Real-space renormalization-group investigation of the randomly dilute q-state Potts model

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The randomly bond-diluted two-dimensional nearest-neighbor q-state Potts ferromagnet and antiferromagnet on the square lattice is studied by renormalization-group methods based on the Migdal-Kadanoff approximate recursion relations. In the bond-diluted ferromagnet Potts model, differential recursion relations yield a phase diagram which is in quantitative agreement with all known results for $q \leq 4$. In the bond-diluted antiferromagnet Potts model, the phase diagram obtained shows that the threshold p^* at which the critical temperature goes to zero depends upon q.

I. INTRODUCTION

During the last few years, considerable theoretical effort has been dedicated to random magnetism, especially in the Ising, Heisenberg, and Potts models (for an excellent review of the latter see Ref. 1). Because of its richness, the Potts model has recently received special attention, in particular the quenched bond-diluted q-state Potts ferromagnet which has been studied within effective-field,[2,3] duality-based,[4-8] and real-space renormalization-group (RG) (Refs. 9-11) approaches. The quenched bond-diluted q-state Potts antiferromagnet model has not been studied as extensively as the ferromagnet model. However the annealed model, which is more tractable to analyze, has been studied exactly by Wu[12] on the decorated square lattice; the critical probability he obtained depends on q.

The purpose of this paper is to study the quenched bond-diluted q-state Potts ferromagnet and antiferromagnet models on a square lattice by using renormalization-group methods based on the Migdal-Kadanoff[13,14] (MK) approximation. In the ferromagnetic (FM) Potts model, the phase diagram obtained is in quantitative agreement with all known results for $q \leq 4$. In the antiferromagnet (AFM) model the phase diagram is similar to that of the annealed model. The present work is organized as follows: In Sec. II we introduce the model and the recursion relations, and in Sec. III we present our main results.

II. MODEL AND RECURSION RELATIONS

Consider a nearest-neighbor q-state Potts model on a d-dimensional cubic square lattice and subject to randomly inhomogeneous pair coupling. The appropriate Hamiltonian (in reduced units) is

$$-\beta H = \pm \sum_{\langle i,j \rangle} K_{ij}(q\delta\sigma_i\sigma_j - 1) \quad (\sigma_1 = 1,2,\ldots,q \; \forall i) , \tag{1}$$

where $+$ $(-)$ corresponds to FM (AFM). The sum runs over all pairs of first-neighboring sites on a square lattice, and K_{ij} is a random variable whose probability law is given by

$$P(K_{ij}) = (1-p)\delta(K_{ij}) + p\delta(K_{ij} - K) \quad \text{with } K > 0 , \tag{2}$$

i.e., each bond either has value K (with probability p) or is absent (with probability $1-p$).

Before constructing the renormalization-group recursive relations, let us associate, with every bond characterized by an arbitrary coupling constant K_{ij}, a convenient variable defined[5] as follows:

$$t_{ij} = \frac{1-\exp(\pm qK_{ij})}{1+(q-1)\exp(\pm qK_{ij})} = f(K_{ij}) , \tag{3}$$

where $-$ $(+)$ corresponds to FM (AFM).

The Migdal-Kadanoff approximate consists in successive contractions by a scale factor b along each of the d Cartesian directions, resulting in a volume contraction by an overall factor b^d. Each contraction involves a bond shifting perpendicular to the contraction and a decimation along the contraction. Schematically,

$$\tilde{K}_\perp = \sum K_\perp , \quad f(\tilde{K}_\parallel) = \prod f(K_\parallel) , \tag{4}$$

or, in terms of the convenient variable $t = f(K)$, which is finite at both $K \to 0$ $(t=0)$ and $K \to \infty$ $[t=1$ for FM and $t = -1/(q-1)$ for AFM];

$$f^{-1}(\tilde{t}_\perp) = \sum f^{-1}(t_\perp) , \quad \tilde{t}_\parallel = \prod t_\parallel . \tag{5}$$

A. Recursion relations for pure systems

When the system is homogeneous, the recursion relation for the coupling along the direction of initial contraction (x) is

$$t' = f(b^{d-1} f^{-1}(t^b)) , \tag{6}$$

while the coupling along the direction of final contraction (y) transforms as

TABLE I. Critical points K_c and thermal exponents y_T of the ferromagnet Potts model on the square ($d=2$) lattice for some values of q. The exact values are from Ref. 1. The xy and yx Migdal-Kadanoff results are calculated for $b=2$.

	K_c				y_T			
q	MK (xy)	MK (yx)	IMK	Exact	MK (xy)	MK (yx)	IMK	Exact
1	0.481	0.241	0.347	0.347	0.612	0.612	0.614	0.75
2	0.609	0.305	0.441	0.441	0.747	0.747	0.754	1.00
3	0.693	0.347	0.503	0.503	0.830	0.830	0.839	1.20
4	0.756	0.378	0.549	0.549	0.890	0.890	0.901	1.50

$$t'=[f(b^{d-1}f^{-1}(t))]^b . \tag{7}$$

We shall refer to these as xy and yx recursion relations, respectively. They have quite different fixed point couplings for integer $b>1$. However, it is easy to interpret b as a continuous variable and to take the limit $b=1+dl$, $dl\to 0^+$ (dl is an infinitesimal). In this $b\to 1$ limit, the xy and yx recursion relations become identical and yield, for $dt\equiv t'-t$,

$$\frac{dt}{dl}=\frac{(d-1)}{q}(1-t)[1+(q-1)t]\ln\left[\frac{1+(q-1)t}{1-t}\right] +t\ln(\pm t) , \tag{8}$$

where $+$ ($-$) corresponds to FM (AFM).

It is interesting to note that for $d=2$ and in the ferromagnetic case the fixed point of (8) is $\exp(2K_c)=1+\sqrt{q}$, which is the exact transition temperature of the Potts model. This indicates that Migdal's approximation obeys the dual symmetry in the limit $b\to 1$.[15]

This remarkable agreement does not, unfortunately, extend to exponents which are given by

$$y_T=2\left[1-\frac{1}{\sqrt{q}}\ln(1+\sqrt{q})\right] .$$

The thermal exponents given by the infinitesimal Migdal-Kadanoff (IMK) are compared with exact values[1] in Table I.

It is well known that a critical value of q exists, $q_c(d)$, depending on the space dimension d, where the phase transition changes order, becoming a first-order one. Exact results[16] give $q_c(2)=4$. The simple MK transformation described above is unsuitable for treating the q-state Potts ferromagnet for $q>q_c$ since it continues to predict a second-order transition.

In the antiferromagnetic case, Eq. (8) exhibits a cutoff value $q_0(d)$ below which a long-range order will arise at low temperature. The IMK approximation yields the values[17] $q_0(2)=2.296$ and $q_0(3)=3.3$. This problem was solved exactly by Wu[12] on the decorated square lattice and he obtained $q_0(2)=2.618$.

As a consequence, the antiferromagnetic model has a phase transition only for $q=2$. This result is consistent with the fact that, for $q\geq 3$, the ground state of the model has a nonzero entropy. We also note that, for $q=2$, the critical temperature coincides with that of the ferromagnetic model.

B. Recursion relations for dilute systems

For the bond-diluted q-state Potts model the analogs of Eqs. (6) and (7) determine each new local coupling $t'_{\alpha\beta}$ in terms of a set of original couplings $\{t_{ij}\}$. If each t_{ij} is independently distributed according to a probability distribution $P(t_{ij})$ given in (2), then the probability distribution $P'(t'_{\alpha\beta})$ for the renormalized coupling is given by[18]

$$P'(t'_{\alpha\beta})=\int\prod_{\langle i,j\rangle}dt_{ij}P(t_{ij})\delta(t'_{\alpha\beta}-t'_{\alpha\beta}(\{t_{ij}\})) . \tag{9}$$

Although, initially, the couplings are either present or absent ($t_{ij}=0,t$) corresponding to the two-peaked distribution (2), they do not remain so under iteration: The initial contraction along the x direction gives an intermediate x coupling

$$\tilde{t}_x=f(\tilde{K})=\begin{cases}t^b, & \text{with probability } p^b ,\\ 0, & \text{with probability } 1-p^b .\end{cases} \tag{10}$$

When b^{d-1} of these intermediate couplings are added together in the succeeding y contraction, the coupling $K'_{\alpha\beta}=f^{-1}(t'_{\alpha\beta})$ is an integral multiple of $f^{-1}(t^b)$,

$$K'_{\alpha\beta}=mf^{-1}(t^b),\quad 0\leq m\leq b^{d-1} \tag{11}$$

with corresponding probability

$$P_m(b)=\binom{b^{d-1}}{m}(p^b)^m(1-p^b)^{b^{d-1}-m} . \tag{12}$$

To render the computations tractable we make an additional approximation at each iteration by forcing the full distribution (11) back to a two-peak form. Namely,

$$P'_{\text{approx}}(t'_{\alpha\beta})=p'\delta(t'_{\alpha\beta}-t')+(1-p')\delta(t'_{\alpha\beta}) . \tag{13}$$

Equating the weights of $t'=0$ of $P'(t'_{\alpha\beta})$ and $P'_{\text{approx}}(t'_{\alpha\beta})$ gives the following recursion relation for p':

$$p'=1-(1-p^b)^{b^{d-1}} , \tag{14}$$

where p' depends on p but not on t.

Equating the average values of t' for $P'(t'_{\alpha\beta})$ and $P'_{\text{approx}}(t'_{\alpha\beta})$ gives the remaining recursion relation, which can be written

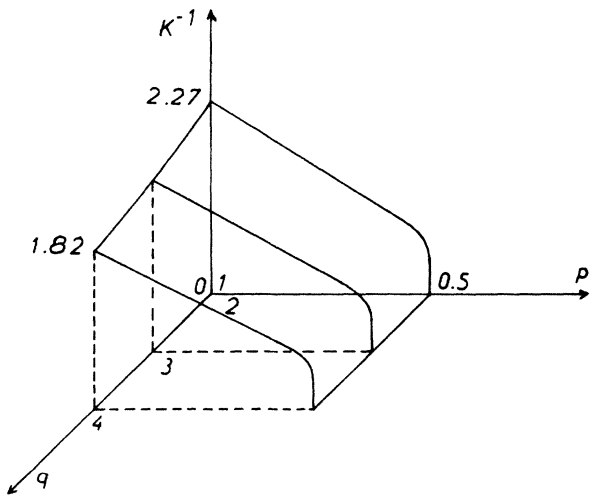

FIG. 1. Phase diagram for ferromagnetic interactions derived from IMK approximation in the space (q,p,K^{-1}) for $q \leq 4$.

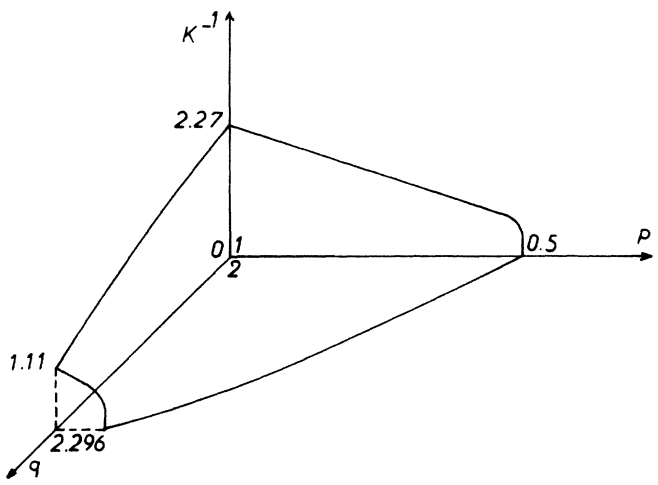

FIG. 2. Phase diagram for antiferromagnetic interactions derived from IMK approximation in the space (q,p,K^{-1}).

$$p't' = \sum_{m=0}^{b^{d-1}} P_m(b) f(mf^{-1}(t^b)) \ . \tag{15}$$

The $b \to 1$ limit yields, for $dp \equiv p' - p$, the recursion relation

$$\frac{dp}{dl} = p \ln(p) - (d-1)(1-p)\ln(1-p) \ . \tag{16}$$

This equation exhibits for $d=2$ the exact bond percolation fixed point $p_c = 0.5$. The associated eigenvalue exponent $y_p = 2[1 - \ln(2)]$ corresponds to a correlation length exponent for percolation $\nu_p = 1.629$ [series (Ref. 19) $\nu_p = 1.34 \pm 0.02$] for any value of q. This concentration is a geometrical percolation threshold which does not depend upon the model and the strength of interactions. However, for quenched dilute systems in general, we have from Griffiths and Lebowitz[20] the exact statement that for finite-range interactions the transition temperature vanishes at a concentration, p^*, greater than the percolation concentration, p_c, corresponding to linkages of that range.

III. RESULTS AND CONCLUSION

A. Phase diagram for ferromagnetic interactions

To obtain the IMK recursion relations, the difficulty with (15) is that b is also a summation limit. We wish, therefore, to cast (11) into a form which allows the sum to be performed. Let

$$X = (1-t^b)/[1+(q-1)t^b], \quad X \leq 1 \ .$$

Then

$$f(mf^{-1}(t^b)) = \frac{1-X^m}{1+(q-1)X^m} = (1-X^m) \sum_{r=0}^{\infty} (-1)^r (q-1)^r X^{rm} \ . \tag{17}$$

The sum over m is a binomial expansion, so

$$p't' = \sum_{r=0}^{\infty} (-1)^r (q-1)^r [(1-p^b+p^b X^r)^b - (1-p^b+p^b X^{r+1})^b] \ . \tag{18}$$

The phase diagram in the space (q,p,K^{-1}), given by the recursion relations (14) and (18) for the ferromagnetic model, is represented in Fig. 1. It is seen that the phase space is divided into two regions: a low-temperature region containing the q axis in which the system is in an ordered phase, and a high-temperature region in which the system is disordered. Note that the concentration p^*, at which transition temperature vanishes, coincides with the percolation concentration, p_c, for all $q \leq 4$. The phase diagram is limited to $q \leq 4$ since the IMK approximation continues to predict a second-order transition for $q > 4$. This phase diagram is similar to that of the quenched diluted Potts model obtained under the effective interaction approximation[3] and under real-space renormalization.[11] The $q=2$ case recovers the results obtained by Jayaprakach *et al.*[21] for the Ising model.

B. Phase diagram for antiferromagnetic interactions

In the antiferromagnetic case Eqs. (17) and (18) can be rewritten as follows:

$$f(mf^{-1}(t^b)) = (X^{-m}-1) \sum_{r=0}^{\infty} (-1)^r (q-1)^{-(r+1)} X^{-rm} \ , \tag{19}$$

$$p't' = \sum_{r=0}^{\infty} (-1)^r (q-1)^{-(r+1)} [(1-p^b+p^b X^{-(r+1)})^b - (1-p^b+p^b X^{-r})^b] \ . \tag{20}$$

The phase diagram given by the recursion relations (14) and (20) is represented in Fig. 2. The phase space is divided into a low-temperature (ordered) region containing the origin, and a high-temperature (disordered) region. We also find the cutoff value $q_0(2)=2.296$ to compare with the value obtained by Wu[12] on the decorated square lattice $q_0(2)=2.618$. This difference is due to the fact that the exact cutoff[12] $[q_0(2)=2.618]$ is given by the point of coordinates $(K^{-1}=0,p^*=1)$ while the cutoff given by IMK corresponds to the value $q_0(2)=2.296$ at which the transition disappears [Eq. (20) has no solution] and this arrives at $K^{-1}=1.11$ for $p=1$, and $p^*=0.915$ for $K^{-1}=0$. Note that p^* depends upon q; for $q=2$, $p^*=p_c=0.5$, and for $q=2.296$, $p^*=0.915$.

To summarize, we have obtained the phase diagram for the diluted Potts model on a square lattice using IMK approximation. In the ferromagnetic case our phase diagram is in quantitative agreement with all known results for $q\leq 4$. In the antiferromagnetic case the phase diagram we obtain is similar to that of the annealed dilution given by Wu.[12]

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