Fractal dimension and scaling behavior of cracks in a random medium: "Frequency-rank" distribution described by generalized random walks

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A dynamical model of cracks in a random medium, specified by the fractal dimension and conformation of the cracks, is proposed. The random medium is composed of many different types of segments. Each segment is sectioned into subsegments in which many microcracks, not directly observable, are randomly arranged. Under an applied stress, the directions of the microcracks partly line up along an easy axis of the stress, and a set of the microcracks forms connected microcracks in the subsegment, and finally the subsegment causes a fracture over the segment. The fractures are counted as cracks in the frequency-rank (FR) distribution. The FR distribution of cracks is characterized by two parameters, corresponding to the fractal dimension D and the "reciprocal temperature" β, respectively.

I. INTRODUCTION

The size distributions in random cuttings of polymers,[1] or the size distribution of cracks[2] in a random medium are interesting problems which give us a new viewpoint in the study of the dynamics of random media. Specifically, size distributions having inverse-power form have been extensively studied in terms of the fractal dimension.[3,4] Recently, from a fractal-dimensional point of view, a model of the cumulative number of cracks has been proposed by Matsushita[5] and a model of fractures has been considered by Takayasu[6] to study the total number of broken sticks. The fractal dimension is a geometrical factor specifying random irregularities and fragmentations. The analysis can be applied to the morphogenesis of a biological complex,[7] or as an interpretation for the reduction law of metabolism, in which geometrical properties are essential.[8]

In simple treatments of the FR distribution of cracks, the behavior on the log-log plot is described by linear curves specified by the fractal dimension. The actual behavior, however, does not always fit the linear curves; it deviates from the linear curves as seen in the Gutenberg-Richter law,[9] or in acoustic cumulative energy counts observed in concrete.[10] Similar deviations are observed in the so-called Zipf law and the Bradford law studied in information science.[11] Specifically, in the behavior of the Zipf law and the Bradford law, one uses an additional parameter to the index, corresponding to the fractal dimension which is introduced, so that one can reproduce the curves of the frequency-rank distribution in "information space" consisting of elements.[12] Recently, Takagi[13] reported that in fluid dynamics, configurations of two-dimensional vortices are expressed by two parameters: the fractal dimension and the temperature. Suzuki[14] has considered transient factors and Takayasu[6] has proposed a concept of differential fractals.

In this paper, we propose a dynamical model, different from the above models,[6,14] and consider the behavior of cracks. Our system is composed of many different types of segments. In the segments, a scaling property of growth for microcracks is assumed. Configurations of cracks having a length specified by the segment are determined by a probability proportional to a total number of the events. The behavior of the FR distribution is described by two parameters; the "fractal dimension" for microcracks and an additional parameter specifying conformations of possible paths for crack propagation. The possible paths are expressed by a recurrence relation of generalized random walks,[15] where jumping probabilities are specified by the fractal dimension representing an occurrence probability of the connected microcracks.

II. MODEL PROCESSES OF FRACTURES

We consider the dynamics of cracks in a random medium. The random medium is composed of many different types of segments. The segments are classified by their sizes and properties for an applied stress σ. The segment E is sectioned into subsegments; these are cubes having a length $L(E)$, see Figs. 1(a) and 1(b). Here we assume that a space of subsegments, that is a "material space," is spanned by many kinds of fundamental elements E_k having lengths l_k ($k=1,2,3,\ldots$),

$$\begin{bmatrix} E_1, E_2, \ldots, E_\nu \\ l_1, l_2, \ldots, l_\nu \end{bmatrix} . \tag{2.1}$$

Under an applied stress σ ($t>0$), some of the fundamental elements become microcracks, still not directly observable ones, and the microcracks grow into "connected microcracks" along a path, composed of the fundamental elements. When tips of the connected microcracks arrive at both ends of the boundary of the subsegment having length $L(E)$ ($\sim t_0$), the subsegment will be fractured. We regard the fracturing of the subsegment as an annihilation of the subsegment. This means that the number of subsegments decreases in the segment under the applied stress σ. Therefore, to compensate the decrease of the number of subsegments in the segment E, the length of the subsegment $L(E)$ increases and depends on "effectively" time t. For the sake of clarity, we write $L(E,t)$ instead of $L(E)$. The fracture expressed by the connected microcracks percolates further, until their tips arrive at both ends of the boundary of a segment ($\sim t_c$). The connected microcracks in the segment have various directions. In the following analysis, we regard the connected microcracks over the segment as "cracks" having minimal lengths with a direction, see Figs. 1(c) and 1(d).

The processes we consider are divided into three stages: I, first stage, $0<t<t_0$; II, intermediate stage, $t_0<t<t_c$; and III, final stage, $t>t_c$. Let M be the total number of segments in the medium composed of different types of segments, $A,B,\ldots,$

$$M=\sum_{E=A,B,C,\ldots} M_E \, , \tag{2.2}$$

where M_E is the number of segments E. Here we assume that total volume of all segments is fixed under the stress σ, and the number of subsegments in the segment E, $G(L(E,t))$, is expressed by a characteristic length $L(E,t)$. The total number of subsegments in the medium is then expressed by

$$\sum_{E=A,B,C,\ldots} M_E G(L(E,t)) \, . \tag{2.3}$$

We suppose that the number of subsegments for all segments is equal to G_0 $[=G(L(E,0))]$ before applying the stress σ ($t=0$). The stress changes some of the microcracks into connected microcracks having the respective directions. To represent the cube containing connected microcracks having a direction i, we use a new subscript i $(=1,2,3,\ldots,k)$, and rewrite the symbol of the length as $L_i(E,t)$, and let $G(L_i(E,t))$ be the number of these subsegments in the segment E. The number of the subsegments in the segment E, independent of i, is then given by

$$G(L(E,t))=\sum_{i=1} G(L_i(E,t)) \, . \tag{2.4}$$

In the subsegment, an occurrence of connected microcracks having the direction i means an onset of fractures, still not directly observable ones, and this situation is expressed by decreasing $G(L_i(E,t))$ in the course of time t.

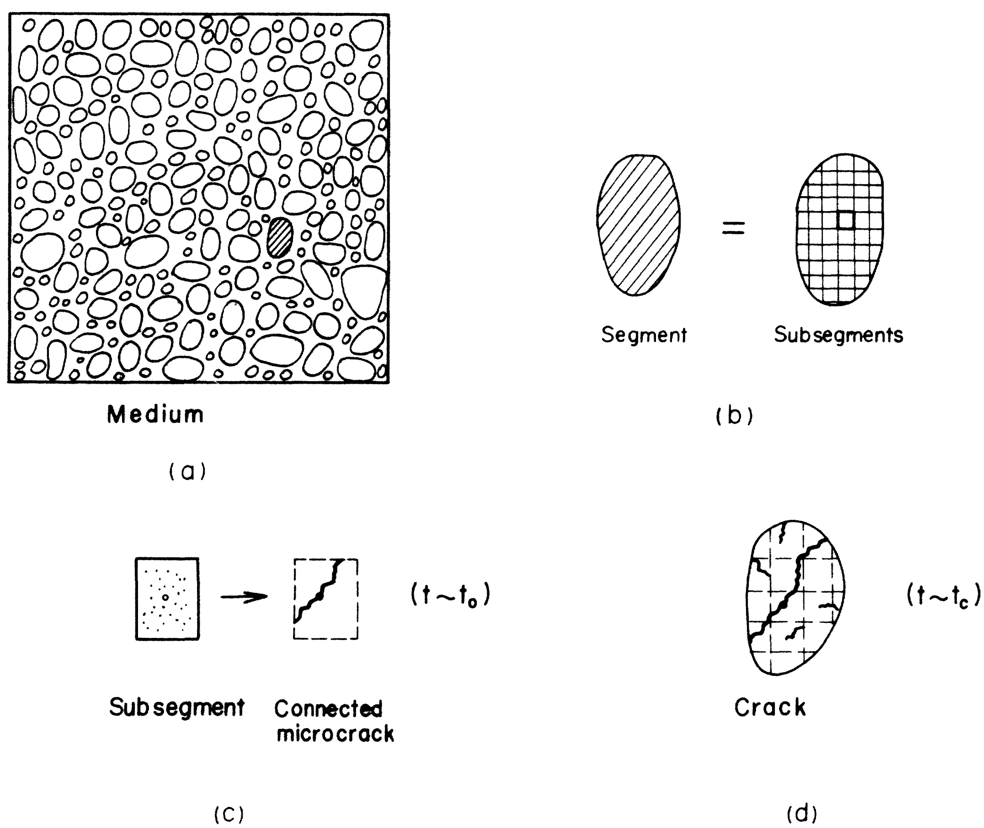

FIG. 1. Random medium and fracture processes. (a) Medium composed of many different types of segments. Background itself may be regarded as huge segment. (b) and (c) The segment is sectioned into subsegments specified by respective lengths $L(E,t)$. Tips of the connected microcracks arrive at both ends of the boundary of subsegments. (d) The segment contains many connected microcracks having various directions.

A. Initial stage ($0<t<t_0$) and intermediate stage ($t_0<t<t_c$)

The applied stress σ changes microcracks into connected microcracks having a direction i. In the initial stage, the microcracks grow into connected microcracks in the subsegment, and tips of the connected microcracks arrive at both ends of the subsegment, that is, of a cube having the length $L_i(E,t)(\sim t_0)$. We assume that a decreasing rate of $G(L(E,t))$ is proportional to an occurrence probability of the connected microcracks having direction i, $m_i(\sigma,t)$, and increasing rate of the length $L_i(E,t)$;

$$-\frac{1}{G(L_i(E,t))}\frac{d\,G(L_i(E,t))}{dt} = \eta m_i(\sigma,t)\frac{1}{L_i(E,t)}\frac{dL_i(E,t)}{dt} \quad (0<t<t_0)\,, \tag{2.5}$$

$$G(L_i(E,0))=G_0,\;\; L_i(E,0)=L_0(E)\,, \tag{2.5'}$$

where η is a positive constant and $L_0(E)$ represents the length of the cube which does not contain any connected microcrack having direction i. The symbol t_0 characterizes when tips of the connected microcrack arrive at both ends of the subsegment. Here the t dependence of the $L_i(E,t)$ is assumed to be given by

$$L_i(E,t)=A_i(E)^{1/\nu}[R(E)]^{t/\nu}\,, \tag{2.6}$$

where $A_i(E)$, ν, and $R_i(E)$ are constants (see the Appendix). Note that the t dependence of $L_i(E,t)$ determines processes in which the connected microcracks specified by $G(L_i(E,t))$ in (2.5) grow at a constant rate when $m_i(\sigma,t)$ is independent of t.

In the intermediate stage ($t_0<t<t_c$), we consider another characteristic time t_c. It characterizes when tips of the connected microcracks arrive at both ends of the boundary of the segment E in the processes. To this end, we introduce a new scale defined by

$$\tilde{L}_i(E,t)=\lambda_i(E)L_i(E,\,t-t_0) \quad (t_0\le t<t_c)\,, \tag{2.7}$$

where $\lambda_i(E)$ has a large value ($\gg 1$) valid only for $t\ge t_0$, and vanishes for $t<t_0$ [see (2.5)]. The connected microcrack over the segment is a minimal "observable" crack having a direction i, and having a length $\tilde{L}_i(E,t)$. To specify the processes in the intermediate stage ($t_0<t<t_c$), we consider a quantity representing a "free volume" for a single crack having the direction i, $\rho(\tilde{L}_i(E,t))$, defined by

$$\rho(\tilde{L}_i(E,t))=\Omega(E)/q(\tilde{L}_i(E,t)) \quad (t_0\le t<t_c)\,, \tag{2.8}$$

where $\Omega(E)$ is the volume of the segment E and $q(\tilde{L}_i(E,t))$ is the number of cracks having the direction i in the segment E. Here the free volume represents a region in which the connected microcracks grow. At $t=t_0$, $q(\tilde{L}_i(E,t))$ is assumed to be equal to q_0 $[=q(\tilde{L}_i(E,0))]$.

Assuming that the dynamics has a scaling property, we put a temporal evolution for ρ as follows:

$$-\frac{1}{\rho}\left[\frac{d\rho}{dt}\right]=\eta m_i(\sigma,t)\frac{1}{\tilde{L}_i(E,t)}\frac{d\tilde{L}_i(E,t)}{dt} \quad (t_0<t<t_c)\,, \tag{2.9}$$

$$\rho(\tilde{L}_i(E,t_0))=\rho_0,\;\; \tilde{L}_i(E,t_0)=\lambda_i(E)L_0(E)\,. \tag{2.9'}$$

Considering that $q(\tilde{L}_i(E,t))$ increases with decreasing number of subsegments, we assume a relation between a "scaled number of subsegments," $G(\tilde{L}_i(E,t))$, and the number of cracks $q(\tilde{L}_i(E,t))$ as follows:

$$q(\tilde{L}_i(E,t))G(\tilde{L}_i(E,t))=f(E)\,, \tag{2.10}$$

where $f(E)$ is an unknown positive function and $G(\tilde{L}_i(E,t))$ is a scaled function so that it has the same form as that specified in (2.5), except for the argument $L_i(E,t)$. A total free volume for cracks in the medium is then expressed by

$$V(t)=\sum_i\sum_E\rho(\tilde{L}_i(E,t))=\sum_i V_i(t)\,. \tag{2.11}$$

With the aid of $V(t)$ and $V_i(t)$, we can define an occurrence probability of the cracks having the direction i, expressed by

$$P(\tilde{L}_i(t))=\frac{V_i(t)}{V(t)} \quad (t_0<t<t_c)\,. \tag{2.12}$$

We consider the case where $m_i(\sigma,t)$ is independent of time t and direction i, writing for it $m_0(\sigma)$. After integrating the Eq. (2.5) over $(0,t)$, where $t_0<t$, we have a solution expressed by

$$G(L_i(E,t))=K_0(E,D)(L_i(E,t))^{-D} \quad [K_0(E,D)=G_0L_0(E)^D,\;\; D=\eta m_0(\sigma)]\,, \tag{2.13}$$

where D is a quantity, corresponding to the fractal dimension. For a specialized case, by integrating (2.9) over (t_0,t), where $t<t_c$, with (2.8) and (2.10), we have

$$G(\tilde{L}_i(E,t))=K_0(E,D)[\tilde{L}_i(E,t)]^{-D}\,, \tag{2.14}$$

where we have put $G(\tilde{L}_i(E,0))=K_0(E,D)[\tilde{L}_i(E,0)]^{-D}$. The result (2.14) is obtained easily by replacement of L_i in (2.13) by $\tilde{L}_i$. Specifically, when $\lambda_i(E)$ and $A_i(E)$ are E-independent quantities, we can rewrite $G(\tilde{L}_i(E,t))$ from (2.6) and (2.7) as follows:

$$G(\tilde{L}_i(E,t))=K_0(E,D)[R(E)^{-(D/\nu)(t-t_0)}]\times(\lambda_i A_i^{1/\nu})^{-D}\,, \tag{2.15}$$

[cf. (2.13)]. Under these conditions, the expression (2.12) becomes a t-independent quantity given by

$$P(\tilde{L}_i)=\frac{\tilde{L}_{i,s}^{-D}}{L_s(D)} \quad \left[\tilde{L}_{i,s}\equiv\lambda_i A_i^{1/\nu},\;\; L_s(D)\equiv\sum_{i=1}\tilde{L}_{i,s}^{-D}\right] \tag{2.16}$$

after using (2.8), (2.10), (2.11), and (2.15). An index "s" denotes that the quantity is t independent over all segments.

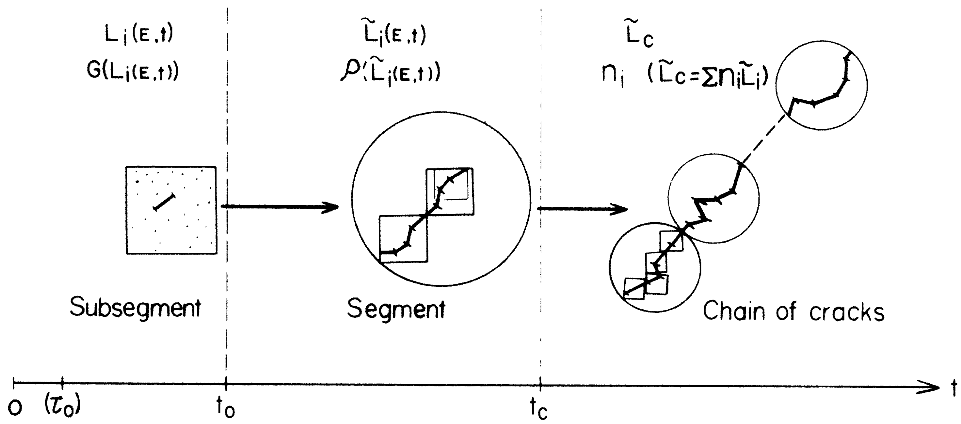

FIG. 2. A scenario for temporal evolution for microcracks in subsegments, and for connected microcracks, cracks in segments, and for chains of cracks in a medium. As to τ_0 see (A2) in the Appendix.

B. Final stage ($t_c < t$)

Following the two preceding states, we consider a final stage in which the cracks grow into chains as shown in Fig. 2. For the final stage, we assume that a temporal evolution of states for the cracks is described by a set of the numbers of cracks and the occurrence probability given by (2.16). The growth patterns of the cracks are controlled by the easiest direction of the stress. To clarify this situation, we introduce the easiest axis denoted by a new index 1, and express less easy directions by the sequence, 2,3, . . . , according to their easiness. Then we can specify a distribution of cracks in the chain patterns by the index i. For simplicity, we use the same notations representing the new index as the subscript of directions. The notation $\tilde{L}_{i,s}$ [$\propto \tilde{L}_i(E,t)$], therefore, denotes a characteristic length with an arrowhead having the index i in a given chain for the set of the cracks (see Fig. 3). Let n_i be the number of cracks, having the index i, and let $W(n_1,n_2,\ldots,n_k;\, N)$ be the probability of conformations composed of the number of cracks having new indices $n_1,n_2,\ldots,n_k$. The chain of the cracks is denoted by a set of the arrow lines heading the respective directions 1,2,3, . . . , k at the elapsed time $t = Nt_c$, where t_c is the unit time introduced in (2.7).

Before describing a general temporal evolution of states for W, we show a simple crack propagation of the chain type pattern ($n_1=2$, $n_2=1$, $n_3=2$), after $N=(t/t_c)=5$ steps. The process is expressed by a recursion relation of generalized random walks,[15]

$$W(2,1,2;5)=P(\tilde{L}_{1,s})W(1,1,2;4)+P(\tilde{L}_{2,s})W(2,0,2;4)+P(\tilde{L}_{3,s})W(2,1,1;4)\ , \qquad (2.17)$$

with the initial condition

$$W(0,0,0;0)=1\ , \qquad (2.18)$$

where $P(\tilde{L}_{i,s})$, ($i=1,2,3$), are defined by (2.16), except for the meaning of the index i. They are the transition probabilities between the states expressed by $W(2,1,2;5)$, $W(1,1,2;4)$, $W(2,0,2;4)$, and $W(2,1,1;4)$, respectively.

For more general crack propagations, the temporal evolution is described by a similar recurrence relation to (2.17) of generalized random walks,

$$W(n_1,n_2,\ldots,n_k;N)=\sum_{i=1}^{k}\sum_{\alpha=\pm 1} P_{N-1}^{\alpha}(n_i,\{n_j'\}\mid n_i-\alpha,\{n_j'\})W(n_1,n_2,\ldots,n_i-\alpha,\ldots,n_k;N)$$
$$(\{n_j'\}=n_1,n_2,n_{i-1},n_{i+1},\ldots,n_k)\ , \qquad (2.19)$$

with the initial condition

$$W(0,0,\ldots,0;0)=1 \qquad (2.20)$$

and the jumping probabilities are defined by

$$P_{N-1}^{\alpha}(n_i,\{n_j'\}\mid n_i-\alpha,\{n_j'\})=\begin{cases} P(\tilde{L}_{i,s}), & (\alpha=+1) \\ 0, & (\alpha=-1)\ . \end{cases} \qquad (2.21)$$

The initial condition (2.20) is a state representing an embryo for the crack propagations in the final stage. From (2.19) with (2.20), we get an expression for W:

$$W(n_1,n_2,\ldots,n_k;N)=\frac{N!}{n_1!n_2!\ldots n_k!}\left[\frac{\tilde{L}_{1,s}^{-D}}{L_s(D)}\right]^{n_1}\left[\frac{\tilde{L}_{2,s}^{-D}}{L_s(D)}\right]^{n_2}\times\cdots\times\left[\frac{\tilde{L}_{k,s}^{-D}}{L_s(D)}\right]^{n_k}$$
$$(\tilde{L}_{i,s})=(\lambda_i A_i)^{1/\nu},\quad L_s(D)=\sum_i(\tilde{L}_{i,s})^{-D}\ , \qquad (2.22)$$

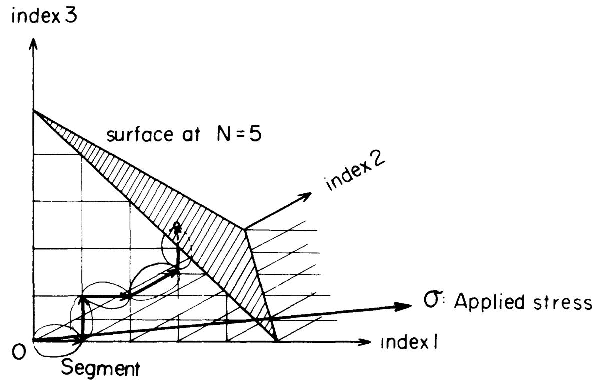

FIG. 3. A multidimensional space ($\nu=3$) spanned by the possible crack propagations specified by directions. The index 1 is the easiest axis for σ. The indices 2 and 3 denote the following easier directions for σ.

The patterns due to the cracks propagations described by the probability $W(n_1,n_2,\ldots,n_k;N)$ are illustrated by chain type diagrams in a k-dimensional space. The k component of the chain-type diagram represents the index of the cracks (see Fig. 3).

III. "FREQUENCY-RANK" DISTRIBUTION

The probability W is proportional to a total number of events, namely onsets of the cracks having the length $\tilde{L}_{i,s}$, and it gives us the possible patterns of crack propagations. Here we suppose that in the frequency-rank (FR) distribution only the processes maximizing W are observed as the cracks having the index i. The index i represents the rank measured from the easiest axis of the applied stress σ. To maximize the probability W, we consider the entropy defined by

$$S=k_0\ln W\ , \tag{3.1}$$

in place of W, where k_0 is a proportionality constant. The maximum of S is studied by varying n_i under the subsidary conditions

$$\sum_{i=1}^{k} n_i=N\ , \tag{3.2}$$

$$\sum_{i=1}^{k} n_i\tilde{L}_{i,s}=\tilde{L}_c\ , \tag{3.3}$$

where $\tilde{L}_c$ represents the total length of all chains in the crack propagations. With the aid of (2.22), the Lagrange multipliers γ for (3.2) and $-\beta$ for (3.3), we have the number $\bar{n}_i$ giving the maximum of S results in

$$\bar{n}_i=e^{\gamma}Q_ie^{-\beta\tilde{L}_{i,s}}\quad [Q_i\equiv(\tilde{L}_{i,s})^{-D}/L_s(D)]\ , \tag{3.4}$$

where we have used $\ln n!=n\ln n-n$. The parameter γ is determined by (3.2);

$$e^{\gamma}=\frac{N}{\sum_{i=1} Q_ie^{-\beta\tilde{L}_{i,s}}}\ . \tag{3.5}$$

The number $\bar{n}_i$ represents the frequency of cracks with the index i. The symbol β is the parameter determined by (3.3); it corresponds to the reciprocal temperature of the Maxwell-Boltzmann distribution for particle systems.

By taking the logarithm of (3.4), and by introducing y for $\ln\bar{n}_i$, we have

$$y(=\ln\bar{n}_i)=\ln[N/Z(\beta,D)\cdot L_s(D)]-Dx-\beta e^{x}$$
$$\left[Z(\beta,D)=\sum Q_ie^{-\beta\tilde{L}_{i,s}},\quad x=\ln\tilde{L}_{i,s}\right]\ . \tag{3.6}$$

The FR distribution expressed by (3.6) is shown in Fig. 4. This curve shows that the inclusion of the term $-\beta e^{x}$ yields a rapid decay of y. Note that D is proportional to the occurrence probability of connected microcracks, see (2.13), while β specifies conformations of a chain by the sets of $n_1,n_2,\ldots,n_k$ in the crack propagation [see (3.3)]. Here note that consecutive series of x_i $(i=1,2,\ldots,)$ in the rank

$$x_1<x_2<x_3<\ldots \tag{3.7}$$

are transformed by the relation $x_i=\ln\tilde{L}_{i,s}$ into

$$\tilde{L}_{i,s}<\tilde{L}_{2,s}<\tilde{L}_{3,s}<\ldots\ . \tag{3.8}$$

Therefore, in a sense, the length $\tilde{L}_{t,s}$ may be regarded as a variable representing energy monitored by acoustic emission. In Fig. 5 we show a curve of Gutenberg-Richter expression.[9] We can find a profile similar to that given by (3.7).

Before closing the section, let us remark on an application of the present model to Horton's law with which we can study a relationship between the streams and their drainages. To this end we reinterpret the letter of index as an order of the streams of the drainage network; let k be the order of the streams, and let L_k^+ $(=\tilde{L}_{k,s})$ be the length of the streams. If we put

$$\ln L_k^+=Ck\,(C:\ \text{constant})\ , \tag{3.9}$$

we can state that the streams having greater order are longer ones. Expressed differently, when the drainage network is regarded as "great cracks," and if we introduce notations

$$L(D)=\sum_k (L_k^+)^{-D},\quad Q_k=(L_k^+)^{-D}/L(D)$$
$$\left[Z=\sum_k Q_ke^{-\beta L_k^+}\right]\ , \tag{3.10}$$

we obtain the number of drainages, n_k, expressed by

$$\ln n_k=A-Bk-R(k)$$
$$\{A=\ln[N/Z\cdot L(D)],\quad B=DC,\quad R(k)=\beta e^{Ck}\}\ . \tag{3.11}$$

Thus we have a "Harton's law" of the drainage network described by patterns of the great cracks in the random medium.

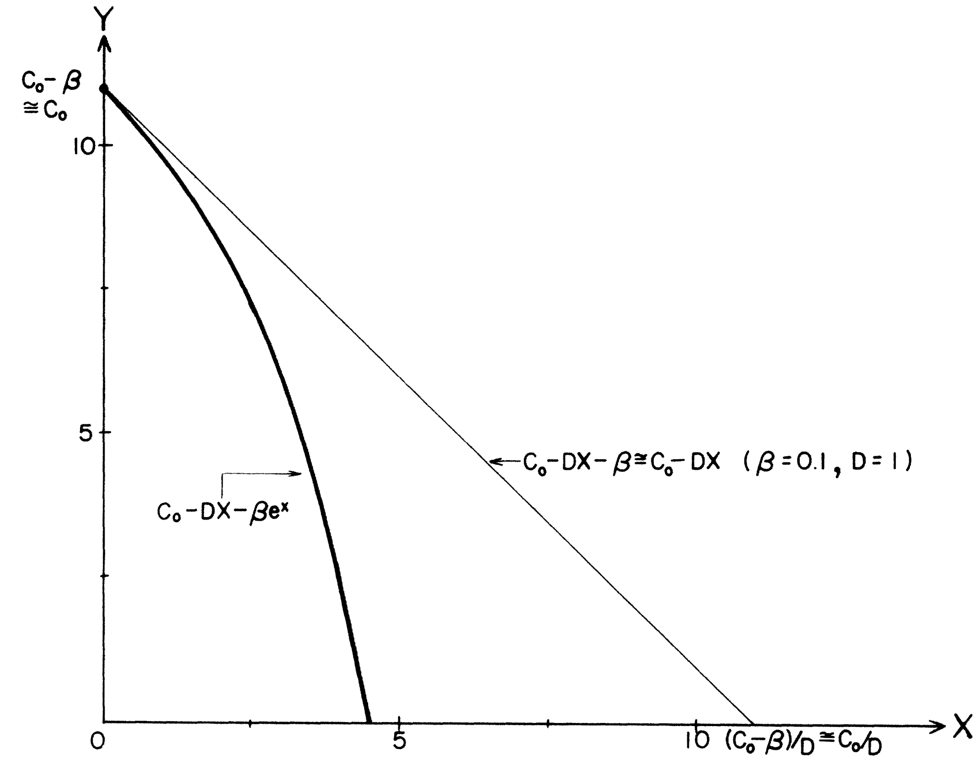

FIG. 4. The frequency-rank (FR) distribution given by (3.6). Here, for simplicity, we put $\ln[N/Z(\beta,D)\cdot L_s(D)]=c_0$, $D=1$, and $\beta=0.1$, respectively.

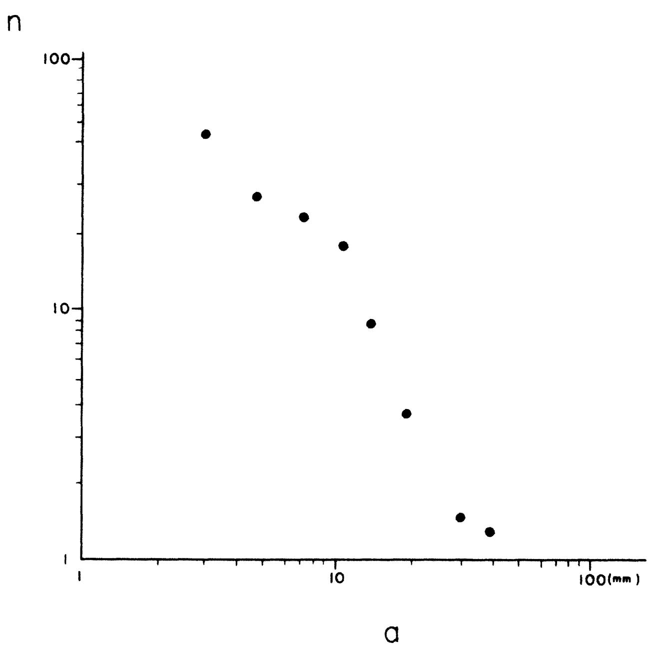

FIG. 5. A curve for Gutenberg-Richter expression observed by Mogi (Ref. 9) in pine resin. Here a is the maximum amplitude of elastic shocks and n $[=n(a)]$ a their number.

IV. CONCLUSION

A simple dynamical model of cracks in a random medium was proposed. The dynamical stages ($0<t<t_0$, $t_0<t<t_c$, and $t_c<t$) are specified by (2.5), (2.9), and (2.19), respectively. The present model yields a frequency-rank (FR) distribution for cracks expressed by their lengths. The behaviors of the FR cracks are specified by two parameters: the fractal dimension D and the additional parameter β, corresponding to the reciprocal temperature for the particle systems. The fractal dimension D is related to the occurrence probability of connected microcracks. The parameter β is one of the Lagrange multipliers for the subsidary conditions (3.2) and (3.3), and it specifies the conformation of chains in a crack propagation, expressed by the sets of the cracks having the respective indices (3.3). The index denotes the easier directions for the applied stress.

In the present model it was found that the linear curve on the log-log plot, determined by the fractal dimension, were modified by the contributions due to the additional parameter β. To set up a more elaborate model process for crack propagations, we can use the continuum limit of (2.19), which yields a higher dimensional Fokker-Planck equation,

$$\frac{\partial W(n_1,n_2,\ldots,n_k:t)}{\partial t}=\left[-\sum_i \frac{\partial}{\partial n_i}K_i^{(1)}(\tilde{L}_{i,s})+\sum_i \frac{\partial}{\partial n_i^2}K_{ii}^{(2)}(\tilde{L}_{i,s})\right]W\,, \tag{4.1}$$

and we can express the solution by the path integral representation, where

$$K_i^{(1)}=\tilde{L}_{i,s}P(\tilde{L}_{i,s})/t_c \ , \tag{4.2}$$

$$K_{ii}^{(2)}=\tilde{L}_{i,s}^2P(\tilde{L}_{i,s})/2t_c \ , \tag{4.3}$$

see Ref. 16.

It is also interesting to note that the present result (3.7) has a form similar to the size distribution of droplet obtained recently by Monte Carlo simulation.[17]

In the present model, the probability $P(\tilde{L}_{i,s})$ is independent of t [see (2.21) with (2.16)], we may regard it as a probability of a bond being occupied on the lattice system. Specifically, in the two-dimensional cases, we can interpret the result n_k as the average number of k clusters per lattice site, $\tilde{n}_k$ expressed by

$$\tilde{n}_k=\sum_e G_{ke}P^k(1-P)^e \ , \tag{4.4}$$

where e is the so-called perimeter and it denotes the number of empty neighbors of a cluster. The symbol G_{ke} represents the number of cluster configurations with size k and perimeter e.[18]

Finally let us remark that there is an interesting application of the present model to microspike movements on the neutral growth cone.[19,20] This will be given in a separate paper.

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APPENDIX: DERIVATION OF EQ. (2.6)

To get the t dependence of $L_i(E,t)$, we consider a total number of possible paths for various "virtual connected microcracks." The connected microcracks are composed of fundamental elements $E_k^{(i)}$, having the length $l_k^{(i)}$ $(k=1,2,\ldots k)$ [see (2.1)];

$$\begin{Bmatrix} E_1^{(i)},E_2^{(i)},\ldots E_\nu^{(i)} \\ l_1^{(i)},l_2^{(i)},\ldots l_\nu^{(i)} \end{Bmatrix} \ , \tag{A1}$$

where we used the symbol i to represent the direction. But in the following argument we omit the symbol i on E_k and l_k, as the need for them does not arise. The lengths l_k's are very short compared with $L_i(E,t)$. After a given time interval t possible combinations of the connected microcracks, composed of fundamental elements $\{E_k\}$, are calculated for the virtual connected microcracks between points 0 and t.

Let $N^{(i)}(E,t)$ be the total number of paths for the various virtual connected microcracks between points 0 and t. The number $N^{(i)}(E,t)$ is calculated by solving a recurrence relation:

$$N^{(i)}(E,t)=N^{(i)}(E,t-l_1\tau_0)+N^{(i)}(E,t-l_2\tau_0) + \cdots +N^{(i)}(E,t-l_\nu\tau_0) \quad (\tau_0<t<t_c) \ , \tag{A2}$$

where τ_0 is a unit of time per length of element. A formal solution of (A2) is expressed by

$$N^{(i)}(E,t)=A_i(E)[R_i(E)]^t \ , \tag{A3}$$

where R_i $[\equiv R_i(E)]$ is a constant determined by

$$1=R_i^{-l_1\tau_0}+R_i^{-l_2\tau_0}+\cdots+R_i^{-l_\nu\tau_0} \ , \tag{A4}$$

and $A_i(E)$ is an arbitrary constant. Note here that $N^{(i)}(E,t)$ is proportional to a "degree of spreads of paths" for the virtual connected microcracks between the origin and a point t in the ν-dimensional space. This fact leads us to introduce a characteristic length $L_i(E,t)$ representing the spreads of the paths in the ν-dimensional space;

$$N^{(i)}(E,t)\sim[L_i(E,t)]^\nu \quad (\tau_0<t<t_c) \ . \tag{A5}$$

Therefore, from (A3) and (A5), we have

$$L_i(E,t)\sim A_i^{1/\nu}[R_i(E)]^{t/\nu} \quad (0<t<t_c) \tag{A6}$$

for $\tau_0\to 0$. We use the characteristic length introduced above as a measure of scale for the subsegment E which contains connected microcracks having direction i for $0<t<t_0$ in (2.6), and also a measure of scale for the possible connected microcracks having the direction (or the index) i for $t_0<t<t_c$ in (2.7).

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