Intermediate statistics for vortices in superfluid films

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A quantization of the classical theory of a two-vortex system in a superfluid film is carried out. This is done by finding the representations of an algebra of observables which respect the fact that the vortices are identical objects. The representations depend on an undetermined parameter θ, which may be interpreted as defining intermediate types of statistics for the vortices. We relate our approach to a previous discussion, where only two values for θ were claimed to be possible.

I. INTRODUCTION

In a two-dimensional world the types of particle statistics are not restricted to the two well-known types: Fermi-Dirac and Bose-Einstein. Theoretically a continuum of *intermediate statistics* appear, which interpolate between the only two types which are possible in three (and higher) dimensions. We originally showed this in an earlier paper,[1] and it has also been discussed by other authors.[2] In the last few years this theoretical possibility has been related to real physical phenomena in quasi-two-dimensional systems. In particular, it has been suggested that systems that demonstrate the fractional quantum Hall effect contain quasiparticle excitations with fractional statistics.[3] Also the quantized system of vortices in thin films of superfluid helium has been examined from this point of view.[4–6]

Chiao, Hansen, and Moulthrop have analyzed the quantization of the classical vortex motion in superfluid films, and draw from this the conclusion that the vortices satisfy a statistic halfway between the fermion and boson cases.[4] However, this result has been disputed in other papers.[5,6] In this paper we reexamine the quantization of the two-vortex system, to study this question. We restrict the discussion mainly to the question of the "correct" quantization of the classical motion, with the effect of the vortices being identical taken properly into account. The microscopic basis for such a quantum description, addressed in Ref. 5, we would only briefly comment on at the end of the paper.

From a particle point of view the vortex system is somewhat unusual, since, although physically it is a two-dimensional system, dynamically it is only one dimensional. This is so because the two orthogonal coordinates in the plane, in a Hamiltonian formulation, are canonically conjugate. Therefore, the general analysis of the wave functions of two-dimensional systems, carried out in Ref. 1, is not so readily applied to the vortex system. For this reason we would rather examine the system by studying the algebra of observables of the quantized system. The identical-particle effect is then introduced by restricting the observables to operators that are *symmetric* under exchange of particle indices.

The conclusion we reach is that the quantized system contains an undetermined parameter, which is naturally interpreted as representing the possible intermediate statistics. This agrees with the canonical quantization of the two-vortex system in polar coordinates, discussed by Chiao *et al.*[4] However, in Ref. 4 a canonical quantization was carried out also in Cartesian coordinates, and this quantization introduced a restriction on the statistics parameter, to only two possible values. From our point of view, the latter quantization scheme is too restrictive. This has to do with the fact that the position variables are not observables in the sense that they respect the identical-particle effect. Another way to state this is that the quantization does not take into account the singularity structure of phase space, and therefore is restricted by the fact that it conserves more symmetries than are actually present in the physical system. And, as discussed in Ref. 1, the presence of the singularities, due to identification of particle coordinates, is really the reason for the appearance of the generalized types of particle statistics, both in one- and two-dimensional systems.

II. THE TWO-VORTEX SYSTEM

Following Ref. 4 we consider a two-vortex system, consisting of two identical vortices, in a thin, incompressible superfluid film. In the approximation where the vortices are treated as pointlike objects, the classical motion of their *relative* coordinates is described by the equations

$$\begin{aligned} \varkappa \frac{dx}{dt} &= 2 \frac{\partial \mathcal{H}}{\partial y} , \\ \varkappa \frac{dy}{dt} &= -2 \frac{\partial \mathcal{H}}{\partial y} , \\ \mathcal{H} &= -\frac{\varkappa^2}{4\pi} \ln[(x^2+y^2)/a^2] . \end{aligned} \tag{1}$$

In these expressions a is a scale parameter and $\varkappa$ is the quantized vorticity of the superfluid.[7] The vorticity is related to the mass m of the atoms in the superfluid, $\varkappa = 2\pi\hbar/m$. (We assume throughout the paper that $\varkappa$ is positive; negative $\varkappa$ corresponds to vortices of opposite circulation.) The system is thus a Hamiltonian one, with the two orthogonal coordinates in the plane, x and y, as canonically conjugate [but only in a limited sense can the system (1) be considered a "classical" one, since Planck's constant already appears through the parameter $\varkappa$]. The Hamiltonian $\mathcal{H}$ is related to the energy H by

$$H = \rho\delta\mathcal{H} \ , \tag{2}$$

with ρ as the density and δ as the thickness of the superfluid film. (**H, then, is the energy relative to the energy for a separation $r = a$ between the vortices.) Therefore the momentum with correct dimension, conjugate to x, is**

$$p_x = \frac{\varkappa\rho\delta}{2} y \ . \tag{3}$$

A standard canonical quantization then gives the following commutation relation for x and y:[4]

$$[x,y] = \frac{2i\hbar}{\varkappa\rho\delta} \ . \tag{4}$$

The relative coordinates x and y are not invariant under interchange of the position of the two vortices:

$$(x,y) \to (-x,-y) \ . \tag{5}$$

However the following quantities, derived from x and y, are invariant,

$$\begin{aligned} A &= \tfrac{1}{8}\varkappa\rho\delta(x^2+y^2) \ , \\ B &= \tfrac{1}{8}\varkappa\rho\delta(y^2-x^2) \ , \\ C &= \tfrac{1}{8}\varkappa\rho\delta(xy+yx) \ , \end{aligned} \tag{6}$$

and therefore represent observables of the system. They have the following commutation relations implied by Eq. (4):

$$\begin{aligned} [A,B] &= i\hbar C \ , \\ [A,C] &= -i\hbar B \ , \\ [B,C] &= -i\hbar A \ . \end{aligned} \tag{7}$$

Our approach is now to consider the commutation relations (7), with the additional positivity constraint $A > 0$, as the fundamental relations which define the quantization of the system. These relations replace the commutation relation (4), which deals with operators that are not necessarily represented within the algebra of observables.

III. QUANTIZATION

The commutator algebra of the operators A, B, and C is identical to the algebra of the group SL$(2,R)$. To quantize the system means to find irreducible representations of this algebra, with inequivalent representations then meaning inequivalent quantizations of the system.

The irreducible representations are most easily found by considering the common eigenvectors $|\alpha,\gamma\rangle$ of A and the Casimir operator

$$\Gamma = A^2 - B^2 - C^2 \ , \tag{8}$$

$$\begin{aligned} A\,|\alpha,\gamma\rangle &= \alpha\hbar\,|\alpha,\gamma\rangle \ , \\ \Gamma\,|\alpha,\gamma\rangle &= \gamma\hbar^2\,|\alpha,\gamma\rangle \ . \end{aligned} \tag{9}$$

If γ is calculated by use of the commutation relation (4), we find $\gamma = -\frac{3}{16}$. However, in the present case, with (7) as the fundamental commutation relation, γ is not restricted to this value.

The operators

$$B_\pm = B \pm iC \tag{10}$$

raising and lowering operators in the spectrum of A, as shown by the commutation relations

$$\begin{aligned} [A,B_\pm] &= \pm\hbar B_\pm \ , \\ [B_+,B_-] &= -2\hbar A \ . \end{aligned} \tag{11}$$

We consequently have

$$B_\pm\,|\alpha,\gamma\rangle = \beta_\pm(\alpha,\gamma)\hbar\,|\alpha\pm 1,\gamma\rangle \ , \tag{12}$$

with

$$\beta_-(\alpha,\gamma) = \beta_+(\alpha-1,\gamma)^* \ . \tag{13}$$

The functions $\beta_\pm$ are determined by the second of the commutation relations in Eq. (11), when we fix their phases by convention,

$$\begin{aligned} \beta_+(\alpha,\gamma) &= \sqrt{\alpha(\alpha+1)-\gamma} \ , \\ \beta_-(\alpha,\gamma) &= \sqrt{\alpha(\alpha-1)-\gamma} \ . \end{aligned} \tag{14}$$

In general, the spectrum of A is unbounded both from below and above. The spectrum is bounded from below only if A has a minimum eigenvalue α_0, where the corresponding eigenvector is annihilated by B_-,

$$\beta_-(\alpha_0,\gamma) = 0 \ . \tag{15}$$

This gives

$$\begin{aligned} \alpha_0 &= \tfrac{1}{2} \pm (\gamma + \tfrac{1}{4})^{1/2} \ , \\ \gamma &= \alpha_0(\alpha_0 - 1) \ . \end{aligned} \tag{16}$$

As shown by the equation, the possible eigenvalues of Γ then are restricted by $\gamma \geq -\frac{1}{4}$. Positivity of A further gives the restriction $\alpha_0 > 0$.

The parameter α_0 now characterizes the inequivalent representations of the algebra (7), and therefore inequivalent quantizations of the system. (The representations may alternatively be characterized by γ, but then there are two inequivalent representations for each γ in the interval $-\frac{1}{4} < \gamma < 0$.) The value of α_0 has physical implications through the eigenvalues a_n of A and the corresponding energies E_n,

$$a_n=(\alpha_0+n)\hbar,\quad n=0,1,2,\ldots,$$
$$E_n=-\frac{\varkappa^2}{4\pi}\rho\delta\ln(8a_n/\varkappa\rho\delta a^2)\ . \tag{17}$$

In the following we shall denote the corresponding eigenvectors by $|n,\alpha_0\rangle$, rather than by $|\alpha,\gamma\rangle$, which has been used above.

IV. THE ϕ-REPRESENTATION

The operator A generates rotations in the plane and thus represents (except for a sign) the relative angular momentum of the two-vortex system. This operator can be given the standard representation

$$A=-\frac{1}{2}\frac{\hbar}{i}\frac{\partial}{\partial\phi}\ , \tag{18}$$

by introducing the following representation $\Psi(\phi)$ for the state vectors $|\Psi\rangle$,

$$\Psi(\phi)=\frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}e^{-2i(\alpha_0+n)\phi}\langle n,\alpha_0|\Psi\rangle\ . \tag{19}$$

In this representation the operators $B_\pm$ get the form

$$B_\pm=e^{\pm i\phi}[A^2-(\alpha_0-\tfrac{1}{2})^2]^{1/2}e^{\pm i\phi}\ , \tag{20}$$

and if we consider the classical limit, where $[A,\phi]\to 0$ and $A\gg\alpha_0$, we find the following expressions for B and C:

$$B=A\cos 2\phi\ ,$$
$$C=-A\sin 2\phi\ . \tag{21}$$

This shows that ϕ indeed represents the polar angle of the relative position of the two vortices. [To be precise $\phi+\pi/2$ is the standard polar angle. The additional angle $\pi/2$ comes from our phase convention in Eq. (14).]

The wave functions $\Psi(\phi)$ satisfy the periodicity condition

$$\Psi(\phi+\pi)=e^{-2i\pi\alpha_0}\Psi(\phi)\ . \tag{22}$$

The parameter $\theta=2\pi\alpha_0$ then corresponds to the parameter which determines the (generalized) statistics of the particles in the case of two identical particles in the plane.[1] In the present case, with the commutation relations (7) as the fundamental relations for the quantized system, there is no restriction on the possible values of θ (except $\theta>0$). This is different from what would happen if (4) were assumed to be the fundamental quantization relation. Then we would have $\gamma=-\frac{3}{16}$, and therefore $\alpha_0=\frac{1}{4}$ or $\frac{3}{4}$, so that the parameter θ would have only two possible values,

$$\theta=\frac{\pi}{2},3\frac{\pi}{2}\ , \tag{23}$$

as was discussed also in Ref. 4.

A naive comparison between the periodicity condition (22) and the corresponding condition for a two-particle system in the plane may lead to the conclusion that the following θ values correspond to the boson and fermion cases,

$$\theta=0 \bmod 2\pi\quad\text{(bosons)}\ ,$$
$$\theta=\pi \bmod 2\pi\quad\text{(fermions)}\ . \tag{24}$$

The values (23) then would be midway between the boson and fermion values. However, as pointed out in Ref. 1, the phase factor which determines the (generalized) statistics of particles in two dimensions, does not only depend on symmetry properties of the wave functions, but in part also depends on the form of observables, like the Hamiltonian. Thus for any value of θ the wave functions may be transformed to a symmetric form, but the θ parameter will then instead appear explicitly in the expression for the observables. Since the dynamics of the two-vortex system is genuinely different from the dynamics of a two-particle system, this introduces an ambiguity in the identification of certain values of θ as corresponding to the boson and fermion cases. We shall return to this point.

One of the characteristic differences between the two-vortex system and the two-particle system in the plane is seen by considering the spectrum of the angular momentum operator A. For the two-particle system, the angular momentum is unbounded both from above and below. An increase in the value of θ lifts the spectrum, but when $\theta\to\theta+2\pi$, each level is simply moved into the position of the next level one step higher up. In fact all observables have a similar periodic dependence on θ, and the physics thus only depends on the phase factor $e^{i\theta}$ which characterizes the statistics of the particles.

For the two-vortex system, however, the spectrum of A is bounded from below. (Negative values of A would correspond to vortices of opposite circulation, $\varkappa\to-\varkappa$.) A change $\theta\to\theta+2\pi$ therefore lifts all the levels into the position of the next level one step higher up, but the spectrum of A is not restored, since the lowest level of the original spectrum now is missing. The expression (20) for the operators $B_\pm$ shows that these operators also have a nonperiodic dependence on θ. As a consequence of this there is a series of dynamically different quantizations for each value of the phase factor $e^{i\theta}$, whereas by the naive argument these should all define particles with the same generalized statistics.

The above discussion points out some similarities and differences between the two-vortex system and a system of two identical particles in two dimensions. However, as already pointed out, the two-vortex system has dynamically the character of a one-dimensional system. Therefore, one may, perhaps more naturally, analyze the system as consisting of two identical particles moving in one dimension. This we will do in the next section.

V. THE x REPRESENTATION

As already stated, we do not consider the variables x and y as representing observables of the system, since they do not respect the identification

$$(x,y)\equiv(-x,-y)\ , \tag{25}$$

which represents the fact that the two vortices are identical. However, if x is restricted to positive values, we may define the observable

$$x=(x^2)^{1/2}=\left[\frac{4}{\varkappa\rho\delta}(A-B)\right]^{1/2} . \tag{26}$$

This is well defined, since $A-B$ is a positive definite operator for $\alpha_0>0$. Interpreting the variable x as describing the configuration space of the system, we note that the restriction of x to the positive half line is the same restriction as found for the relative motion of two identical particles in one dimension.[1] The (xy) plane, with the identification (25), indeed, is identical to the phase space (of the relative motion) of two identical particles moving on a line.

We may now introduce a set of basis vectors $|x\rangle$ and the corresponding x representation $\Psi(x)$ of the state vectors $|\Psi\rangle$ by the relations

$$\begin{aligned}&\frac{4}{\varkappa\rho\delta}(A-B)|x\rangle=x^2|x\rangle ,\\ &\langle x|x'\rangle=\delta(x-x') ,\\ &\Psi(x)=\langle x|\Psi\rangle .\end{aligned} \tag{27}$$

The x representation of the operators $A,B,$ and C can be found from the commutation relations (7). For C we find

$$e^{-i\epsilon C}x^2e^{i\epsilon C}=e^{-\epsilon\hbar}x^2 , \tag{28}$$

with ϵ as a parameter for the transformations generated by C. This implies

$$e^{i\epsilon C}|x\rangle=N(\epsilon,x)|e^{-\epsilon\hbar/2}x\rangle , \tag{29}$$

with $N(\epsilon,x)$ as a normalization factor. This factor is determined from the normalization of the $|x\rangle$ vectors, and choosing $N(\epsilon,x)$ to be real we find

$$N(\epsilon,x)=e^{-\epsilon\hbar/4} . \tag{30}$$

Equation (29), in the x representation, then reads

$$(e^{i\epsilon C}\Psi)(x)=e^{\epsilon\hbar/4}\Psi(e^{\epsilon\hbar/2}x) . \tag{31}$$

Expanding this to first order in ϵ we find the following expression for C:

$$C=-\frac{i\hbar}{4}\left[x\frac{d}{dx}+\frac{d}{dx}x\right] . \tag{32}$$

The operator

$$y^2=\frac{4}{\varkappa\rho\delta}(A+B) \tag{33}$$

has the following commutator with x^2:

$$[y^2,x^2]=-\frac{32i\hbar}{(\varkappa\rho\delta)^2}C . \tag{34}$$

Writing y^2 as

$$y^2=\left[\frac{2\hbar}{\varkappa\rho\delta}\right]^2\left[-\frac{d^2}{dx^2}+f\right] , \tag{35}$$

we get from Eq. (34),

$$[f,x^2]=0 , \tag{36}$$

which implies $f=f(x^2)$.

The third of the commutation relations implied by (7) can be written as

$$[y^2,C]=-i\hbar y^2 . \tag{37}$$

With C given by Eq. (32), this gives the following equation for f,

$$x\frac{df}{dx}+2f=0 . \tag{38}$$

This equation has the solution

$$f=\frac{\lambda}{x^2} , \tag{39}$$

with λ as a parameter which is undetermined by the commutation relations (7). The expression for y^2 then is

$$y^2=\frac{4\hbar^2}{(\varkappa\rho\delta)^2}\left[-\frac{d^2}{dx^2}+\frac{\lambda}{x^2}\right] , \tag{40}$$

and for A and B this gives the following x representations:

$$\begin{aligned}A&=\frac{1}{8}\left[-\frac{4\hbar^2}{\varkappa\rho\delta}\frac{d^2}{dx^2}+\frac{4\hbar^2}{\varkappa\rho\delta}\frac{\lambda}{x^2}+\varkappa\rho\delta x^2\right] ,\\ B&=\frac{1}{8}\left[-\frac{4\hbar^2}{\varkappa\rho\delta}\frac{d^2}{dx^2}+\frac{4\hbar^2}{\varkappa\rho\delta}\frac{\lambda}{x^2}-\varkappa\rho\delta x^2\right] .\end{aligned} \tag{41}$$

The parameter λ, which introduces a $(1/x^2)$-"potential" in the expressions for A and B, is in fact directly related to the parameter α_0. This can be seen by evaluating the Casimir operator

$$\Gamma=\frac{\hbar^2}{4}(\lambda-\tfrac{3}{4}), \tag{42}$$

which gives

$$\begin{aligned}\lambda&=4\gamma+\tfrac{3}{4}\\ &=4(\alpha_0-\tfrac{1}{4})(\alpha_0-\tfrac{3}{4}) .\end{aligned} \tag{43}$$

In the same way as for the ϕ representation, the parameter $\theta=2\pi\alpha_0$ can be interpreted as defining intermediate types of statistics, interpolating between the boson and fermion cases. To see this we consider the asymptotic behavior of the eigenfunctions of A as x approaches the singular point $x=0$,

$$\Psi(x)\approx x^{2\alpha_0-1/2} . \tag{44}$$

This x dependence, which shows that the *analytic continuation* of $\Psi(x)$ has a branch point singularity at $x=0$, also implies the following symmetry property when the wave function is continued to the negative real axis,

$$\Psi(-x)=e^{i(\theta-\pi/2)}\Psi(x) . \tag{45}$$

Interpreting symmetric functions as describing bosons and antisymmetric functions as describing fermions, we then have

$$\theta=\frac{\pi}{2}\text{mod}2\pi \quad \text{(bosons)}\ ,$$
$$\theta=3\frac{\pi}{2}\text{mod}2\pi \quad \text{(fermions)}\ , \tag{46}$$

while other values of θ define something in between. The curious fact now is that the interpretation (46) does not agree with the corresponding "two-dimensional" interpretation (24). In particular, the two special values $\theta=\pi/2$ and $\theta=3\pi/2$, singled out by Chiao *et al.*[4] [Eq. (23)], in the present interpretation are the boson and fermion cases, respectively. The relation (43) between λ and α_0 now offers an explanation of what is special about these values of θ. For $\theta=\pi/2$ and $\theta=3\pi/2$ the parameter λ vanishes. The singular potential λ/x^2 then disappears and the wave functions $\Psi(x)$ have a nonsingular behavior at $x=0$.

VI. CONCLUDING REMARKS

The approach we have chosen for the quantization of the two-vortex system shows that an undetermined parameter θ is present, which may be interpreted as representing possible "intermediate statistics" for the system. We have found no restriction on the possible values of θ (except $\theta>0$), since we have allowed a singular behavior of the wave functions at the singular point $x=0$ of the configuration space.

We have pointed out that there is a certain ambiguity in identifying what values of θ that correspond to the boson and fermion cases. This is related to the fact that the distinction between the effects of "statistics" and "dynamics" is not completely unambiguous in one and two dimensions. In the x representation the values $\theta=\pi/2$ and $\theta=3\pi/2$ seem most natural identified as describing bosons and fermions. The singular potential λ/x^2 then vanishes and the wave functions can be continued to negative x, to give symmetric and antisymmetric wave functions, respectively. On the other hand, in the ϕ representation it seems more natural to consider $\theta=0$ and $\theta=\pi$ as the θ values for bosons and fermions. The spectrum of the angular momentum operator A is then correct (for $A>0$), as compared with the relative angular momentum of a two-particle system in the plane.

The presence of an undetermined parameter θ represents the ambiguity which in principle is always present in the quantization of a classical theory. Often this ambiguity is reduced, or eliminated, by requiring certain symmetries to survive the quantization. As already noticed the canonical quantization rule (4) is more restrictive than (7), which we have used for the quantization. However, since the point $x=y=0$ is a singular point already at the classical level, there seems to be no reason to respect *translational* invariance in the quantization. This is done by the commutation relation (4), while (7) only respects the *rotational* symmetry in the (x,y) plane. Let us also point out that there are other ways to quantize the system, in its x-space representation, than the one we have used here. In Ref. 1 intermediate types of statistics were introduced by the boundary condition

$$\frac{d\Psi}{dx}=\mu\Psi \quad \text{for } x=0\ ,$$

where μ is an interpolating parameter characteristic for the system. If we had adopted this rule here, and assumed $\lambda=0$, this would have given a different way of interpolating the spectrum of A between the boson and fermion cases. The reason for ruling out this quantization scheme here is that it does not respect the rotational symmetry in the (x,y) plane. This symmetry is present at the classical level, and there seems to be no good reason why the quantization should break it.

However, the question of what is the correct quantum description of the vortex motion for a physical superfluid film, like He^4, cannot be decided at the level where the vortices are treated as pointlike objects. At this level, for example, the value of θ is completely undetermined. To determine the physical value of θ, or more generally to determine what is really the correct quantum description of the vortex system, one should consider the lower "microscopic" level, where the atomic structure of the superfluid is taken into account. This is the approach taken by Haldane and Wu.[5] Our conclusions agree with much of the general discussion in their paper, including the point that the statistics parameter θ cannot be determined from the long distance ("classical") properties of the theory. But the way they relate the statistics parameter to the Berry phase[8] of the localized vortex states we do not find totally convincing. Let us point to the fact that the Berry phase has significance for the dynamics of the system only when the system is constrained to move in the submanifold of states for which the Berry phase is calculated. For adiabatic motion this is obtained when the motion is slow on the time scale set by the energy splitting of the states of the manifold. In the present case the vortex motion cannot be considered as slow in this respect, since the motion is induced by the same term in the Hamiltonian which lifts the energy degeneracy of the localized vortex states. Adiabatic motion, therefore, has to be induced by some external potential, which controls the motion of the vortices. But then the connection to the statistics parameter which is associated with the *free* vortex motion is no longer so clear.

In our opinion the core effects discussed in Ref. 5 mainly have to do with corrections to the vortex-vortex interaction. Such corrections can be included in the expression for the Hamiltonian, and they do not necessarily have to make the θ parameter (which appears only implicitly through the spectrum of r^2) ill defined. However, since the θ dependence in the Hamiltonian (1) is equivalent, for large r, to the presence of a θ-dependent $1/r^2$ potential, such other corrections may very well dominate the θ effect, and make it difficult to see it as a real observable effect. Let us therefore finally suggest that a further examination of the microscopic basis for the quantized vortex motion would be of interest, in order to see if it really is possible to ascribe a well-defined value to the statistics parameter θ of the vortex system.

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