Gap-exponent evolution near the upper critical dimension of directed percolation

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Exact series expansions from perimeter polynomials are used for direct estimation of the gap exponent for $d=3$ to $d=6$ in fully directed site percolation. The critical concentrations are estimated at $p_c=0.303\pm0.0015$ $(d=4)$, $p_c=0.231\pm0.003$ $(d=5)$, and $p_c=0.187\pm0.002$ $(d=6)$. The approach to $d=5$ (the upper critical dimension) is also discussed.

INTRODUCTION

Directed percolation, where a fixed orientation of one (or more) lattice directions rules the cluster connectivity and buildup, is a model which has received some attention ever since the first numerical studies were published by Blease.[1] It is considered to belong to the same universality class as Reggeon field theory[2-4] and possesses two correlation-length exponents as opposed to undirected percolation. However, for a different universality class, with such distinctive features, the amount of numerical work on it (both on Monte Carlo and exact series studies) has remained rather limited. This is only partly due to the fact that the most significant advances on the elucidation of the percolation phase transition were made on undirected systems and must also be related to the initial difficulties in formulating the Monte Carlo simulations correctly.[5]

Numerical work has thus relied mostly on series, and the critical exponents β and γ have been estimated in two and three dimensions.[6-8] The susceptibility exponent γ was also estimated by Blease,[1] using expansions for hypercubic bond percolation, before Obukhov[9] established $d_c=5$ as the upper critical dimension for the problem ($d_c=6$ for undirected percolation), but a reanalysis of the data by standard methods (mostly Padé approximants)[2] left the earlier set of estimates substantially unaltered.

Recent efforts have tried to get a more complete picture through the simultaneous study of both dominant and subdominant (confluent) singularities,[4,7] on the basis of a range of various lattice series for two and three dimensions. The overall conclusion seems to be that, beyond $d=2$, some cautious narrowing of Blease's[1] original uncertainties was possible,[4,6] although the gain was still modest when compared with the series extension effort. In this paper, we continue our work using exact series expansions[4,6] and focusing on four, five, and six dimensions. The interest in these is not merely academic; four-dimensional directed percolation has been proposed as a phenomenological model for the evolution of three-dimensional galaxies,[10] but a study of these higher dimensionalities is essential for the correct understanding of the approach to mean-field behavior at the upper critical dimension.

Most studies on dimensions greater than three must rely on a knowledge of low-density perimeter polynomials,[11] which summarize the total number per site of connected clusters at fixed size s with perimeter t: These polynomials obey the standard sum rule

$$p=\sum_{s,t} g_{st}p(1-p)^t \tag{1}$$

and they can be used to estimate not only the "thermal" exponent γ, but also the magnetic exponent δ and the cluster-size exponent τ.[12] The alternative route of transfer-matrix derivations followed by Blease in his initial studies,[1] and extensively used by de'Bell and Essam,[7,8] looses much of its efficiency for dimensions higher than two, while the recursion relations implicit in the Markov properties of fully directed site percolation greatly improve the exhaustive cluster enumeration for $d=4$, $d=5$, and $d=6$.

For site percolation, four-dimensional (4D) hypercubic data are available to order $s=10$ (Ref. 11) and the present work uses additional data (to order $s=13$) and the first twelve and ten polynomials on the 5D hypercubic and 6D hypercubic site problems, respectively. From these, low-density cluster-size moments can be expanded as

$$m^{i+j}(p)=\sum_{s,t} s^i t^j g_{st} p^s(1-p)^t , \tag{2}$$

where $i,j\geq 0$. The dominant and subdominant critical behavior of these moments is

$$m^{i+j}(p)\sim |p-p_c|^{-[\gamma+(i+j-1)\Delta]}[1+B(p_c-p)^{\Delta_1}] , \quad p\to p_c^- \tag{3}$$

with B depending on the expansion. To obtain gap-exponent series $\delta(p)$ from these, denote

$$\langle s^i t^{k-i}\rangle = m^{i+j}(p) \tag{4}$$

and

$$\delta(p)=\langle s^i t^{k-i}\rangle / \langle s^n t^{k-n-1}\rangle , \tag{5}$$

with $k>1$, $0\leq n\leq k-1$. The critical behavior of $\delta(p)$ follows from Eqs. (3) and (5):

$$\delta(p)\sim |p_c-p|^{-\Delta} . \tag{6}$$

A great number of combinations is possible for $\delta(p)$ and direct estimation of the gap exponent can be made

TABLE I. δ series for the simple-cubic problem.

	$\langle s^2\rangle/\langle s\rangle$	$\langle st\rangle/3\langle s\rangle$	$\langle s^3\rangle/\langle s^2\rangle$	$\langle ts^2\rangle/3\langle s^2\rangle$
1	1	1	1	1
2	6	4	12	8
3	24	13	48	23
4	72	39	114	67
5	216	108	486	224
6	588	303	978	551
7	1656	$795.3\overline{3}$	3738	$1621.3\overline{3}$
8	4262	$2151.6\overline{6}$	7264	$4229.6\overline{6}$
9	11 682	5416	27 996	10 934
10	28 680	14 354	41 664	27 524
11	77 904	35 056	212 730	77 742
12	183 768	92 387	206 292	159 791
13	503 832	218 441	1 531 968	529 831
14	1 129 128	$583\,188.3\overline{3}$	760 026	$950\,478.3\overline{3}$
15	3 217 400	$1\,316\,299.6\overline{6}$	11 808 764	$3\,372\,225.6\overline{6}$
16	6 640 944		−4 525 674	

TABLE II. δ series for the 4D hypercubic problem.

	$\langle s^2\rangle/\langle s\rangle$	$\langle st\rangle/4\langle s\rangle$	$\langle s^3\rangle/\langle s^2\rangle$	$\langle s^3t\rangle/4\langle s^3\rangle$
1	1	1	1	1
2	8	6	16	24
3	44	28.5	92	10.5
4	188	121.5	340	931.5
5	788	486	1772	−2 628
6	3056	1909.5	5992	33 202.5
7	11 966	7204	27 206	−124 193
8	44 056	27 181.5	87 752	611 491.5
9	167 480	99 306	407 296	1 938 732
10	596 880	365 489	1 128 572	−57 530 842
11	2 229 608	1 307 821.75	5 756 052	809 681 308.75
12	7 764 914	4 738 329.75	14 306 406	−8 614 238 882.25
13	28 747 794	16 674 709.25	76 487 978	81 196 025 539.25
14	97 931 090		172 873 708	

TABLE III. δ series for the 5D hypercubic problem.

	$\langle s^2\rangle/\langle s\rangle$	$\langle s^3\rangle/\langle s^2\rangle$	$5\langle s^2\rangle/\langle t\rangle$	$\langle st\rangle/\langle t\rangle$
1	1	1	1	1
2	10	20	11	9
3	70	150	83	61
4	390	760	493	355
5	2110	4810	2753	1925
6	10 710	22 880	14 331	10 007
7	54 160	125 960	73 419	50 413
8	262 410	577 280	361 227	249 221
9	1 284 680	3 133 940	1 778 239	1 209 945
10	6 058 670	13 306 100	8 482 635	5 813 357
11	29 067 530	73 362 590	40 781 282	27 598 005
12	134 712 730	298 900 170	190 765 472	130 211 850
13	637 844 040	1 639 822 650		

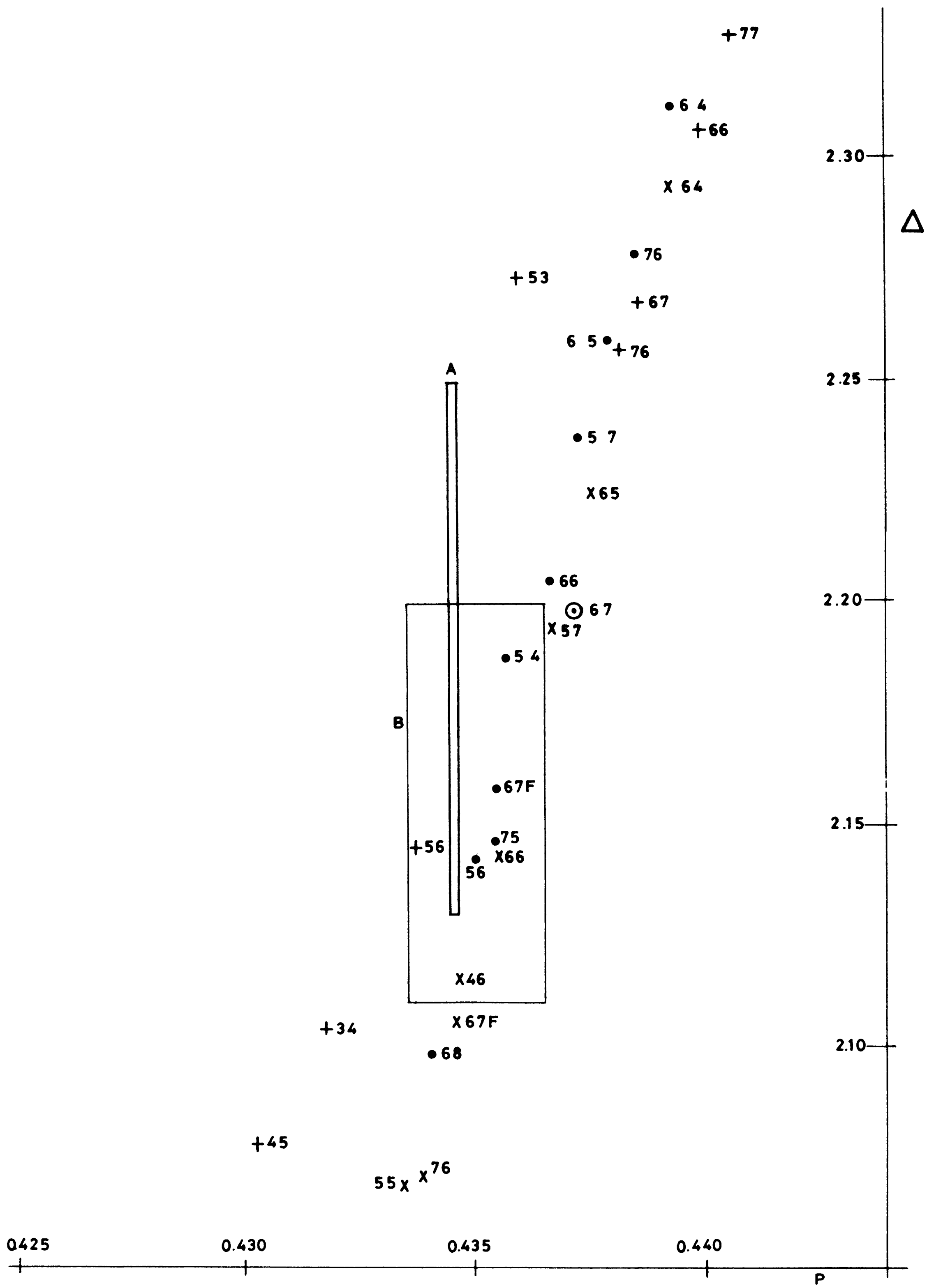

FIG. 1. Padé patterns from various $\delta(p)$ simple cubic series according to the combinations described in Eqs. (4) and (5): $\times$ denotes poles from series for $\langle st\rangle/\langle s\rangle$; $\bullet$ denotes poles from $\langle s^2\rangle/\langle s\rangle$, $\odot$ denotes poles from $\langle s^4\rangle/\langle s^3\rangle$; $+$ denotes poles from $\langle s^3\rangle/\langle s^2\rangle$; critical concentrations are marked on the x axis; Δ values on the vertical axis. A indicates the central p_c and the Δ estimate found by combination of Adler and Duarte's γ (Ref. 4) and the β from Ref. 7. B indicates the same combination with Duarte and Ruskin's γ (Ref. 5). $ij = i$ degree of the numerator; j degree of the denominator.

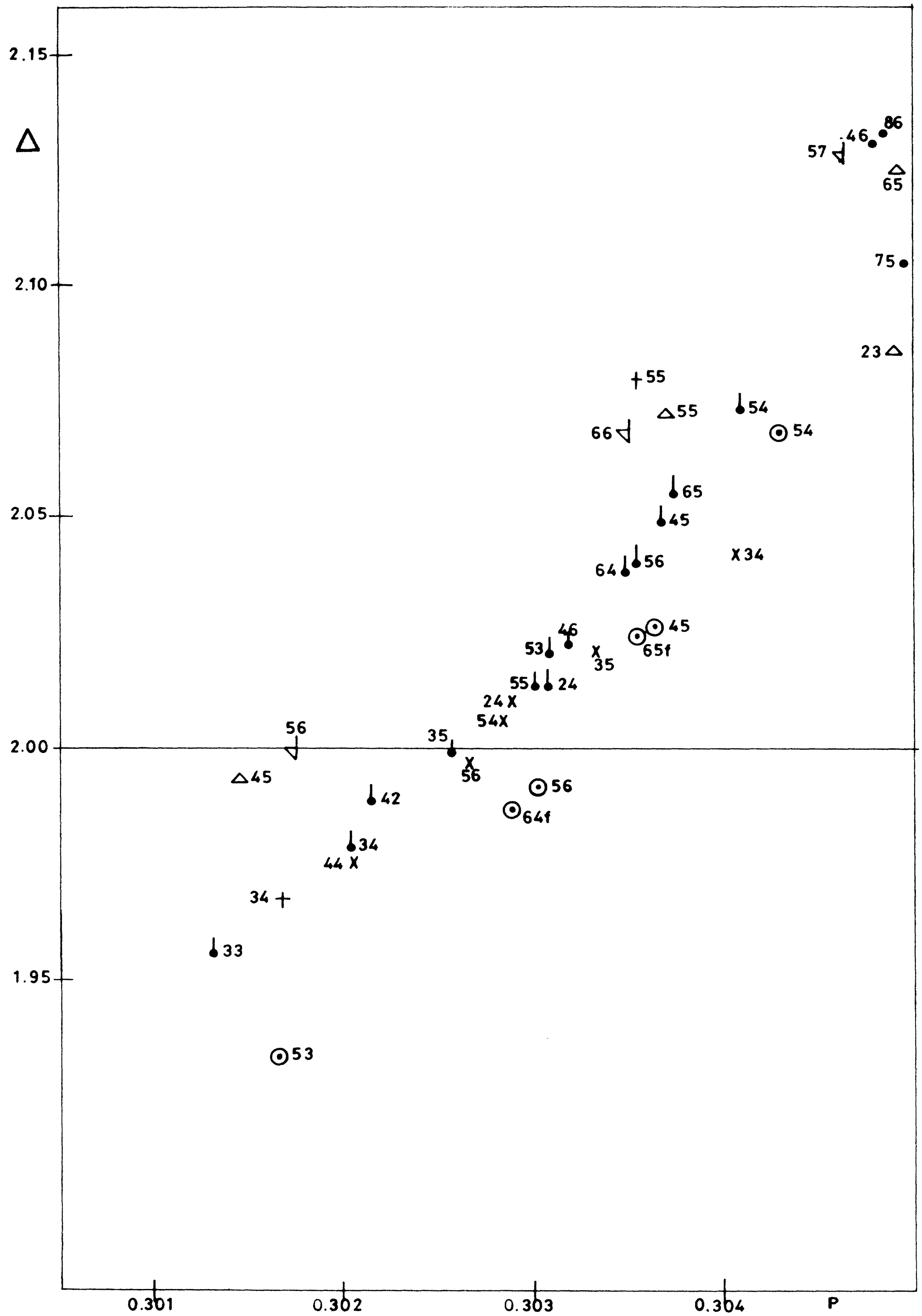

FIG. 2. Padé patterns from various $\delta(p)$ series for the 4D hypercubic site problem: ● denotes poles from series for $\langle s^2\rangle/\langle s\rangle$; Δ denotes poles from $\langle ts^2\rangle/\langle s^2\rangle$; ⊣ denotes poles from $\langle s^2\rangle/\langle s^2\rangle$; ⫰ denotes poles from $\langle st\rangle/\langle t\rangle$, ⊙ denotes poles from $\langle st\rangle/\langle s\rangle$; × denotes poles from $\langle s^2\rangle/\langle t\rangle$; † denotes poles from $\langle ts^3\rangle/\langle s^3\rangle$ (axes like in Fig. 1).

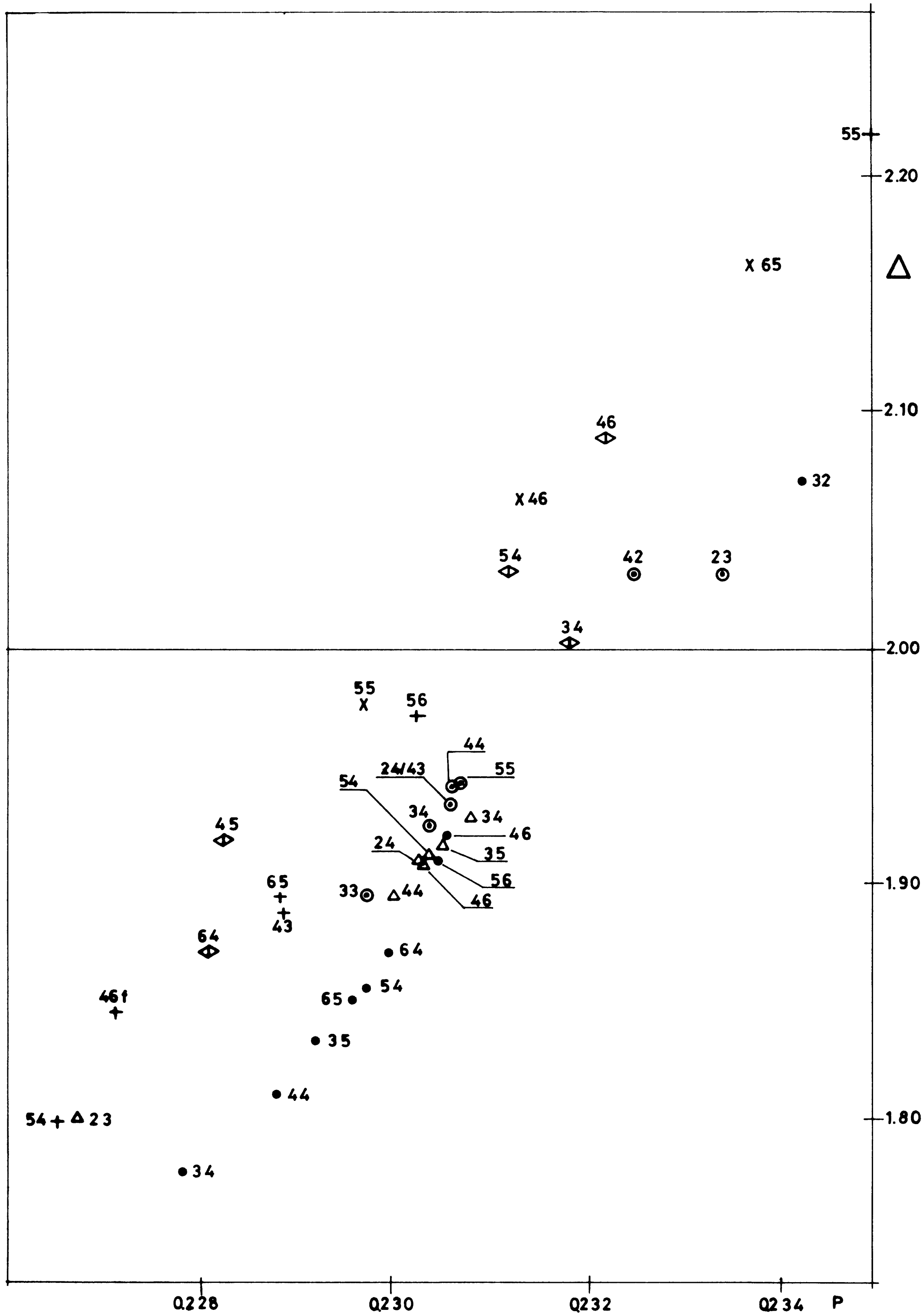

FIG. 3. Padé patterns from various $\delta(p)$ series for the 5D hypercubic site problem: † denotes poles from $\langle s^3\rangle/\langle^2\rangle$; ● denotes poles from $\langle s^2\rangle/\langle s\rangle$; ◇ denotes poles from $\langle ts^2\rangle/\langle s^3\rangle$; × denotes poles from $\langle s^4\rangle/\langle s^3\rangle$; ⊙ denotes poles from $\langle st\rangle/\langle s\rangle$; △ denotes poles from $\langle s^2\rangle/\langle t\rangle$ (axes like in Fig. 1).

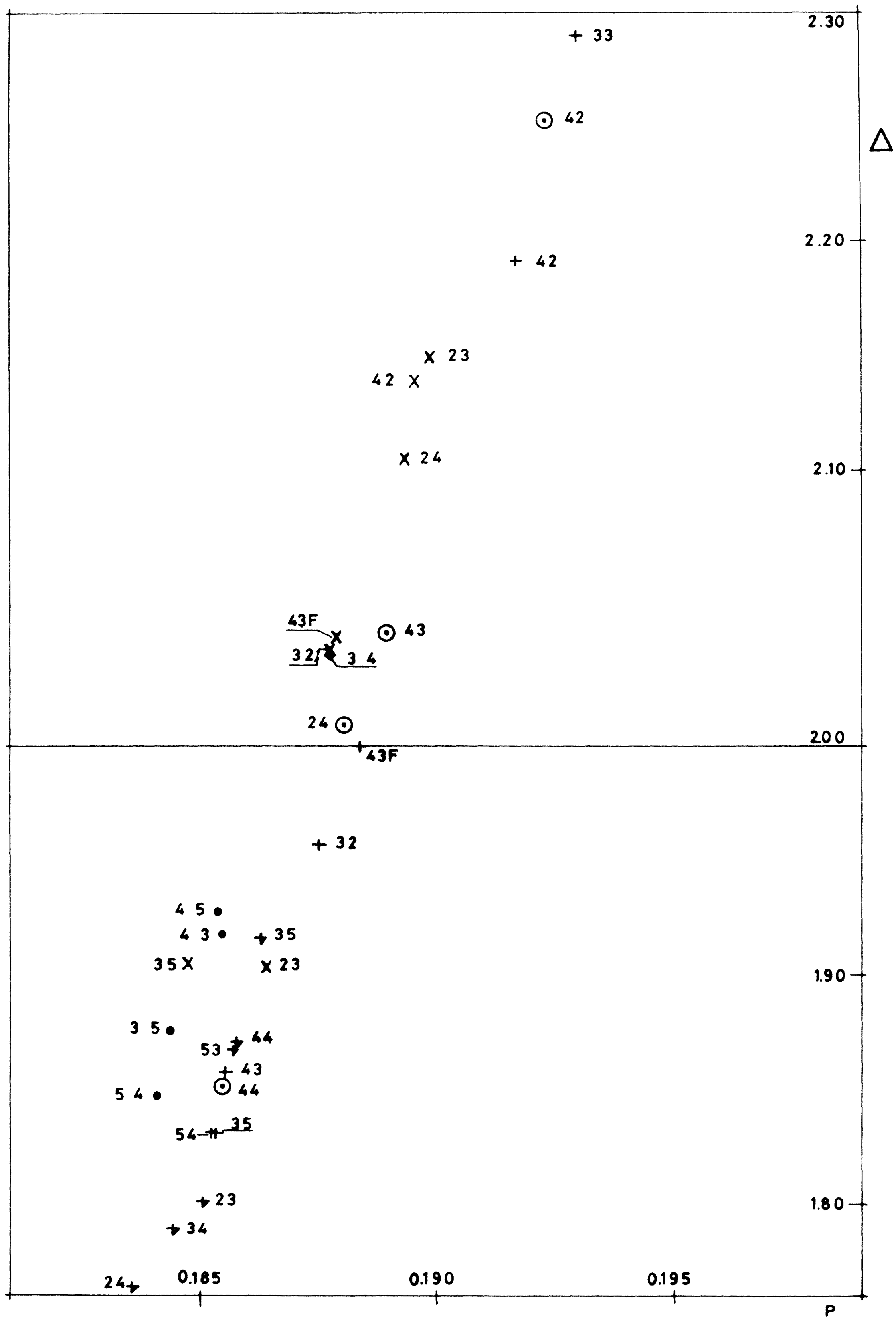

FIG. 4. Padé patterns from various $\delta(p)$ series for the 6D hypercubic problem: $+$ denotes poles from $\langle s^2\rangle/\langle s\rangle$; $\odot$ denotes poles from $\langle s^2\rangle/\langle t\rangle$; $\times$ denotes poles from $\langle ts^2\rangle/\langle s^2\rangle$; ✢ denotes poles from $\langle ts\rangle/\langle s\rangle$; ● denotes poles from $\langle s^3\rangle/\langle s^2\rangle$ (axes like in Fig. 1).

TABLE IV. δ series for the hypercubic 6D problem.

	$\langle s^2\rangle/\langle s\rangle$	$6\langle s^2\rangle/\langle t\rangle$	$\langle ts^2\rangle/6\langle s^2\rangle$	$\langle ts\rangle/6\langle s\rangle$	$\langle s^3\rangle/\langle s^2\rangle$
1	1	1	1	1	1
2	12	13	20	10	24
3	102	117.5	162.5	77.5	222
4	702	849.5	1097.5	532.5	1434
5	4622	5782.75	7895	3465	10 782
6	29 292	37 004.75	49 332.5	21 847.5	65 784
7	182 052	$232\,404.041\overline{6}$	$314\,833.3\overline{3}$	$134\,148.3\overline{3}$	433 122
8	1 094 932	$1\,412\,418.541\overline{6}$	$1\,911\,924.1\overline{6}$	$811\,284.1\overline{6}$	2 550 764
9	6 583 932	8 537 844.1875	11 749 442.5	4 830 827.5	16 294 452
10	38 625 912	50 446 016.1875	68 539 517.5	28 508 022.5	91 489 314
11	227 501 922				577 571 682

from multiple Padé patterns. Traditionally, Δ values have always been determined from γ (based on low-density series) and β estimates (requiring high-density expansions for the percolation probability). There are however several significant difficulties in the derivation of fixed perimeter polynomials required in this expansion of the percolation probability (that determines the β exponent). Low-density perimeter polynomials are a very good and far longer alternative, since they automatically give higher moments of the cluster-size distribution [$i+j>1$ in Eq. (2)], and lead to direct estimations of $\gamma+\Delta$, $\gamma+2\Delta$, etc. From these, as shown in Eqs. (4)–(6), estimation of the gap exponent can be made on a whole range of alternative $\delta(p)$ series. This direct approach keeps the uncertainty intervals within reasonable limits, furnishes acceptably precise values for the critical concentration and clarifies the approach to the upper critical dimensionality $d_c=5$, as will be shown below.

RESULTS AND DISCUSSION

For the simple-cubic site problem we have used the first fifteen polynomials; the second and third moments of the cluster-size distribution (with exponents $\gamma+\Delta$ and $\gamma+2\Delta$, respectively) are too ill-behaved to throw any light on the existing γ ranges. The present gap exponent series (given in Table I) have a significantly improved Padé pattern (Fig. 1) which agrees with the central estimates $\Delta\in|2.15-2.18|$ and $p_c=0.435$, but gives no sound reasons to decide on either the Duarte and Ruskin[6] or the Adler and Duarte[4] exponent ranges for γ.

The four-dimensional series are notoriously difficult to analyze (Table II and Fig. 2). The dispersion of pole locations does not even exclude the mean-field value $\Delta=2.0$ from an untutored estimate range. The various combinations of moments seem to exclude a critical concentration value below 0.3015. When our estimate range of $p_c=0.303\pm0.0015$ is used on the susceptibility series, the γ estimates are not inconsistent with Blease's $\gamma=1.230\pm0.005$ (probably too precise a claim) but they extend well beyond this value. No β estimates are available for this problem—no high-density expansions have been derived beyond $d=3$, and the cluster-size moments show that mean-field values are not yet reached at $d=4$, regardless of the very small deviation of the Δ range (from Fig. 2).

For $d=5$, the upper critical dimensionality is reached, although logarithmic confluent corrections must be assumed. Padé poles (Fig. 3, and from the series in Table III) cluster below the mean-field value $\Delta=2.0$, but their locations fall well within a band $p_c=0.231\pm0.003$; this is a definite improvement on the analyses of cluster size moments. Biassed estimates obtained from these tend to be slightly lower than those due to gap exponent series (the same is also valid for $d=6$). Blease's[1] value (same as $d=4$) is clearly a misprint: the apparent Padé value should be $\gamma=1.119\pm0.005$,[2] markedly different from the mean-field value.

Beyond the upper critical dimension, the shorter series for $d=6$ hypercubic site percolation exhibit no significantly improved behavior, although once again the present series have the edge on their raw moment equivalents $p_c=0.187\pm0.002$ seems a reasonable result (Table IV and Fig. 4).

It is course, the superposition of all alternative series that lends greater consistency to the estimations and avoids the cumulative effect of separate uncertainties: In the present problem the latter virtually impair any sensible indirect guesses for the gap exponent (for $d\geq3$). We summarize our studies in Table V.

TABLE V. p_c and Δ estimates.

d	Δ exponent	p_c
3	2.15–2.18 (central value)	0.435 (central value)[a]
4	$2.02^{+0.04}_{-0.02}$	0.303 ± 0.0015
5	2.0[b]	0.231 ± 0.003
6	2.0[b]	0.187 ± 0.002

[a]Uncertainty ranges as in Refs. 6 and 4.
[b]Mean-field value.

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