,j,k)=\tilde q_iY_{ij}X_{ik}X_{jk}^{-1}.$$

Then

$$F_-(r)=\tfrac{1}{2}\sum_{l=1}^{\infty}\sum_{j=0}^{l}I(k,j), \tag{6}$$

$$F_+(r)=F_-(r)+\frac{\langle\sigma_0\sigma_r\rangle^2_{I,T_c^-}}{\langle\sigma_0\sigma_r\rangle^2_{I,T_c^+}}\tfrac{1}{2}\sum_{l=2}^{\infty}\sum_{j=1}^{l-1}\sum_{m,n}I(k,j)A_mA_n, \tag{7}$$

where $k+j=l$,

$$I(k,j)=(-1)^l\frac{2}{\pi}\int_{-\infty}^{\infty}df_1\cdots\int_{-\infty}^{\infty}df_{2l}\Gamma(1,\ldots,2l)[Z_1(1,2j)+Z_1(2l,2j+1)+Z_2(2l,2j)+Z_2(1,2j+1)$$
$$+Z_3(2j,2l,1)+Z_3(2j+1,1,2l)], \tag{8}$$

and the operator A_n acts on the *formal* expression (8), where for $A_n(q)$ set

$$df_n=dq_n\delta(|q_n|-\infty)\quad\text{and}\quad\tilde q_n=0. \tag{9}$$

The summation over m,n in (7) is over $1\le m\le 2j$ and $2j+1\le n\le 2l$, or $1\le m,n\le 2j$, and $|m-n|$ even, or $2j+1\le m$, $n\le 21$, and $|m-n|$ even. And we have omitted a constant term which gives the dispersion form of the two-point Ising correlation.

The function $I(k,j)$ is the basic unit in this calculation and contributes to the $2l$th exponential order to $F_-(r)$. A sketch of a set for $k=1$, $j=2$ is given in Fig. 1. Each diagram contains two sets of σ_0, σ_r, and $\delta\varepsilon$, with $\delta\varepsilon$ close to one of the spins; a line starts and ends at each of the $\delta\varepsilon$'s and visits the spins for an arbitrary number of times. Denote it as $\omega_{mn}^{\pm}(k,j)$, where $2k,2j$ is the total number of long segments in each part of the diagram, and m,n the number of short segments. The superscript $+,-$ denotes the case that the $\delta\varepsilon$'s are in positions 0^- or r^+, 0^+ or r^-. For example, I_{21} in Fig. 1 is $\omega_{21}^-(1,2)$. The integral representing each diagram can be obtained using the following rules.

(i) Write $-2^{3-l_m-l_n}(-2\pi)^{1-t}\prod_{h=1}^{t}\int_{-\infty}^{\infty}dp_hP_h^{-1}$, where $l_m=1$, 0 for m even, odd; and t is the total number of segments. Write a factor $\exp(-rP_h)$ for every long segment p_h where $P_h\equiv(1+p_h^2)^{1/2}$. Diagrams related by right-left or up-down reflection are considered as one diagram.

(ii) Attach a variable to each segment and assign a direction to each line. For every σ-vertex from segments p_1 to p_2, write $(P_2+sP_1)/i(p_1-p_2-i\epsilon)$, where the prescription $\epsilon\to0^+$. For a $\delta\varepsilon$ vertex, write $(1+ip_1)^{1/2}\times(1+ip_2)^{1/2}+s(1-ip_1)^{1/2}(1-ip_2)^{1/2}$. The constant $s=-1$ if the angle is acute, and $s=+1$ if the angle is obtuse.

(iii) Write $\delta(p_1-p_2+p_3-p_4)$ if segments p_1 and p_3 (p_2 and p_4) point toward (out of) the $\delta\varepsilon$'s. Write $(\sum_{h=1}^{4}s_hP_h)^{-1}$ where every $s_h=+1$ for $\omega_{mn}^{+}(k,j)$, and $s_h=+1,-1$ if the segment p_h is short, long for $\omega_{mn}^{-}(k,j)$.

The above diagrammatic rules are typical and can be extended to any perturbation calculation where one is required to compute certain integrals over all space for the

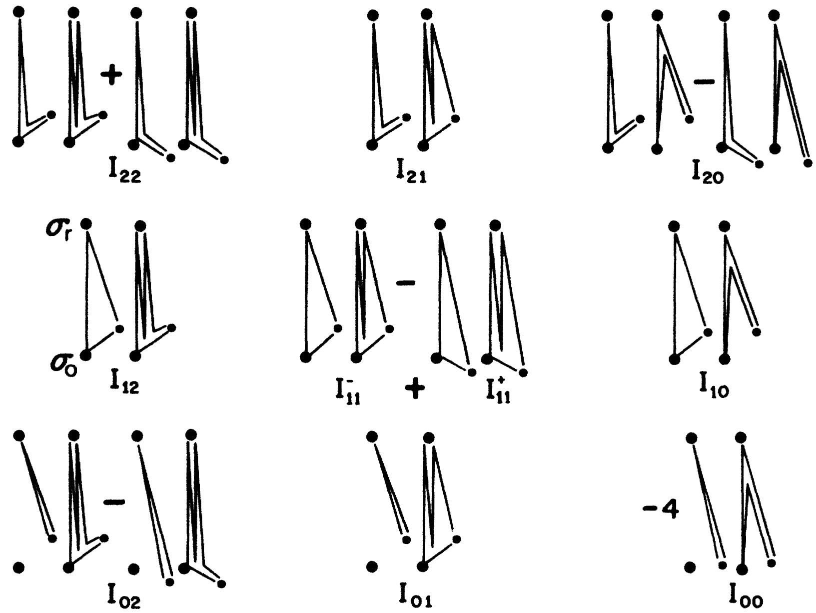

FIG. 1. The diagrams in $I(1,2)$ is a basic set which satisfies rotational invariance in the $O(K_4)$ term of the electric correlation function. Number of segments between the spins determines the exponential order and that of the short ends the number of necessary integrations.

product of Ising correlation functions such as formulas (1) and (2): the two sets of σ_0, σ_r, $\delta\varepsilon$ are needed because of the two factors of the three-point function in (2); rules (i) and (ii) come from the individual Ising correlation, and rule (iii) accounts for the effect of $\int_{-\infty}^{\infty} d\mathbf{r}_1$. And since the x and y dependence in general has the form $\exp(ix\sum_h \pm p_h)$ and $\exp(y\sum_h \pm P_h)$ locally, the operation $\int_{-\infty}^{\infty}\mathbf{r}$ always produces factors of $\delta(\sum_h \pm p_h)$ and $(\sum_h \pm P_h)^{-1}$ as in rule (iii) above.

Each of the diagram satisfies local rotational invariance, but has discontinuity as two operators cross along the y direction. The set of diagrams $I(,kj)$ is the minimal set which has a globally rotationally invariant sum in the calculation. This can be seen by examining the invariance properties of the Ising correlation functions which can be roughly stated as follows: A symmetry is preserved by adding to a diagram another which is obtained by lifting a relevant σ vertex.

To obtain (8) we need to evaluate $m+n$ integrations for each I_{mn} in Fig. 1. The computation is independent of k,j and can be described in the following. Introduce a symmetric change of variables consistent with $\delta(p_1-p_2+p_3-p_4)$ and rationalize the denominator of $(\sum_{h=1}^4 s_h P_h)^{-1}$, the two factors from rule (iii) above. The integrand of I_{22} can be separated into terms having branch points on one complex half-plane only, and the contour of integration for this variable can be closed on the poles on the other half-plane. This can also be done for part of I_{21} and I_{12}; the other parts nearly cancel the residues from I_{22} except for some pole contributions, and their sum is given by the residues of these poles. Such near cancellation happens repeatedly and follows closely the route for preserving symmetries among a set of diagrams discussed above. What remains after all the cancellations are a few terms from $I_{22}+I_{21}+I_{12}$ which are related to poles from the denominator $(\sum_h s_h P_h)^{-1}$. One is intrigued by the simplicity of the result and wonders if there is some underlying theory which may enable one to pick out what remains without any detailed computation. While this is still an open question, we remark that it is possible to express the computation in a simple form.[19]

The high-temperature result[19] is closely related to the low-temperature case, as the expression in (7) suggests; this we obtain not by repeating the procedure for obtaining $F_-(r)$ as one would have proceeded given (2) and the formulas for Ising correlations in Ref. 5. Indeed the same procedure for the $T \to T_c^+$ case will involve calculating the rapidly growing number of diagrams. Instead we exploit the close relation between the low- and high-temperature Ising correlation functions which has not been realized before, namely, the diagrams for the T_c^+ case can be considered as the T_c^- diagrams with deleted segments. Then we need only to study the commutation relations between the operation of deleting a segment and those used in the calculation for $I(k,j)$, such as evaluating the residue of a pole. It is found that the effects of noncommuting operations simply eliminate the possible ambiguity in the final result, and we only need to add in (9) the part $\tilde{q}_n = 0$ to the operator for deleting a segment. Therefore, the result for T_c^+ can be obtained from the T_c^- case without much effort. That this simple approach works not only makes such perturbation calculation practical but also suggests some underlying regularities yet to be understood.

We discuss above the rules for the scaling limit calculation for a first-order perturbation; it is clear that generalization to higher order is straightforward, and lattice generalization can also be done since the Ising correlations on the lattice are known.[5] This will lead to exact results valid for a large portion of phase spaces.

[1]P. Bak and J. von Boehm, Phys. Rev. B **21**, 5297 (1980); W. Selke and M. E. Fisher, Z. Phys. B **40**, 71 (1980); W. Selke and P. M. Duxbury, *ibid.* **57**, 49 (1984).

[2]H. W. J. Blöte and M. P. Nightingale, Physica A **134**, 274 (1985).

[3]R. J. Baxter, Phys. Rev. Lett. **26**, 832 (1971); Ann. Phys. (N.Y.) **70**, 193 (1972); R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic, London, 1982); G. E. Andrews, R. J. Baxter, and P. J. Forrester, J. Stat. Phys. **35**, 193 (1984).

[4]A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, J. Stat. Phys. **34**, 763 (1984); Nucl. Phys. **B241**, 333 (1984); Vl. S. Dotsenko, J. Stat. Phys. **34**, 781 (1984); Nucl. Phys. **B253**, [FS11], 54 (1984); D. Friedan, Z. Qiu, and S. Shenker, Phys. Rev. Lett. **52**, 1575 (1984); J. L. Cardy, Nucl. Phys. **B240**, [FS12], 514 (1984); **B270**, 186 (1986).

[5]R. Z. Bariev, Physica A **83**, 388 (1976); **93**, 354 (1978).

[6]L. P. Kadanoff, Phys. Rev. Lett. **23**, 1430 (1969).

[7]E. Barouch, B. M. McCoy, and T. T. Wu, Phys. Rev. Lett. **31**, 1409 (1973); C. A. Tracy and B. M. McCoy, *ibid.* **31**, 1500 (1973); T. T. Wu, B. M. McCoy, C. A. Tracy, and E. Barouch, Phys. Rev. B **13**, 316 (1976); M. Sato, T. Miwa, and M. Jimbo, Publ. Res. Inst. Math. Sci. **14**, 223 (1977); **15**, 201 (1979); **15**, 577 (1979); **15**, 871 (1979).

[8]M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform,* SIAM Studies in Applied Mathematics (SIAM, Philadelphia, 1981).

[9]A. Luther and I. Peschel, Phys. Rev. B **12**, 3908 (1975); D. B. Creamer, H. B. Thacker, and D. Wilkinson, Phys. Rev. D **21**, 1523 (1980); M. Imada, Prog. Theor. Phys. **68**, 527 (1982).

[10]L. Faddeev, Sov. Sci. Rev. Math. Phys. C **1**, 107 (1980).

[11]B. M. McCoy and M. L. Yan, Nucl. Phys. **B257**, [FS14], 303 (1985).

[12]W. J. Camp and M. E. Fisher, Phys. Rev. Lett. **26**, 73 (1971); Phys. Rev. B **6**, 946 (1972); W. J. Camp, *ibid.* **6**, 960 (1973); **7**, 3187 (1973).

[13]L. F. Ko and B. M. McCoy, Phys. Rev. Lett. **56**, 2645 (1986); the delta-function in their result (15b) should not be there.

[14]F. Y. Wu, Phys. Rev. B **4**, 2312 (1971); see R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic, London, 1982).

[15]J. Ashkin and E. Teller, Phys. Rev. **64**, 178 (1943).

[16]L. P. Kadanoff and F. J. Wegner, Phys. Rev. B **4**, 3989 (1971).

[17]L. F. Ko and B. M. McCoy, Phys. Lett. A **118**, 287 (1986).

[18]F. J. Baxter and S. B. Kelland, J. Phys. C **7**, L403 (1974).

[19]L. F. Ko, Nucl. Phys. B (to be published).