Classes for growth kinetics problems at low temperatures

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We consider the determination of universality classes for growth-kinetics problems. We find that many of these problems can be classified into four basic groups characterized by different low-temperature behavior. The classification is based on the study of the scaling laws obeyed by each system by means of a differential renormalization-group equation of the Callen-Symanzik type. Examples are given showing how the classification of particular growth-kinetics problems can be achieved in practice from analysis of numerical data.

I. INTRODUCTION

The growth of order in systems subjected to a quench from a temperature T_I above an ordering temperature T_c to a temperature T below T_c has been widely investigated.[1] One important question to be addressed in this context is whether classes[2] exist akin to the universality classes in critical phenomena. Current research has focused on the identification of the relevant parameters. It has been understood for some time, for example, that conversation laws can influence the growth kinetics. But the role of other possible factors such as the number of phases that can coexist at low temperature,[3] the role of continuous symmetry,[4] the lattice structure, the range of interactions, the hardness of the domain walls,[5] and other factors, is much less clear. In particular, the role that the temperature plays has not been adequately studied yet. We discuss in this paper the relationship among temperature, scaling, and renormalization-group (RG) structure for quenches to low temperatures.

It appears that growth kinetics problems can be catalogued into at least four basic families which are characterized by their different low-temperature behavior. The simplest and best understood class is where the temperature is strictly an irrelevant variable at low temperatures. Examples of this type are the spin-flip kinetic Ising (SFKI) model on most lattices (square, triangular, hypercubic, . . .)[6] or the antiferromagnetic spin-exchange kinetic Ising (SEKI) model on these lattices.[7] It is well known that the growth law is then given by the Lifshitz-Cahn-Allen (LCA) result[8] $L(t)=Bt^{1/2}$, where $L(t)$ is the characteristic domain size and B is a weakly temperature-dependent constant. The second, third, and fourth classes, which include all obvious[9] experimental systems, are less trivial. They involve kinetics which freeze[10] for quenches to zero temperature. For class 2 systems the freezing is due to local defects with activation energies independent of the domain size $L(t)$. Examples of this class include the ferromagnetic spin-exchange kinetic Ising model, the spin-flip kinetic Ising model on a hexagonal lattice, and the $q>2$ Potts model on a square lattice with a nonconserved order parameter.[3] Classes 3 and 4 include those cases where the freezing involves a collective behavior dependent on L. In particular, these classes involve activated processes where the activation energies depend on L. An apparent example of a class 3 system is the random-field Ising model,[11] while dilute ferromagnets[12] and spin glasses[13] are candidates for class 4 behavior. In previous work[10,14,15] we had included the ferromagnetic spin exchange Ising model in class 3. Recent Monte Carlo work by Amar *et al.*[16] convincingly shows that this assignment was incorrect and we demonstrate here that it belongs instead in class 2.

We do not claim that these classes exhaust all the possibilities. Indeed they are restricted to Ising-like systems and other variations may be found for these and other types of systems. However these classes do appear to be rather general.

We are interested here in how these families fit into a general theoretical structure. As we indicate below, the unifying theme is that each of these systems displays scaling and self-similar behavior *if* one uses the appropriate variables. Clarification of this point requires a generalization of the renormalization group structure introduced in earlier work.[10] One can also ask about the practical ramifications of our discussion here. It has been apparent for some time that direct brute-force measurements of growth laws are a very subtle business. There are numerous examples of both numerical simulations and experiments which obtain values for exponents n smaller than those expected from the accepted phenomenology (Lifshitz, Cahn, and Allen[8] or Lifshitz and Slyozov,[17] for example). This has been particularly true for class 2 systems where the growth kinetics is very slow for low temperatures. In the case of the Potts model on a square lattice, Grest *et al.*[3] had to go to very large systems and very long times in order to see a crossover from an effective exponent $n\approx 0.42$ to the LCA result $n=\frac{1}{2}$. Similarly, in the SEKI model on a square lattice there were numerous inconclusive studies (including our own earlier study) before the large scale study of Ref. 16. The practical problem in these class 2 systems is the existence of a slow transient associated with the zero-temperature freezing which makes the analysis of the asymptotic region difficult. In Sec. III of this paper we discuss in detail how to distinguish class 1 and class 2 systems from an analysis of numerical results. We use as examples the SFKI model on a square lattice (class 1), and on a hexagonal lattice (class 2), and the SEKI model on a square lattice (class 2). At present there is primarily theoretical evidence for class 3 and 4 systems. This is discussed further in Sec. IV.

II. RENORMALIZATION-GROUP STRUCTURE

A. Growth laws—phenomenology

As discussed in the Introduction, the low-temperature behavior is quite different among the four types of growth kinetics classes. In particular, for class 1 systems one finds that the characteristic domain size $L(t)$ is essentially independent of temperature for quenches to low temperatures and $L(t)$ is given by a power-law behavior in time:

$$L(t)=L_0+Bt^n , \tag{2.1}$$

where L_0 and B are only weakly temperature dependent. Systems in class 2 show a typical behavior

$$L(t,T)=L_0+B(t/\tau_0(T))^n , \tag{2.2}$$

where again L_0 and B are weakly temperature dependent as $T\to 0$, while τ_0 has the activated form

$$\tau_0(T)=\tau_1 e^{E_0/T} , \tag{2.3}$$

where τ_1 is weakly temperature dependent as $T\to 0$ and the activation energy E_0 is determined by local barriers. The exponent n is assumed to be independent of temperature in both cases.

The situation for classes 3 and 4 systems is more involved. The key physics in these cases is that there are barriers to growth with energies dependent on the characteristic length L:

$$E(L)=(L-L_0)^{1/m}/A , \tag{2.4}$$

where A is only weakly temperature dependent. It is then argued that the times necessary to overcome such barriers are of the activated form:

$$t\approx\tau_1 \exp[E(L)/T] . \tag{2.5}$$

If we naively invert this, we obtain the growth law:

$$L(t,T)=L_0+[AT\ln(t/\tau_1)]^m \tag{2.6}$$

and one has a logarithmic time dependence. Again the length L_0 is associated with the long-time freezing for quenches to zero temperature. The particular case $m=1$ corresponds to class 3 systems, while $m\neq 1$ corresponds to class 4 systems. The reason for this distinction will become clear below.

B. General renormalization-group analysis

Growth-kinetics problems show scaling and self-similar behavior.[18] It therefore seems attractive to develop a renormalization-group approach for such problems in order to classify the various types of growth. In previous work we have described an RG (Refs. 10 and 19) approach which works well in describing the case of a nonconserved Ising order parameter. It now appears that we must generalize our approach in order to describe a broader class of growth kinetics problems. There is a large amount of emperical data that indicates that at long times the order-parameter correlation function $C(\mathbf{x},t,T)$ will satisfy a scaling law

$$C(\mathbf{x},t,T)=F(\mathbf{x}/L) . \tag{2.7}$$

Implicit in the above expression is that $L(t,T)$ is large compared to any other length in this system. Let us assume that there exists another length[20] in the problem, $\zeta(T)$, which may be large enough to compete with $L(t,T)$ over some intermediate time regime. For temperatures below but near the critical temperature, the equilibrium correlation length $\xi(T)$ will be large and one can identify $\zeta(T)=\xi(T)$. If such a length exists, one must modify (2.7) to read

$$C(\mathbf{x},\tau,\zeta)=F(\mathbf{x}/L,\zeta/L) . \tag{2.8}$$

In writing (2.8) we have introduced a scaling time $\tau=\tau(t,T)$ which may differ from the "natural" time t. We showed in earlier work[10] that a scaling form of this type follows if the correlation functions satisfy a self-similarity relation

$$C(\mathbf{x},\tau,\zeta)=C(\mathbf{x}/b,\Delta\tau,\zeta/b) , \tag{2.9}$$

where all lengths are rescaled by a factor $b>1$ and time is rescaled by a factor $\Delta(b)$, and if the length L in (2.8) satisfies

$$L(\tau,\zeta)=bL(\Delta\tau,\zeta/b) . \tag{2.10}$$

How can this phenomenology be quantified? In Appendix A we discuss in more detail how to obtain the time rescaling factor Δ and the scaling results of the type given by (2.9) and (2.10). Here we simplify things somewhat by considering only the characteristic length $L(\tau,\zeta)$. If L is a monotonically increasing function of τ then it is sensible to *define* a quantity $\tau'(\tau,\zeta)$ such that

$$L(\tau,\zeta)=bL(\tau',\zeta/b) . \tag{2.11}$$

One can, in principle, compute L for the two temperatures T_1 and T_2, and determine b via

$$b=\zeta(T_1)/\zeta(T_2) , \tag{2.12}$$

where $T_1>T_2$, and then match $L(\tau,\zeta_1)$ and $bL(\tau',\zeta_2)$ to determine $\tau'(\tau,\zeta)$. This is what is done in standard[21] real-space renormalization-group–type calculations.

We present here a local Callen-Symanzik–type formulation which avoids many of the matching problems encountered in the method described above. The RG equation is obtained by differentiating both sides of (2.11) with respect to τ, ζ, and b. We have

$$\left[\frac{\partial L}{\partial \tau}\right]_{\zeta} = b\left[\frac{\partial L'}{\partial \tau'}\right]_{\zeta'}\left[\frac{\partial \tau'}{\partial \tau}\right]_{\zeta'} , \tag{2.13a}$$

$$\left[\frac{\partial L}{\partial \zeta}\right]_{\tau} = b\left[\frac{\partial L'}{\partial \tau'}\right]_{\zeta'}\left[\frac{\partial \tau'}{\partial \zeta}\right]_{\tau} + b\left[\frac{\partial L'}{\partial \zeta'}\right]_{\tau'}\left[\frac{\partial \zeta'}{\partial \zeta}\right]_{\tau'} , \tag{2.13b}$$

$$0 = L' + b\left[\frac{\partial L'}{\partial \tau'}\right]_{\zeta'}\left[\frac{\partial \tau'}{\partial b}\right]_{\tau,\zeta} + b\left[\frac{\partial L'}{\partial \zeta'}\right]_{\tau'}\left[\frac{\partial \zeta'}{\partial b}\right]_{\tau,\zeta} , \tag{2.13c}$$

where $L' = L(\tau', \zeta/b)$ and $\zeta' = \zeta/b$. We can use (2.13a) and (2.13b) to eliminate the partial derivatives of L' with respect to ζ' and τ' in (2.13c) to obtain

$$\left[b\frac{\partial}{\partial b} + \zeta\frac{\partial}{\partial \zeta} + D\tau\frac{\partial}{\partial \tau}\right]\tau'(\tau,\zeta,b) = 0 , \tag{2.14}$$

where

$$D = \frac{1 - n_\zeta}{n_\tau} \tag{2.15a}$$

and n_ζ and n_τ are the logarithmic derivatives

$$n_\zeta = \frac{\zeta}{L}\frac{\partial L}{\partial \zeta} , \tag{2.15b}$$

$$n_\tau = \frac{\tau}{L}\frac{\partial L}{\partial \tau} . \tag{2.15c}$$

The boundary condition associated with (2.14) is

$$\tau'(\tau,\zeta, b=1) = \tau . \tag{2.16}$$

The time-rescaling parameter Δ mentioned above is then defined by

$$\tau'(\tau,\zeta,b) = \Delta(\tau,\zeta,b)\cdot\tau . \tag{2.17}$$

Equations (2.14) and (2.16) then become

$$\left[b\frac{\partial}{\partial b} + \zeta\frac{\partial}{\partial \zeta} + D\tau\frac{\partial}{\partial t} + D\right]\Delta(\tau,\zeta,b) = 0 \tag{2.18a}$$

and

$$\Delta(\tau,\zeta, b=1) = 1 . \tag{2.18b}$$

Equations (2.18) are our fundamental RG equations. They are of the same standard RG form as found in Ref. 22. The information specific to the system is contained in $D(\tau,\zeta)$ and the problem reduces to a determination of D. As discussed in Appendix B, we find a fixed point[23,24] if $\lim_{\tau\to\infty} D(\tau,\zeta) = D(\zeta)$ and $D(\zeta) > 0$ for all ζ. Once D is known, the time-rescaling parameter Δ is obtained by solving (2.18), and the low-temperature form for $L(t)$ follows then from a solution of (2.11).

C. Class 1 systems

Class 1 systems are very simple since the natural variables (t,T) can be taken to be the scaling variables (τ,ζ) for low temperatures. In this case we expect that ζ can be chosen to be the equilibrium correlation length for all $T < T_c$ and for low temperatures $\zeta \sim \xi \sim T$. Consequently the fixed point is simple since

$$\lim_{t\to\infty} D(t,T) = D^* \tag{2.19}$$

for all $T < T_c$ and D^* is independent of T. It is shown in Appendix B that if D^* is independent of T, then

$$\Delta = b^{-D^*} \tag{2.20}$$

and (2.11) has the solution

$$L(t) = Bt^{1/D^*} . \tag{2.21}$$

It is appropriate here to briefly review the nature of the fixed point found in the discussion above. In all of the growth-kinetics problems of interest there is the critical fixed point associated with quenches to $T = T_c$. This fixed point can influence the growth behavior for intermediate times for quenches below but near T_c. However, since the critical fixed point is unstable to perturbations away from T_c, the system will eventually cross over to a low-temperature growth-kinetics fixed point.[25]

Consider a system, quenched to a final temperature T, on a long length scale l and at a long time t after the quench. Suppose we rescale distances by a factor $b > 1$, $l^{(1)} = l/b$, and times are rescaled to $t^{(1)} = \Delta(b,T,t)t$. Then, for T reasonably low, the RG flows for temperature go as $T^{(1)} = T/b$ (since the correlation length ξ is typically proportional to T for low temperatures and $\xi' = \xi/b$). Then, after several (m) iterations, l, t, and T have scaled to smaller values. If, however, the initial l and t are sufficiently large, one eventually scales to a regime where $\Delta(b, T^{(m)}, t^{(m)}) \approx \Delta(b,0,\infty) = b^{-1/n}$ plus corrections which become even smaller as one increases t and lowers T. Physical observables are invariant under the transformation $t^{(1)} = \Delta(b)t, l^{(1)} = l/b$ for those l, t, and T where $\Delta = b^{-1/n}$, and one has therefore found a fixed point. A key point here is that for class 1 systems Δ is at most a weak function of temperature for all $T < T_c$. One concludes, as far as the asymptotic growth properties are concerned, that it is advantageous to set $T = 0$ and work with quenches to zero temperature. One has a "zero-temperature fixed point." When we say that the temperature is irrelevant for class 1 kinetics, we are referring to the fact that the growth at zero temperature is just as for nonzero temperatures if one waits long enough and looks on long enough length scales.

D. Class 2 systems

The scaling form (2.11) with $\tau = t$ and $\zeta = \xi$ is not appropriate for classes 2 and 4. If we write (2.11) with $\tau = t$ and $\zeta = \xi(T)$, we make the assumption that temperature enters into the long-time and distance behavior of domain growth only through the interfacial width which is proportional to the equilibrium correlation length ξ. This is a physically relevant effect. Under renormalization of lengths, $\xi \to \xi' = \xi/b$, since $\xi = \xi(T)$ can be inverted to give $T = T(\xi)$, one can obtain the temperature recursion relation

$$T' = T'(T,b) < T , \tag{2.22}$$

which drives the system toward a zero-temperature fixed point and sharper interfaces. However, if we take $\tau = t$ and $\zeta = \xi(T)$ in (2.11), it is assumed that this is the only influence of temperature on the growth kinetics. For class 2 and 4 systems this is incorrect. As we now explain, the temperature may also enter the analysis in other ways which are not naturally included in the scaling form above.

Class 2 systems freeze when quenched to $T=0$, but show power-law growth laws for $T>0$. A possible form for the growth law is given by (2.2). In this case the activation energy E_0, in (2.3), is determined by local barriers. Since these barriers do not depend on the global size $[\sim L(t)]$ of surrounding droplets or domains, τ_0 should not rescale when one rescales lengths. Thus the temperature T appearing in $\tau_0(T)$ should not be rescaled under RG transformations. This can be seen from (2.2) for the case $t \gg \tau_0(T)$ where L_0 can be ignored. If one assumes that (2.2), (2.3), and (2.10) hold, and that temperature renormalization is given by (2.21), one can solve for Δ, with the result

$$\Delta = b^{-1/n} \exp\left[E_0 \left[\frac{1}{T'} - \frac{1}{T} \right] \right] .$$

Since $T' < T$, one has, for low enough T, that $\Delta > 1$ which is unphysical since $t' > t$ is contrary to causality for a macroscopic ordering system.

The resolution of this problem is straightforward: one must choose a new unit of time

$$\tau = t/\tau_0 \tag{2.23}$$

and rewrite (2.2) as

$$L(\tau) = L_0 + B\tau^n . \tag{2.24}$$

We can still identify $\zeta = \xi(T)$, which will not be relevant for very low temperatures. With these identifications, class 2 systems are mapped onto class 1 systems. Thus the RG structure becomes identical with that for class 1 systems when the appropriate scaling variables are used.

The difficult part of dealing with class 2 systems is the identification of $\tau_0(T)$. Suppose we are ignorant of $\tau_0(T)$ and proceed as if we had a class 1 system, what is the signature that would enable us to discover that we have a class 2 system? Consider first the relationship between

$$\tilde{D} = \frac{1 - n_T(T,t)}{n_t(T,t)} , \tag{2.25}$$

where T and t are the independent variables, and (2.15a), where $\zeta = T$ and τ are the independent variables. We have that

$$n_\tau(\zeta, \tau) = n_t(T,t) \tag{2.26}$$

but

$$n_T(T,\tau) = \frac{T}{L} \frac{\partial}{\partial T} L(T,\tau) = n_T(T,t) + n_t(T,t)\kappa , \tag{2.27}$$

where

$$\kappa = \frac{T}{\tau_0} \frac{\partial \tau_0}{\partial T} . \tag{2.28}$$

Inserting (2.26) and (2.27) in (2.25) one obtains

$$\tilde{D} = D + \kappa . \tag{2.29}$$

Suppose, for low temperatures, that τ_0 is of the form (2.3). Inserting (2.3) in (2.28) gives

$$\kappa = -E_0/T . \tag{2.30}$$

Since one expects, in the scaling regime, that $D \to D^* = 1/n$, one has, using the form (2.30) for κ, that

$$\tilde{D} = \frac{1}{n} - E_0/T \tag{2.31}$$

and for low enough temperatures $\tilde{D}$ becomes negative. This is physically unacceptable and indicates that the naive scaling analysis in terms of the natural variable t does not lead to a fixed point.

E. Class 3 systems

For class 3 and 4 systems there are barriers to growth with energies dependent on L as given by (2.4). This in turn leads to the growth law (2.6). Consider first the case $m=1$. Assuming $\zeta = AT$ for low temperatures, (2.6) can be written as

$$L(t,T) = L_0 + \zeta \ln(t/\tau_1) . \tag{2.32}$$

Inserting (2.32) into (2.10) and solving for Δ, one obtains

$$\Delta = e^{-L_0(1-b)/\zeta} . \tag{2.33}$$

This strong temperature dependence of Δ in the case of logarithmic growth was discussed in Ref. 10. From the point of view of the Callen-Symanzik formulation, (2.32) leads to

$$D(T) = \frac{L_0}{\zeta} \tag{2.34}$$

which is strongly temperature dependent as $T \to 0$. As shown in Appendix B, solving (2.18a), given (2.34), leads back to (2.33). A key characteristic difference between class 1 and 2 and class 3 systems is that D^* and Δ are strongly temperature dependent for class 3 systems but not for classes 1 or 2.

F. Class 4 systems

When $m \neq 1$, one must be more careful in choosing the scaling variables. If we define

$$\zeta = (AT)^m \tag{2.35}$$

and

$$\ln\tau = [\ln(t/\tau_1)]^m , \tag{2.36}$$

then (2.6) again has the same form as (2.32). Thus the choice of scaling variables (2.35) and (2.36) maps a class 4 system onto a class 3 system. One then finds directly that (2.33) and (2.34) follow for class 4 systems.

We can turn this analysis around and ask for the solutions of the new scaling law:

$$L(\tau, \zeta) = bL(\Delta\tau, \zeta/b) \tag{2.37}$$

with Δ given by (2.33). We easily find a general solution

$$L(\tau)=\zeta f(\tau e^{L_0/\zeta}) . \tag{2.38}$$

Note that this form differs qualitatively from that of class 2 systems where

$$L(t)=f(te^{-E_0/T}) . \tag{2.39}$$

If Eq. (2.38) is to remain valid for low temperatures and small ζ then the function f must be logarithmic for large arguments,

$$f(x)=f_0 \ln x , \tag{2.40}$$

which returns the logarithmic behavior for long times. We note that class 4 systems map onto class 3 in much the same way as class 2 maps onto class 1 by a rescaling of the variables.

G. Behavior of $\tilde{D}$ using natural variables

If we work with natural variables t and T and compute n_t, n_T, and $\tilde{D}$, what are the signatures of class 3 and 4 behavior? Assuming the forms (2.1), (2.2), and (2.6), one obtains the quantities $n_t(t,T)$, $n_T(t,T)$, and $\tilde{D}$ in terms of the natural variables t and T as shown in Table I. For class 3 systems for long times and fixed nonzero temperature, $n_t(t,T)$ vanishes as $1/\ln t$, $n_T(t,T)$ goes to 1, and $\tilde{D}$ goes to L_0/AT for long times. For class 4 systems, $\tilde{D}$ decreases as $-\ln t$ for long times. A key point is that the natural variables (t,T) are the scaling variables for class 3 systems while one must use the variables τ and ζ defined by (2.35) and (2.36) for class 4 systems.

In comparing the four classes in terms of the natural variables t and T, the quantity $\tilde{D}$ shows quite different behavior. For class 4 systems, $\tilde{D}$ is unbounded for long times. Instead $\tilde{D}$ decreases as $-\ln t$ for long times. For class 1 systems the long time limit for $\tilde{D}$ is positive and finite ($=1/n$) as $T\to 0$, while it is negative and decreasing as $T\to 0$ for class 2 systems. For class 3 systems, $\tilde{D}$ is positive and increasing as $T\to 0$.

III. MONTE CARLO STUDIES

A. Introduction to Monte Carlo studies

In this section we use direct numerical Monte Carlo studies to establish the behavior of class 1 and 2 systems defined in the last section. We study critical quenches for three different systems. The first example is the single spin-flip kinetic Ising (SFKI) model defined on a square lattice. The growth kinetics of this model has been well studied[6,10] and is understood in some detail. We demonstrate here that it falls into class 1. The second system we study is the SFKI model on a hexagonal lattice. We find that the lattice structure can affect the zero-temperature growth kinetics, although not the long-time behavior for $T>0$, and that this system belongs in class 2. The third example is the spin-exchange kinetic Ising model (SEKI) on a square lattice. In this case, we will see the role the conservation of the order parameter plays in the growth kinetics and that this system also belongs in class 2.

Let us specify our Monte Carlo dynamics in more detail. In each case we quench the system from a completely disordered state (infinite temperature) to a final temperature $T<T_c$. We assume a nearest-neighbor Ising Hamiltonian with an exchange interaction J. We shall be interested in flip or exchange operators with a flip or exchange probability of the Metropolis form:

$$W=\begin{cases} e^{-\Delta E/T}, & \Delta E>0 \\ 1, & \Delta E\le 0 , \end{cases} \tag{3.1}$$

where ΔE is the change of the energy of the system if the flip or exchange is made. We also study the Glauber form for the flip or exchange probability

$$W=\tfrac{1}{2}[1-\tanh(\Delta E/2)] . \tag{3.2}$$

This second form seems more physical in the case where ΔE is zero, but is expected to drive systems to equilibrium more slowly. We shall use the Metropolis form for all three systems and, in addition, compare the difference between Metropolis and Glauber forms in the spin exchange case.

While we could choose to study various observables and measures[26] of the domain growth, we have found it convenient to study the quantities $R_M(t,T)$ defined in Appendix A. These quantities, discussed in some detail in Ref. 10, give one a measure of the order which grows out to a length scale M. Unlike other measures, such as the $\mathbf{q}=0$ component of the structure factor or the first zero in the pair correlation function, this quantity can be used for both conserved and nonconserved order parameters. In addition, we have found, in the flip case, for example, that the fluctuations in this quantity are smaller than for the $\mathbf{q}=0$ component of the structure factor.

B. Spin-flip dynamics on a square lattice

Consider first spin-flip dynamics on a square lattice. All the computations[27] reported here were on a 80×80

TABLE I. List of the behavior of the logarithmic derivatives, n_t, n_T, $\tilde{D}$ [defined by (2.25)], and D [defined by (2.15a)] for class 1–4 systems.

	n_t	n_T	$\tilde{D}$	D
Class 1	$n(1-L_0/L)$	0	$[n(1-L_0/L)]^{-1}$	$[n(1-L_0/L)]^{-1}$
Class 2	$n(1-L_0/L)$	$-n(1-L_0/L)\kappa$	$[n(1-L_0/L)]^{-1}+\kappa$	$[n(1-L_0/L)]^{-1}$
Class 3	AT/L	$1-L_0/L$	L_0/AT	L_0/AT
Class 4	$\dfrac{mAT(L-L_0)^{(m-1)/m}}{L}$	$m(1-L_0/L)$	$\dfrac{(1-m)L+mL_0}{mAT(L-L_0)^{(m-1)/m}}$	$\dfrac{L_0}{(AT)^m}$

system. Temperatures are measured in terms of the variable $y=e^{-4K}$ where $K=J/k_BT$. We have investigated zero temperature and $y=0.01$, 0.02, and 0.03. The block correlation functions $R_M(t,T)$ were computed over the range $M=10-25$ for times from 1 to 20 Monte Carlo steps per spin (MCS) and we have averaged over at least 600 runs in the analysis. In Fig. 1 we show $M^3R_M(t,T)$ versus M for $T=0$. We see that M^3R_M can very well fit to the linear form[10]

$$M^3R_M(t,T)=\chi(t,T)M+\chi_1(t,T) , \tag{3.3}$$

with the coefficient of determination >0.9999. If the scaling holds (after some transient time), $\chi(t,T)$ must be identified with the characteristic domain size $L(t,T)$ via

$$\chi(t,T)=L^2(t,T) \tag{3.4}$$

and $\chi_1(t,T)$ has the form

$$\chi_1(t,T)=-c(t,T)L^3(t,T) , \tag{3.5}$$

where $c(t,T)$ is weakly dependent on time and temperature for long times. The above relations (3.3)–(3.5) should be valid over the regime where $L(t)<M$. Our results for χ and χ_1, determined by fitting M^3R_M over the range $10\le M\le 25$, are presented in Figs. 2 and 3 in terms of L and c defined by (3.3)–(3.5).

We see from Fig. 2 that there is a very small variation in L with temperature. In particular L is not monotonic in temperature within the error bars on the data. Thus our most consistent conclusion is that $n_T=0\pm0.1$ over the time range $t>3$ MCS. In Fig. 3 we see that $c(t)$ is a very weak function of temperature and, in agreement with scaling, approaches a long time limit ($=0.63\pm0.03$) apparently independent of temperature (within fluctuations in the data) for low temperatures. In Fig. 4 we plot $D(t,T)$ versus time and see that our results are consistent with

$$D^*=\lim_{t\to\infty} D(t,T)=2 . \tag{3.6}$$

Various fits of the data for $L(t)$ to (2.1) (three parameters fits, fits with $n=\frac{1}{2}$, and fits with $L_0=0$) lead to the conclusions that $n=\frac{1}{2}$, $L_0=0$, and $B=2.2\pm0.1$ independent of temperature.

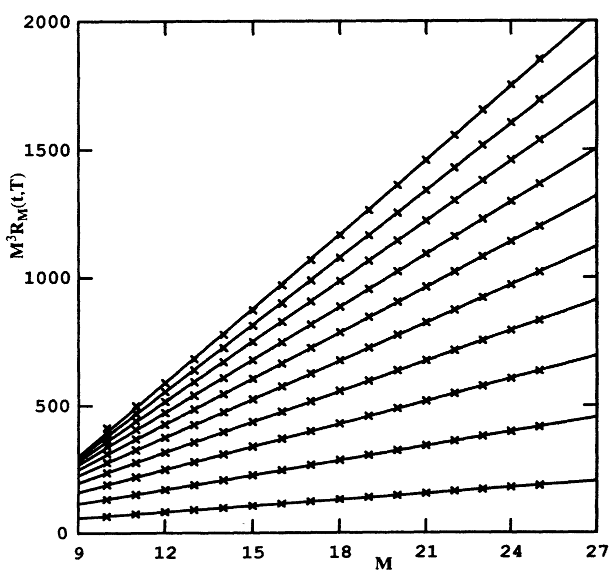

FIG. 1. $M^3R_M(T)$ vs M for quenches to final temperature $T=0$. Crosses are the data points and straight lines are the least-square linear fit. From bottom to top, the data correspond to times from $t=2$ to $t=20$ at time steps $\Delta t=2$. All times are in Monte Carlo steps per spin (MCS) and length in units of lattice spacing.

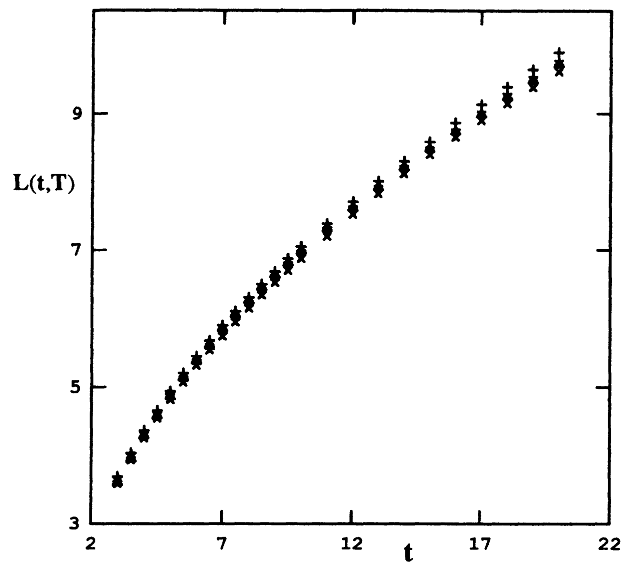

FIG. 2. $L(t,T)$ vs t for quenches to $y=0$ (pluses), 0.01 (asterisks), 0.02 (crosses), and 0.03 (triangles).

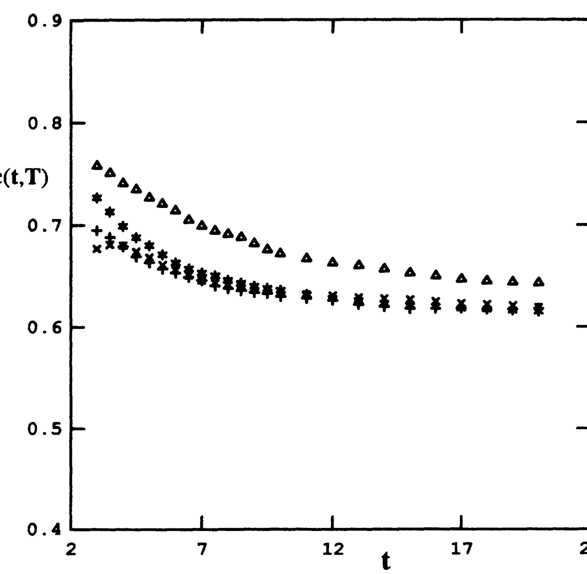

FIG. 3. $c(t,T)$ vs t for quenches to $y=0$ (pluses), 0.01 (asterisks), 0.02 (crosses), and 0.03 (triangles).

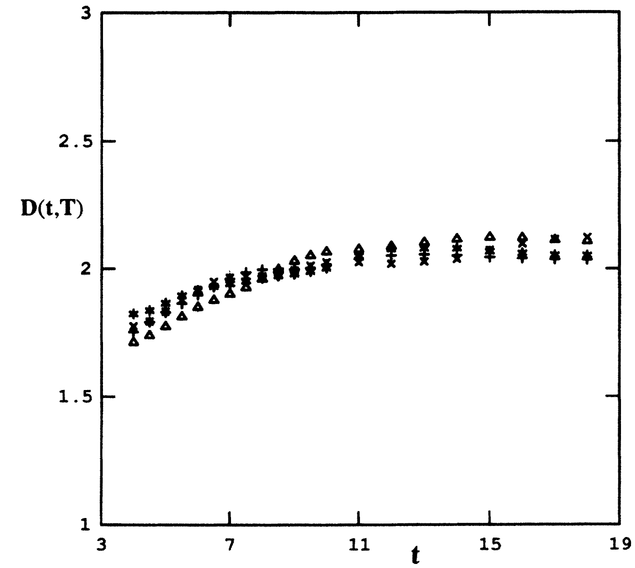

FIG. 4. $D(t,T)$ for quenches to $y=0$ (pluses), 0.01 (asterisks), 0.02 (crosses), and 0.03 (triangles).

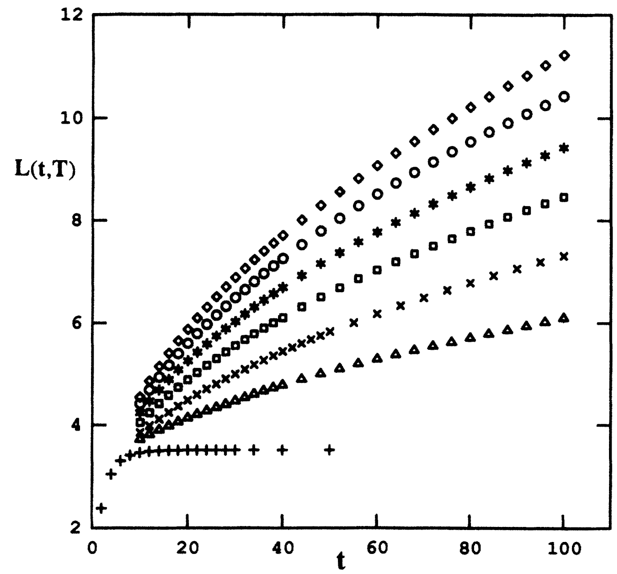

FIG. 5. $L(t,T)$ vs t for quenches (from bottom to top) $y_H=0$, 0.01, 0.02, 0.03, 0.04, 0.055, and 0.0718.

C. Spin-flip dynamics on a hexagonal lattice

The low-temperature growth kinetics of the SFKI model on a hexagonal lattice is quite different from that on a square lattice. Due to local defects, the system freezes for quenches to zero temperature, and for quenches to $T>0$, the growth shows a strong temperature dependence. However we have found that the data can still be analyzed using the relations (3.3)–(3.5). Thus, we can use the same method as in Sec. III B to extract $L(t,T)$ and $c(t,T)$.

In this case we measure temperatures using the variable $y_H=e^{-2K}$. We have runs for an 80×80 system for quenches to $y_H=0$, 0.01, 0.02, 0.03, 0.04, 0.055, and 0.0718. We have also used a 100×100 system for $y_H=0.04$, 0.055, and 0.0718 to perform a limited check for finite-size effects. The data are consistent within statistical errors. We have computed $R_M(t,T)$ over the range $M=10$–26 for times from 10 to 100 MCS and averaged over at least 500 runs. The extracted quantities $L(t,T)$ are shown in Fig. 5 for all seven values of temperature. In Fig. 6, we show $c(t,T)$ versus t for all seven values of temperature. The c's are weak functions of time and temperature and approach a common long-time, temperature-independent constant $c(\infty,T)\approx 0.64$ which is very close to the value found for the square lattice. Thus we have strong evidence for the scaling form (2.7) at low temperatures.

We now proceed to analyze the data using the methods discussed in Sec. II. Consider first quenches to zero temperature. $L(t,T=0)$ versus t is given by the lowest curve in Fig. 5. The system freezes after about 15 MCS (this is not a class 1 system). The frozen value of L, $L_0(T=0)=3.5\pm 0.1$. It is the existence of this length, which makes an analysis of the growth kinetics of this problem and classes 2–4 more generally, more difficult than class 1 problems.

For quenches to $T>0$, we plot the logarithmic time derivative n_t versus t in Fig. 7. We see for all the temperatures investigated that this quantity is a slowly increasing function of time. These data would indicate that this system is a class 2 system rather than a class 3 or 4 system (which would show n_t as a slowly decreasing function of time) if there is no longer-time cross over. As indicated in Sec. II we expect that the variation of n_t with time will be strongly influenced by L_0. If $L(t,T)$ is of the form (2.2), then we can obtain an estimate of L_0 and n by plotting[28] n_t versus $1/L$ and fitting the data to a straight line. The values of n and $L_0(T)$ are shown in Table II. We note here that it is consistent to extrapolate these

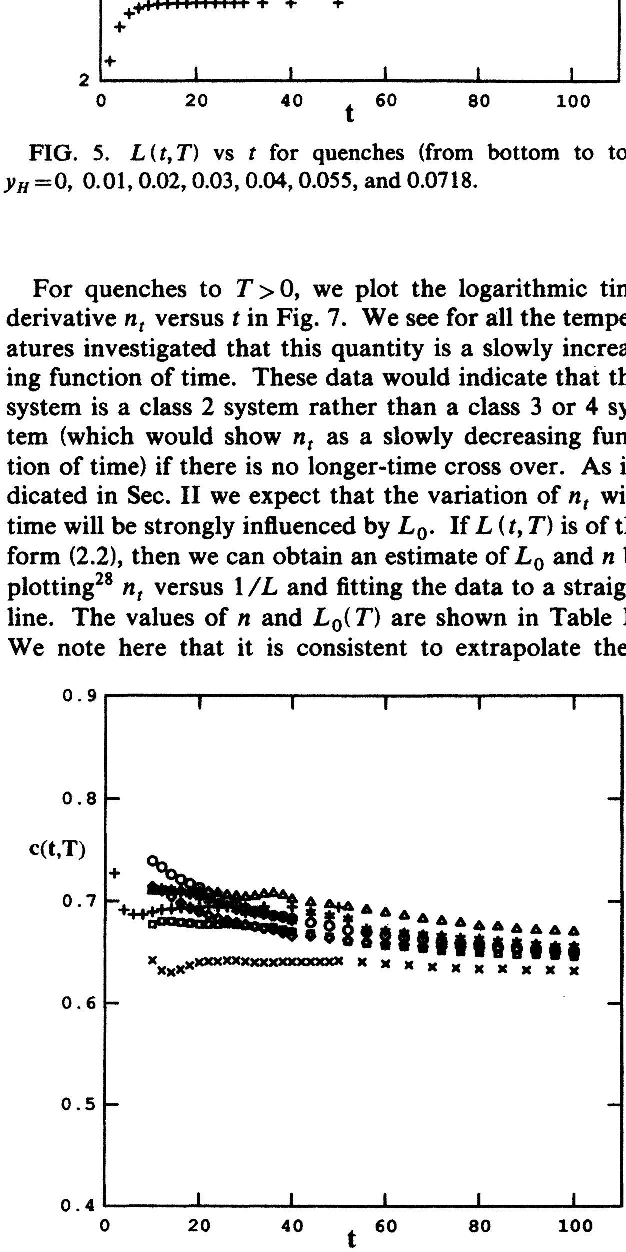

FIG. 6. $c(t,T)$ vs t for quenches to $y_H=0$ (pluses), 0.01 (triangles), 0.02 (crosses), 0.03 (squares), 0.04 (asterisks), 0.055 (circles), and 0.0718 (diamonds).

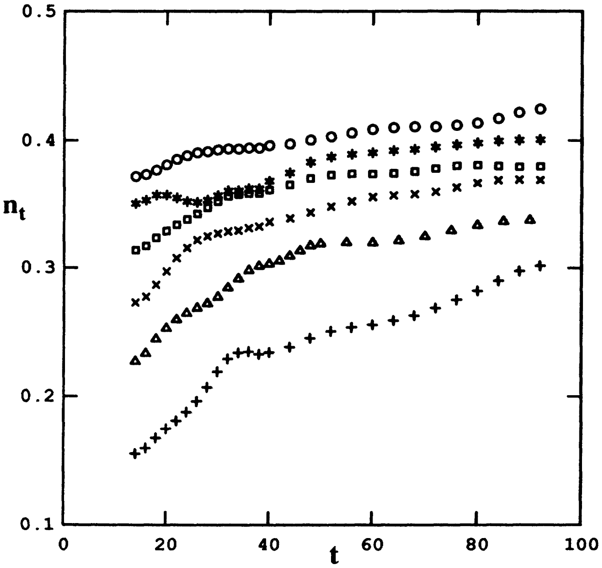

FIG. 7. n_t vs t. From bottom to top, the symbols correspond to $y_H=0.01$ to 0.0718.

values to L_0 to the frozen value obtained from the direct calculation of $T=0$.

These results indicate that we have a class 2 system and the data should be fit to the form (2.2). Our data do not extend to far enough times to obtain a reliable three parameter fit because $L(t,T)$ changes relatively little over the regime studied. Taking all of the evidence together one is led to the conclusion[29] that $n=\frac{1}{2}$. We have then fit our data for $L(t,T)$ to (2.2) with $n=\frac{1}{2}$ and normalize τ_0 such that $B=1$. The results for L_0 and $\tau_0^{-1/2}$ are shown in Table III, together with their uncertainty. We see that L_0 is a decreasing function of increasing temperature and that τ_0^{-1}, as expected for a class 2 system, vanishes as $T\rightarrow 0$. We find that a fit of the form

$$\tau_0^{-1}=ay_H(1+by_H) \tag{3.7}$$

for low temperatures, gives a good fit to the data for $\tau_0^{-1/2}$ with $a=14.5$ and $b=-1.5$. As expected, L_0 shows a weak temperature dependence and can be fit to the form $L_0(T)=3.5-2.4k_BT/J$.

Given these values for L_0 and τ_0 we can define the characteristic length

$$\tilde{L}=L-L_0 \tag{3.8}$$

which we can plot versus $\tau=t/\tau_0$ as shown in Fig. 8. We

TABLE II. Results for L_0 and n extracted from linear fits to n_t vs $1/L$ for the spin-flip hexagonal case.

y_H	L_0	n
0.01	2.8±0.3	0.55±0.05
0.02	2.2±0.3	0.50±0.05
0.03	1.9±0.3	0.48±0.05
0.04	1.5±0.3	0.46±0.05
0.055	1.2±0.3	0.46±0.05
0.0718	1.1±0.3	0.46±0.05

TABLE III. Results for L_0 and $\tau_0^{-1/2}$ obtained by fitting data for $L(t,T)$ to (2.2) with $n=\frac{1}{2}$ for the spin-flip hexagonal case.

y_H	L_0	$\tau_0^{-1/2}$
0.0	3.5±0.1	0.0
0.01	2.5±0.1	0.36±0.02
0.02	2.2±0.1	0.51±0.02
0.03	2.1±0.1	0.64±0.02
0.04	2.0±0.1	0.75±0.03
0.055	1.8±0.1	0.86±0.03
0.0718	1.6±0.1	0.96±0.03

see that all of the temperatures fall on the same curve given by

$$\tilde{L}=\tau^{1/2} \tag{3.9}$$

and shown as a continuous curve in the Fig. 8. This is confirmation that the exponent is $\frac{1}{2}$. It is clear that in this region $\tilde{L}$ shows no temperature dependence and the associated quantity given by (2.15) is simply

$$D^*=2 \tag{3.10}$$

and we have a fixed point.

In contrast, let us look at the quantity $\tilde{D}$ given by (2.25). In Fig. 9, we plot $\tilde{D}$ versus t for the four available temperatures. As expected for a class 2 system, $\tilde{D}$ becomes negative at long times for sufficiently low temperatures. This is in qualitative agreement with the asymptotic form (2.31) for class 2 systems.

D. The spin-exchange kinetic Ising model

We turn now to spin-exchange (also known as Kawasaki) dynamics. We restrict ourselves to a square

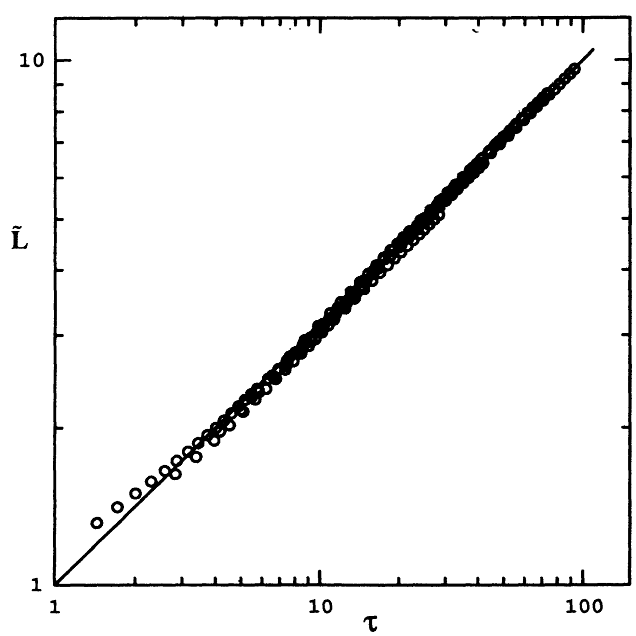

FIG. 8. ln-ln plot of $\tilde{L}$ vs τ. Circles are the data points for six different temperatures and the solid line is the function $\ln\tilde{L}=\frac{1}{2}\ln\tau$.

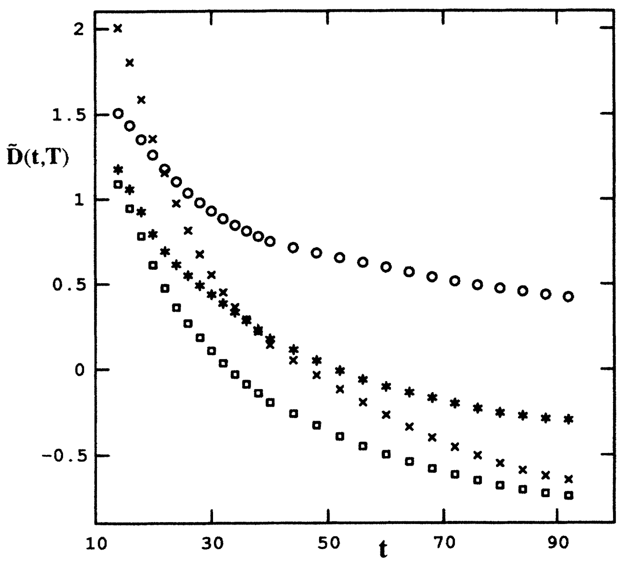

FIG. 9. $\tilde{D}(t,T)$ vs t for $y_H=0.02$ (crosses), 0.03 (squares), 0.04 (asterisks), and 0.055 (circles).

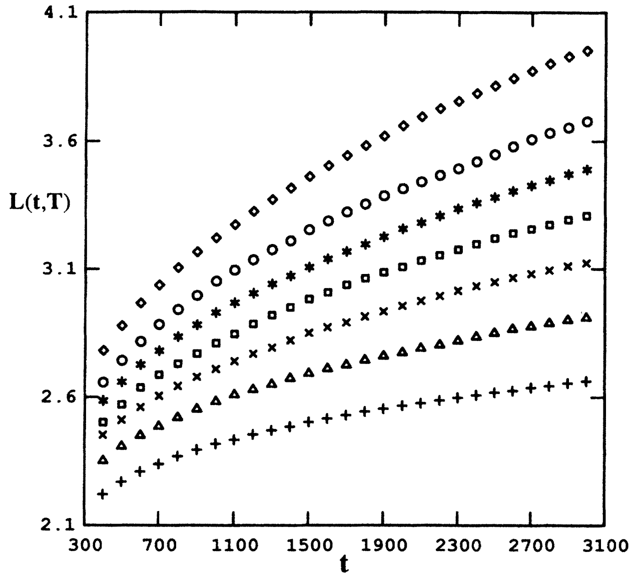

FIG. 11. $L(t,T)$ vs t (from bottom to top) quenches to $y=0.005$, 0.01, 0.02, 0.025, 0.03, and 0.04.

lattice since it is this case which has been most studied and which, one thinks, contains the basic physics. We shall concentrate on the Metropolis form of the exchange probability first and return to the Glauber form later.

All calculations reported here were carried out on a 40×40 system. However, since we restrict ourselves to block sizes of a maximum value of 19 and the largest value of $L(t,T)$ is about 4, we do not expect finite-size effects to play a role. Indeed our results, as discussed below, agree quantitatively with calculations carried out for much larger systems.

We have investigated this system for $y=e^{-4K}=0$, 0.005, 0.01, 0.015, 0.02, 0.025, 0.03, and 0.04. In Fig. 10 we show the logarithmic derivative $n_M(t,T) = -\frac{1}{3}\partial \ln R_M(t,T)/\partial \ln M$ for $y=0.02$ and several times as a function of M. We find that $n_M(t,T)=1$ to very good accuracy, which means that $M^3R_M(t,T)$ is independent of M within statistical error. This conclusion holds for all other temperatures studied and allows us to extract $L^3(t,T)$ by averaging $M^3R_M(t,T)$ over the appropriate range of M which moves slowly to larger M values as time proceeds. The extracted lengths $L(t,T)$ are shown in Fig. 11 for all seven temperatures. We can now follow the analysis in Sec. III C.

Consider first quenches to zero temperature. We show in Fig. 12 a plot of $M^3R_M(t)$ versus M for various times

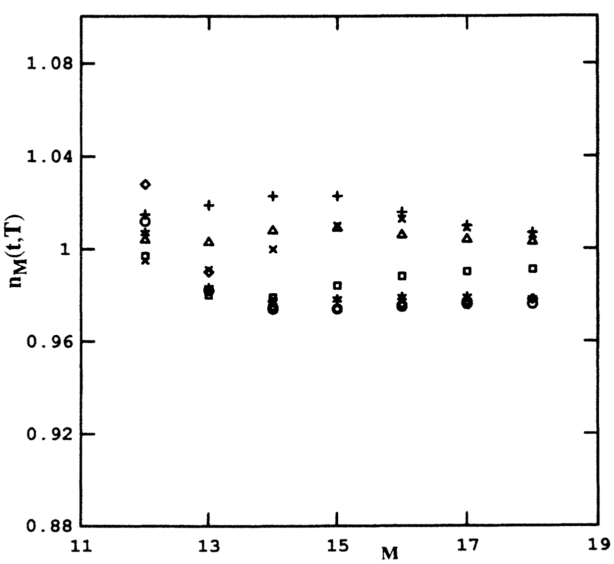

FIG. 10. $n_M(t,T)$ vs M for $y=0.02$, for times $t=600$ (pluses), 1000 (triangles), 1400 (crosses), 1800 (squares), 2200 (asterisks), 2600 (circles), and 3000 (diamonds) MCS.

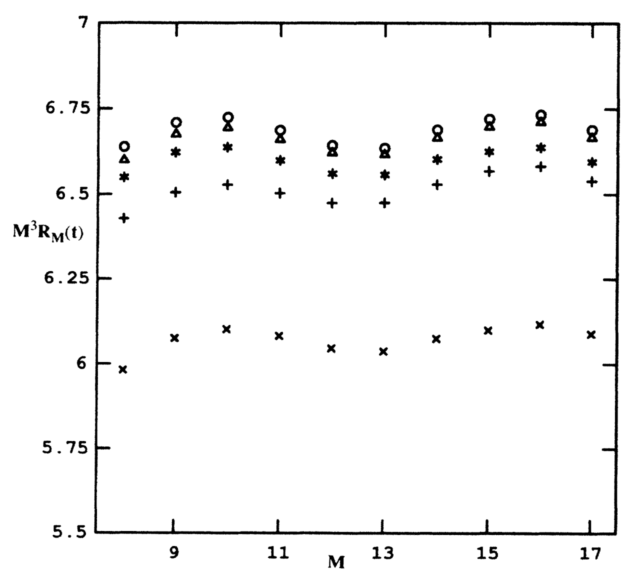

FIG. 12. $M^3R_M(t)$ vs M for quenches to zero temperature. From bottom to top, symbols correspond to $t=50$ (crosses), 100 (pluses), 150 (asterisks), 200 (triangles), and 600 (circles) MCS.

(time steps of 50 MCS from 50 to 600 MCS) after a quench from a completely disordered state. There are two important observations. The first is that the system freezes after about 200 MCS (this is not a class 1 system), and the second is that $M^3R_M(t,T)$ is roughly independent of M over the range of M investigated. Averaging M^3R_M over M for 7 to 17, we find that the frozen value of L is $L_0(0)=1.89\pm0.02$.

The logarithmic time derivative n_t is plotted versus t in Fig. 13. We see that for all the temperatures investigated, this quantity is a slowly increasing function of time. This would indicate that this system is again a class 2 system rather than a class 3 system. As in the hexagonal spin-flip case, if $L(t)$ is of the form (2.2), then we obtain an estimate of L_0 and n by fitting n_t to $n(1-L_0/L)$. The fitted values of n and L_0 are listed in Table IV. Because of the local fluctuations of the data, the linear extrapolations are not very accurate. A similar analysis for the SEKI model on a square lattice by Huse[24] and Amar *et al.*[16] also shows some considerable scatter in the data, but one is led to the conclusion that $n=\frac{1}{3}$. We have then fit our data for $L(t,T)$ to (2.2) with $n=\frac{1}{3}$ and normalized τ_0 such that $B=1$. We give in Table V the values we have extracted for L_0 and $\tau_0^{-1/3}$ using this method. We see that L_0 is again a decreasing function of increasing temperature and that τ_0^{-1}, as expected for a class 2 system, vanishes as $T\to0$.

If the Lifshitz-Slyozov theory[17] is correct, then $\tau_0^{-1}\sim D_s$ where D_s is the spin diffusion coefficient. However, it is expected that $D_s\to$const as $T\to0$ which does not agree with the freezing of the system in the current case. Huse[24] has suggested that $\tau_0^{-1}\sim\lambda$ the spin conductivity. This quantity goes[16] as y^2 for low temperatures. We find that the low-temperature form

$$\tau_0^{-1}=ay^2(1+by) \tag{3.11}$$

gives a good fit to the data for $\tau_0^{-1/3}$ with $a=4.2$ and

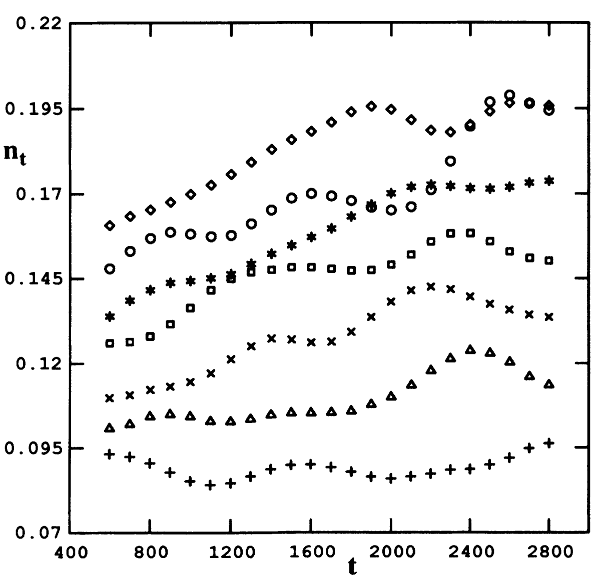

FIG. 13. n_t vs t for $y=0.005$ to 0.04. Symbols are the same as those in Fig. 11.

TABLE IV. Results for L_0 and n extracted from linear fits to n_t vs $1/L$ for the SEKI model.

y	L_0	n
0.005	0.3±0.8	0.1±0.1
0.010	1.5±0.3	0.23±0.06
0.015	1.7±0.2	0.30±0.04
0.02	1.5±0.2	0.28±0.04
0.025	1.7±0.2	0.34±0.04
0.03	1.6±0.2	0.33±0.04
0.04	1.5±0.2	0.32±0.04

$b=-7.5$. As expected, L_0 shows a weak temperature dependence and can be fit to the form $L_0=1.9-9.2y$.

Given these values for L_0 and τ_0 we can define the characteristic length

$$\tilde{L}=L-L_0 \tag{3.12}$$

which we can plot versus $\tau=t/\tau_0$ as shown in Fig. 14. We see that for large enough τ all of the temperatures fall on the curve given by

$$\tilde{L}=\tau^{1/3} \tag{3.13}$$

and shown as a continuous curve in the Fig. 14. With $\tilde{L}$ given by (3.13),

$$D^*=3 \tag{3.14}$$

and we have a fixed point. The quantity $\tilde{D}$ given by (2.25), shown in Fig. 15, decreases with time. However, the growth is very slow since τ_0 is very large and one must go to longer times to see $\tilde{D}$ become negative.

Note that the temperature dependence of L_0 found here does not agree with that predicted by Huse[24] from his phenomenological extension of the Lifshitz-Slyozov theory. He suggests that L_0 blows up exponentially as $T\to0$. We find that L_0 does increase as $T\to0$, but it saturates at the value controlled by the zero-temperature freezing. For the two temperatures studied by Amar *et al.*,[16] we obtain a ratio $L_0(0.3T_c)/L_0(0.5T_c)=1.15$ in very good agreement with their result of 1.18 and in contrast with the result of 5.33 proposed by Huse.

We have also studied[30] the Glauber form for the exchange probability for $T=0$ and 4 nonzero temperatures. The major difference between the Glauber and Metropolis operators is that the former drives the system to equi-

TABLE V. Results for L_0 and $\tau_0^{-1/3}$ obtained by fitting data for $L(t,T)$ to (2.2) with $n=\frac{1}{3}$ for the SEKI model.

y	L_0	$\tau_0^{-1/3}$
0.0	1.89±0.02	0.0
0.005	1.87±0.05	0.055±0.003
0.01	1.84±0.05	0.074±0.003
0.015	1.75±0.05	0.095±0.003
0.02	1.70±0.05	0.112±0.003
0.025	1.64±0.05	0.128±0.004
0.03	1.62±0.05	0.142±0.004
0.04	1.56±0.05	0.167±0.004

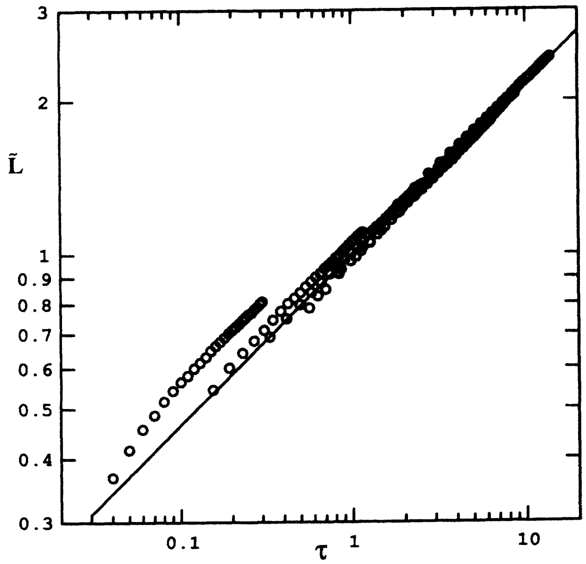

FIG. 14. ln-ln plot of $\tilde{L}$ vs τ. Circles are the data points for seven different temperatures and the solid line is the function $\ln\tilde{L}=\frac{1}{3}\ln\tau$.

librium more slowly since configurations with $\Delta E=0$ exchange with 50% probability, not 100% as for the Metropolis form. For quenches to $T=0$, the freezing time in the Glauber case is about 250 MCS, compared to about 200 MCS for the Metropolis case. The freezing value of the typical domain size in the Glauber case is about 2% smaller than that in the Metropolis case. We have carried out for the Glauber case an analysis similar to that carried out for the Metropolis algorithm. The data can still be fit to the form (2.2). The parameter τ_0^{-1} in the Gauber case is, however, smaller than that in the Metropolis case. The quantity τ_0 can again be fit to the form $\tau_0^{-1}=ay^2(1+by)$ with $a=3.4$ and $b=-11.7$. At the same low temperature, τ_0 in the Glauber case is 1.2

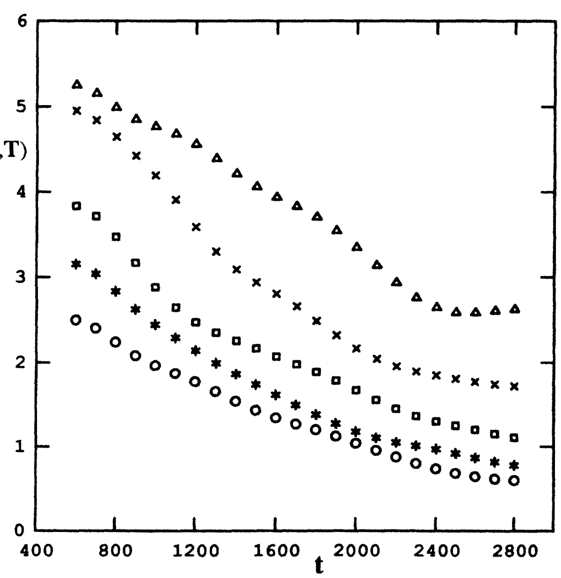

FIG. 15. $\tilde{D}(t,T)$ vs (from top to bottom) $y=0.01$, 0.015, 0.02, 0.025, and 0.03.

times of that in the Metropolis case. This means that the growth processes are much slower in the Glauber than in the Metropolis case. This is why the Glauber case is more difficult to analyze.

IV. CLASS 3 AND 4 SYSTEMS

In this section we briefly discuss the existing evidence indicating possible physical realizations of class 3 or 4 systems. All of these systems are disordered Ising-like systems with impurities providing a pinning mechanism responsible for the zero-temperature freezing. The key question in each case is whether the pinning impurities provide an $L(t,T)$ dependent activation energy barrier.

Our first example is the random-field Ising model (RFIM). A fairly recent review is given in Ref. 11. It is indicated there that there is theoretical agreement that the dynamics is governed by the presence of activated barriers and that for $2<d<5$ the activation energy is given by[31] $E\approx h^2L(t)/J$, where h is the strength of the random fields and J the strength of the coupling. This expression is of the form (2.4) with $m=1$. Such an activation barrier will lead to a logarithmic behavior for $L(t,T)$. The numerical support for this logarithmic law has recently been reviewed and investigated by Anderson.[32] He finds "strong support" given to the "Villain-Grinstein-Fernandez theory[31] of logarithmic growth." The experimental situation remains extremely controversial.[33,34]

One of the important aspects of class 3 and 4 systems is the existence of a new control parameter, governing the degree of disorder, which can be varied and which can lead to very large values for L_0 and $\zeta=AT$ in (2.6). In the case of the RFIM, theory predicts that both L_0 (Ref. 35) and A (Ref. 31) go as h^{-2} in three dimensions. Thus by making h sufficiently small one can make L_0 arbitrarily large. For very large L_0, one expects an early-time growth governed by the Lifshitz-Cahn-Allen law $L\sim t^{1/2}$ until $L\sim L_0$.

The second example is the Ising model with random quenched impurities. Theoretical work by Huse and Henley[12] in this case indicates that the kinetics is dominated by the presence of energy barriers which impede the movement of domain walls by continuous deformation. These activation energies are of the form (2.4). A first-order ϵ expansion gives $m=1.82$ for $d=3$, and numerical results give $m=4$ for $d=2$. The long-time simulation results of Grest and Srolovitz[36] show that the system freezes at zero temperature with a value of L_0 inversely proportional to the square root of the impurity concentration (playing the same role as h in the random-field problem). For quenches to $T>0$, their work suggests a logarithmic law, but they are unable to determine the power m.

Our next example is the short-range spin glass. There has been a recent proposal by Fisher and Huse[13] that in these systems the low-frequency dynamics is dominated by large-droplet excitations and that the formation of these is of the form (2.4) with $\theta\leq m\leq d-1$ where θ is estimated to be 0.2 at $d=3$.

V. CONCLUSION

The purpose of this paper has been to develop a general framework within which one can understand the scaling and self-similarity of growth kinetics problems. Two key elements in this development have been the identification of the appropriate scaling variables and the existence of the slow transient L_0 for classes 2–4. Both of these points revolve around the role of temperature in growth-kinetics problems and whether it is a relevant or irrelevant variable. For class 1 systems temperature is strictly irrelevant with regard to the longest time behavior. For class 2 systems part of the temperature is relevant in that one must choose a temperature dependent time scale if one is to obtain the appropriate scaling variables. For class 3 systems the time and temperature are relevant variables, while for class 4 systems one must introduce modified variables ζ and τ defined by Eqs. (2.35) and (2.36) in order to find a fixed point.

The situation for class 1 and 2 systems seems relatively clear since we have several examples of each which behave as one would expect. It would also be interesting to investigate systems with more internal degrees of freedom to understand if they fit naturally into the development outlined here. We intend to extend this analysis to systems not described by kinetic Ising model dynamics. In particular we have recently studied Langevin models with both conserved and nonconserved dynamics using both numerical and analytical methods.[37] A careful discussion of the temperature dependence of the growth kinetics of these systems at low temperatures is underway.

The situation with respect to class 3 and 4 systems is much less well established. The numerical and experimental problems are severe in both because L_0 can be very large for systems with very weak disorder and one must wait a time $\sim L_0^2$ before one can obtain characteristic lengths $L \sim L_0$. Only for subsequent times can one probe the logarithmic behavior. This makes it extremely difficult to carry out any asymptotic analysis unless one can arrange to make L_0 reasonably small.

ACKNOWLEDGMENTS

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APPENDIX A

In this appendix we present the definition of the correlation functions that we use in the numerical calculations of Sec. III.

In a hypercubic lattice of the size N^d let us consider[10] a block containing M^d sites ($M < N$). Let us denote by $\sigma(\mathbf{R})$ the Ising spin under consideration at site $\mathbf{R}$. Consider the block magnetization

$$m_M = M^{-d} \sum_{\mathbf{R} \in \text{block}} \sigma(\mathbf{R}) , \qquad \text{(A1)}$$

where the sum extends over the block. We define the block correlation function $R_M(t)$ as

$$R_M(t,T) = [\langle m_M^2 \rangle_t - \langle m_M^2 \rangle_0]/A , \qquad \text{(A2)}$$

where A is a suitable normalization factor. It is often convenient to choose A such that $R_M(t,T) \to 1$ for $T \to \infty$.

In the rescaling regime we must have, using the scaling variables introduced in Sec. II:

$$R_M(\tau,\zeta) = f(L/M, \zeta/M) , \qquad \text{(A3)}$$

where f is a scaling function, L a characteristic domain size, and ζ is a temperature-dependent length. The scaling regime at a given time will include, in general, a wide band of M values $L < M < N$. The function L is obtained by studying the M dependence of R_M as is discussed in Sec. III.

The renormalization-group equations can be formulated in terms of $R_M(\tau,\zeta)$. If we rescale the lengths by a factor $b > 1$ and introduce $M' = M/b$, $\zeta' = \zeta/b$, then we can define $\tau'(M,M',\tau,\zeta)$ via

$$R_M(\tau,\zeta) = R_{M'}(\tau',\zeta') . \qquad \text{(A4)}$$

The RG equations in the scaling regime are derived by differentiating both sides of (A4) with respect to M, τ, ζ, and b, and then eliminating the partial derivatives of $R_{M'}(\tau',\zeta')$. The procedure is analogous to that in deriving (2.14). We obtain, after some algebra,

$$\left[M\frac{\partial}{\partial M} + b\frac{\partial}{\partial b} + \zeta\frac{\partial}{\partial \zeta} + D\tau\frac{\partial}{\partial \tau} \right] \tau'(M,b,\tau,\zeta) = 0 , \qquad \text{(A5)}$$

where

$$D \equiv -\frac{1}{\tau(\partial/\partial\tau) R_M(\tau,\zeta)} \left[M\frac{\partial}{\partial M} + \zeta\frac{\partial}{\partial \zeta} \right] R_M(\tau,\zeta) . \qquad \text{(A6)}$$

Introducing Δ in the usual way (see Sec. II), we can transform (A5) into

$$\left[M\frac{\partial}{\partial M} + b\frac{\partial}{\partial b} + \zeta\frac{\partial}{\partial \zeta} + D\tau\frac{\partial}{\partial \tau} + D \right] \Delta = 0 \qquad \text{(A7)}$$

with $\Delta(M, b=1, \tau, \zeta) = 1$. In the scaling regime where (A3) holds, D is independent of M and (A6) reduces to (2.15a). Then Δ is also independent of M and (A7) reduces to (2.18).

APPENDIX B

We consider in this appendix the solution of (2.18), which can be rewritten as

$$\sum_{i=1}^{3} a_i(\mathbf{x}) \frac{\partial \psi}{\partial x_i} = a_0(\mathbf{x}) \qquad \text{(B1a)}$$

and

$$\psi(0,x_2,x_3) = 0 , \qquad \text{(B1b)}$$

where $\mathbf{x}$ represents the vector (x_1,x_2,x_3) with $x_1 \equiv \ln b$,

$x_2 \equiv \ln\zeta$, $x_3 \equiv \ln\tau$, $\psi \equiv \ln\Delta$, and $a_1 = a_2 = 1$, $a_3 = D(x_2, x_3) = -a_0$. Equations (B1) can now be solved by the method of characteristics. Let us introduce functions of a scalar parameter s such that

$$\frac{dx_i(s)}{ds} = a_i(\mathbf{x}(s)) \ , \tag{B2a}$$

$$x_i(0) = x_i \ . \tag{B2b}$$

Solving (B1) and (B2), we have the final solution

$$\psi(\mathbf{x}) = \int_0^{-x_1} ds D(x_2(s), x_3(s)) \ . \tag{B3}$$

When D is independent of τ and $\zeta (D = D^*)$ the integral in (B3) is trivial and we recover (2.20). In order to have a fixed point it is required that $D^* > 0$, otherwise we would have $\Delta > 1$ which is physically unacceptable.

Another case of interest is when D depends on ζ only. One then has, after some algebra,

$$\Delta(\zeta) = \exp\left[-\int_{\zeta/b}^{\zeta} \frac{d\zeta'}{\zeta'} D(\zeta')\right] \ . \tag{B4}$$

Consider now a ζ dependence of the form [see, for example, (2.34)]

$$D = \beta + L_0/\zeta \ . \tag{B5}$$

Evaluation of the integral in (B4) then yields (2.33):

$$\Delta = b^{-\beta} \exp\left[-\frac{L_0(b-1)}{\zeta}\right] \ . \tag{B6}$$

For $b = 1 + \epsilon$ and small ϵ one has from (B6) that

$$\Delta = 1 - \epsilon D(\zeta) + O(\epsilon^2) \ . \tag{B7}$$

Thus if $D(\zeta)$ were negative for any ζ, one would have, for small enough ϵ, $\Delta > 1$ which is unphysical.

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[2]The existence of such classes has been implicitly assumed by a number of workers.

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