# Analytical representation of impedance using the Boltzmann equation for nonlinear two-terminal device behavior

Clifford M. Krowne
*Electronics Technology Division, Naval Research Laboratory, Washington, D.C. 20375-5000*
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A nonlinear time-domain equation is derived which describes the electromagnetic field and electron transport behavior in an $n^+$-$n$-$n^+$ structure. From this governing equation, closed-form representations for impedance are obtained. One form provides an impedance evaluatable continuously in a complex-frequency domain. The other form provides an impedance at fundamental and harmonic frequencies. The second representation is dependent upon a sampling time interval. Electron transport in the approach obeys the Boltzmann transport equation, reexpressed in velocity-moment equations.

## I. INTRODUCTION

An extensive literature exists on the numerical solution of the Boltzmann transport equation leading to carrier distribution function, density, velocity, and energy and electric field information. Similarly, a large literature exists for determining device terminal characteristics such as current density and voltage based on treatments of the Boltzmann equation. Recently, interest in developing efficient analytical techniques or computationally fast numerical methods for studying elemental and compound semiconductors has occurred. Some of this interest has been encouraged by size reduction of devices for both very-large-scale-integration VLSI and microwave- and/or millimeter-wave applications, with a few dimensions going into the submicron regime. Use of Legendre or Legendre-Hermite polynomial expansions for the carrier distribution function in the development of a fully numerical solution to the one-dimensional (1D) Boltzmann transport equation has been examined for nonstationary modeling of III-V compound semiconductor materials and devices.[1] Spatial and time behavior for electron density, average carrier velocity and energy, and device terminal voltage and current were obtained. An analytical technique for finding high-field transport parameters in semiconductors using Hermite functions to represent the carrier distribution function has been presented.[2] A two-term Hermite polynomial expansion was employed for steady-state transport in a spatially homogeneous electric field. Electron distribution function, mobility, valley occupation, and drift velocity were found for GaAs. Under the assumption of small perturbations to the system, a closed-form analytical formula for the electron distribution function was found as a solution to the 1D Boltzmann transport equation.[3] Steady-state and time-dependent linear responses were described for spatially inhomogeneous compound semiconductor systems using the relaxation time approximation. Electron distribution function data was given for sinusoidal doping variation as in a superlattice and $n^+$-$n^-$-$n^+$ GaAs structures. An exact 1D numerical solution technique to the Boltzmann transport equation for electrons has been formulated enlisting a state equation approach for integral multiples of the optical phonon energy.[4] Steady current flow occurred creating a steady-state nonequilibrium problem. Electron distribution function, temperature, and potential were found for Si material typical of a metal-oxide-semiconductor field-effect transistor (MOSFET) channel. Utilizing a two-valley III-V semiconductor model, electron valley distribution functions were determined from the Boltzmann transport equation.[5] A relaxation time approach had been employed for the steady-state solution. Velocity-field curves were obtained for GaAs and InP up to 40 kV/cm. The Boltzmann transport equation was solved in a linear fashion providing a closed-form analytical expression for the electron distribution function.[6] This relaxation time method yielded an admittance which contains contact terms (contact conductance and capacitance) for an $n^+$-$n$-$n^+$ device. The internal $n$-region admittance in a circuit representation has only a real conductance component, possibly denying such a modeled device the ability to display negative differential conductance.

A nonequilibrium, time varying study of an $n^+$-$n$-$n^+$ device was conducted to assess negative differential conductance behavior.[7] The study used both an analytical moments solution approach[8] to the Boltzmann transport equation in order to consider many different boundary conditions as well as a Monte Carlo simulation approach. The analytical moment approach is attractive because its numerical implementation runs at least a thousand times faster than a comparable Monte Carlo computer run and can provide useful numerical results, and in principle may have admittance expressed as a closed-form expression. Because it is perturbational, large signal admittance or impedance determination is beyond this analytical model's capability. However, it is possible, under the same constraint of using displaced Maxwellian distribution functions, to obtain a closed-form impedance representation under large signal conditions.

Such a representation is derived here. Reduction of the electromagnetic field equation and the carrier transport equations into a compact nonlinear time-domain equation is undergone in Sec. II. Transformation of the nonlinear equation from the time domain into the complex frequency $s$ domain is discussed in Sec. III. The continuous transform employed to map into the $s$ plane enables a closed-form impedance representation to be found. Impedance can be evaluated at any complex frequency value. By sampling only a finite time interval of device operation, a discrete Fourier transform can be employed to operate on the nonlinear time domain equation. The result is that another closed-form impedance representation is obtained in Sec. IV. Impedance must be calculated at a fundamental frequency, or multiples of it. This harmonic representation is therefore compatible with or similar in philosophy to other standard harmonic analysis techniques like the fast Fourier transform technique.

## II. DERIVATION OF GOVERNING NONLINEAR TIME-DOMAIN EQUATION

A single electron gas model is invoked to obtain a set of equations derived as moments of the 1D Boltzmann transport equation. Such a model is utilizable for elemental (e.g., Si or Ge) or compound semiconductors (e.g., GaAs or InP) if proper consideration is taken to determine a single electron effective mass $m^*$ and temperature $T_e$ based on conduction-band edge minima occupation. This approach, of course, ignores recombination and/or generation and impact ionization effects. Using the relaxation time and displayed Maxwellian distribution function approximations, along with a parabolic conduction band, allows the first three moment equations of the Boltzmann transport equation to be written as[8,9]

$$\frac{\partial n}{\partial t}+\frac{\partial(nv)}{\partial x}=0\ , \tag{1a}$$

$$\frac{\partial(nv)}{\partial t}+\frac{\partial(nv^2)}{\partial x}=\frac{q}{m^*}nE-\frac{1}{m^*}\frac{\partial(nk_bT_e)}{\partial x}-\frac{nv}{\tau_p}\ , \tag{1b}$$

$$\frac{\partial w}{\partial t}+\frac{\partial(vw)}{\partial x}=qnvE-\frac{\partial(vnk_bT_e)}{\partial x}-\frac{w-w_e}{\tau_w}\ . \tag{1c}$$

These equations are, respectively, the electron density $n$, electron ensemble momentum $nv$, and electron ensemble energy $w$ conservation equations. Here $v$ is the average electron ensemble velocity, and $q$, $k_b$, $\tau_p$, $\tau_w$, $w_e$, and $E$ are, respectively, the electron charge, Boltzmann's constant, electron momentum and energy relaxation times, electron ensemble equilibrium energy, and electric field. The right-hand side (rhs) of (1a) is zero because of the single gas model.

Coupling of the transport equations to Maxwell's field equations is accomplished by Poisson's equation

$$\frac{\partial E}{\partial x}=\frac{q}{\epsilon}(n-n_d)\ , \tag{2}$$

where $\epsilon$ and $n_d$ are, respectively, the dielectric constant and donor doping density. Current density $J$ at any $x$ is

$$J(x,t)=qnv+\epsilon\frac{\partial E}{\partial t}\ , \tag{3}$$

composed of a particle component and a displacement component. It is easy to show that $J$ is a constant in space (but not in time) and represents the terminal current when evaluated at any position within the device. Solving for $n$ using (2), taking its partial time derivative, and placing this into (1a) gives

$$\frac{\epsilon}{q}\frac{\partial^2 E}{\partial t\partial x}+\frac{\partial(nv)}{\partial x}=0 \tag{4}$$

which is easily integrated to produce (3). Device terminal voltage $V_d$ is expressed as

$$V_d=-\int_0^l E(x)dx \tag{5}$$

with the positive terminal at $x=l$.

Placing the particle momentum $nv$ from (3) into the momentum moment equation (1b) generates the single equation

$$\frac{\partial J}{\partial t}-\epsilon\frac{\partial^2 E}{\partial t^2}+q\frac{\partial(nv^2)}{\partial x}=\frac{q^2}{m^*}nE-\frac{q}{m^*}\frac{\partial(nk_bT_e)}{\partial x}-\frac{J}{\tau_p}+\frac{\epsilon}{\tau_p}\frac{\partial E}{\partial t}\ . \tag{6}$$

Equation (6) can be integrated over the interval $(0,l)$ to yield

$$l\frac{dJ}{dt}+\epsilon\frac{d^2V_d}{dt^2}+qnv^2\Big|_0^l=\frac{q^2}{m^*}\int_0^l nE\,dx-\frac{q}{m^*}nk_bT_e\Big|_0^l-J\int_0^l\tau_p^{-1}+\epsilon\int_0^l\tau_p^{-1}\frac{dE}{dt}dx \tag{7}$$

by enlisting (5). The first integral on the rhs of (7) can be found if $n_d=\text{const}$. Density-field product is written as

$$nE=\frac{1}{2}\frac{\epsilon}{q}\frac{\partial E^2}{\partial x}+n_dE \tag{8}$$

employing (2) and (5). Integrating (8) over space and inserting into (7) produces

$$l\frac{dJ}{dt}+\epsilon\frac{d^2V_d}{dt^2}+qnv^2\Big|_0^l=\frac{q\epsilon}{2m^*}E^2\Big|_0^l-\frac{q^2n_d}{m^*}V_d-\frac{q}{m^*}nk_bT_e\Big|_0^l-J\int_0^l\tau_p^{-1}dx+\epsilon\int_0^l\tau_p^{-1}\frac{dE}{dt}dx\ . \tag{9}$$

For slow $\tau_p$ spatial variation, (9) reduces to

$$\epsilon\frac{d^2V_d}{dt^2}+\frac{\epsilon}{\tau_p}\frac{dV_d}{dt}+\frac{q^2n_d}{m^*}V_d=\left[\frac{q\epsilon}{2m^*}E^2-qnv^2-\frac{q}{m^*}nk_bT_e\right]\Bigg|_0^l-\frac{Jl}{\tau_p}-l\frac{dJ}{dt}\ , \tag{10}$$

noting (5). A more convenient form can be found for (10) by defining

$$A(t)=\frac{1}{m^*}\left[\frac{\epsilon}{2}E^2-m^*nv^2-nk_bT_e\right]\Bigg|_0^l . \qquad (11)$$

$A(t)$ is proportional to the difference between the stored electric field energy and the kinetic energy at the device ends. The kinetic energy consists of the drift energy plus the thermal energy of the heated electron gas. Rewriting (10) with the help of (11), gives

$$\epsilon\frac{d^2V_d}{dt^2}+\frac{\epsilon}{\tau_p}\frac{dV_d}{dt}+\frac{q^2n_d}{m^*}V_d=qA(t)-\frac{l}{\tau_p}J-l\frac{dJ}{dt} , \qquad (12)$$

the nonlinear time-domain governing equation. The nonlinearities in the device behavior are contained in the rhs of (12), namely in $A(t)$.

## III. IMPEDANCE BASED ON A CONTINUOUS TRANSFORM

A transform pair can be defined in the Laplace fashion as follows:

$$f(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}\hat{f}(s)e^{st}ds , \qquad (13a)$$

$$\hat{f}(s)=\int_{-\infty}^{\infty}f(t)e^{-st}dt . \qquad (13b)$$

In order that any physical time variable $f(t)$ possess a transform $\hat{f}(s)$, it is required that

$$\sigma_m<\sigma<\sigma_n , \qquad (14)$$

when

$$\lim_{t\to\infty}f(t)=f_me^{\sigma_m t} , \qquad (15a)$$

$$\lim_{t\to-\infty}f(t)=f_ne^{\sigma_n t} . \qquad (15b)$$

Since there are several different $f(t)$, it is necessary that some region exist in $\sigma$ space so that at least one $\sigma$ value satisfies every individual statement of the form (14). Assuming that such a $\sigma$ exists, every physical variable by (13b) has a transform which can be continuously evaluated over the complex part of $s$ space where

$$s=\sigma+j\omega . \qquad (16)$$

Transform operations on the governing equation (12) require that transforms of both $df/dt$ and $d^2f/dt^2$ exist:

$$\int_{-\infty}^{\infty}\frac{df(t)}{dt}e^{-st}dt=s\hat{f}(s) . \qquad (17)$$

Equation (17) is a direct consequence of (15) and the existence requirements for the transform of the original physical variable:

$$\int_{-\infty}^{\infty}\frac{d^2f(t)}{dt^2}e^{-st}dt=s^2\hat{f}(s) . \qquad (18)$$

Equation (18) holds only if

$$\sigma_{m1}<\sigma<\sigma_{n1} \qquad (19)$$

when

$$\lim_{t\to\infty}\frac{df(t)}{dt}=f_{m1}e^{\sigma_{m1}t} , \qquad (20a)$$

$$\lim_{t\to-\infty}\frac{df(t)}{dt}=f_{n1}e^{\sigma_{n1}t} . \qquad (20b)$$

For the case where derivative and limit operators can be interchanged, (19) and (20) reduce to forms similar to (14) and (15) with $\sigma_{n1}=\sigma_n$ and $\sigma_{m1}=\sigma_m$.

Examine the governing equation (12). Application of the transform (13b) to (12) requires that (18) be applied to the first left-hand side (lhs) term, and (17) be applied to the second lhs term and the last rhs term:

$$\left[\epsilon s^2+\frac{\epsilon}{\tau_p}s+\frac{q^2n_d}{m^*}\right]\hat{V}_d(s)=q\int_{-\infty}^{\infty}A(t)e^{-st}dt-l\left[s+\frac{1}{\tau_p}\right]\hat{J}(s) . \qquad (21)$$

The integral term on the rhs will persist even at low terminal temperatures because as $T_e\to 0$, $x=0,l$, only the thermal (or pressure) component of $A(t)$ drops out in (11).

Impedance $\hat{Z}(\Omega\cdot m^2)$ is defined as the ratio of the device terminal potential to the terminal current density.

$$\hat{Z}(s)=-\frac{\hat{V}_d(s)}{\hat{J}(s)} . \qquad (22)$$

Putting (21) into the (22) form generates the impedance expression

$$\hat{Z}(s)=\frac{l}{\epsilon}\frac{\left[s+\frac{1}{\tau_p}\right]}{\left[s^2+\frac{s}{\tau_p}+\omega_p^2\right]}-\frac{q}{\epsilon}\frac{\int_{-\infty}^{\infty}A(t)e^{-st}dt}{\left[s^2+\frac{s}{\tau_p}+\omega_p^2\right]\hat{J}(s)} . \qquad (23)$$

In (23) the plasma frequency was defined as

$$\omega_p^2=\frac{q^2n_d}{m^*\epsilon} . \qquad (24)$$

The first part of $\hat{Z}(s)$ in (23), the linear part $\hat{Z}_1(s)$, is similar in form to an impedance component[8,10] which never has a negative resistance property for real frequencies $\omega$. Real and imaginary parts of $\hat{Z}_1(s)$ are expressible as

$$\mathrm{Re}[Z_1(s)]=\frac{l}{\epsilon}\frac{\sigma^3+2\sigma^2/\tau_p+\sigma(\omega^2+\omega_p^2+1/\tau_p^2)+\omega_p^2/\tau_p}{(\sigma^2+\sigma/\tau_p+\omega_p^2-\omega^2)^2+(2\sigma\omega+\omega/\tau_p)^2} , \qquad (25a)$$

$$\mathrm{Im}[Z_1(s)]=\frac{l\omega}{\epsilon}\frac{\omega_p^2-\omega^2-\sigma^2-2\sigma/\tau_p-1/\tau_p^2}{(\sigma^2+\sigma/\tau_p+\omega_p^2-\omega^2)^2+(2\sigma\omega+\omega/\tau_p)^2} . \tag{25b}$$

As (25a) shows, $\mathrm{Re}(Z_1)>0$ and the first term of $\hat{Z}(s)$ can never display negative resistance in the $s$ plane. Impedance $Z_1$ therefore has positive resistance, and is inductive when $\mathrm{Im}(Z_1)>0$. The susceptibility is positive when the square of the plasma frequency exceeds the other numerator terms in (25b).

Focus below will be directed toward the second term of $\hat{Z}(s)$, that is

$$\hat{Z}_2(s)=\hat{Z}(s)-\hat{Z}_1(s) . \tag{26}$$

$\hat{Z}(s)$ will have negative resistance only if $\hat{Z}_2(s)$ has negative resistance and it exceeds the positive resistance of $\hat{Z}_1(s)$. $\hat{Z}_2(s)$ is a nonlinear term, explicitly dependent upon the terminal current density $\hat{J}(s)$. Impedance $\hat{Z}_2(s)$ can be put in a more convenient arrangement through absorption of the denominator factors by the integrand,

$$\hat{Z}_2(s)=-\frac{q}{\epsilon}\int_{-\infty}^{\infty}A(t)C(t)dt , \tag{27}$$

$$C(t)=\frac{e^{-st}}{(s^2+s/\tau_p+\omega_p^2)\hat{J}(s)} . \tag{28}$$

Real and imaginary components of $\hat{Z}_2(s)$ are given as

$$\hat{Z}_{2r}(s)=\mathrm{Re}(\hat{Z}_2)=-\frac{q}{\epsilon}\int_{-\infty}^{\infty}A(t)\mathrm{Re}[C(t)]dt , \tag{29a}$$

$$\hat{Z}_{2i}(s)=\mathrm{Im}(\hat{Z}_2)=-\frac{q}{\epsilon}\int_{-\infty}^{\infty}A(t)\mathrm{Im}[C(t)]dt , \tag{29b}$$

because $A(t)$ is a real physical observable quantity.

Equations (29) cannot be reduced further unless $\hat{J}(s)$ is specified. Thus let a test current density be imposed on the device, consisting of a step current plus a sinusoid turned on at $t=0$:

$$J(t)=J_0u(t)+J_1\sin(\omega_0 t)u(t) , \tag{30a}$$

$$\hat{J}(s)=\frac{J_0}{s}+\frac{J_1\omega_0}{s^2+\omega_0^2} , \tag{30b}$$

$$u(t)=\begin{cases}1, & t>0\\ 0, & t\le 0 .\end{cases} \tag{30c}$$

Placing (30) into (28),

$$\mathrm{Re}[C(t)]=\frac{e^{-\sigma t}}{(e^2+f^2)(g^2+h^2)}\{[c(eg-fh)+d(fg+eh)]\cos(\omega t)+[d(eg-fh)-c(fg+eh)]\sin(\omega t)\} , \tag{31a}$$

$$\mathrm{Im}[C(t)]=\frac{e^{-\sigma t}}{(e^2+f^2)(g^2+h^2)}\{[d(eg-fh)-c(fg-eh)]\cos(\omega t)-[c(eg-fh)+d(fg+eh)]\sin(\omega t)\} . \tag{31b}$$

Here

$$c=\sigma(\sigma^2+\omega_0^2-\omega^2)-2\sigma\omega^2 , \tag{32a}$$

$$d=2\omega\sigma^2+\omega(\sigma^2+\omega_0^2-\omega^2) , \tag{32b}$$

$$e=\sigma^2+\sigma/\tau_p+\omega_p^2-\omega^2 , \tag{32c}$$

$$f=2\omega\sigma+\omega/\tau_p , \tag{32d}$$

$$g=(\sigma^2+\omega_0^2-\omega^2)J_0+\omega_0 J_1 , \tag{32e}$$

$$h=2\sigma\omega J_0 . \tag{32f}$$

Inserting the above real component formula for $C(t)$ into the integral impedance relationship (29a),

$$\hat{Z}_{2r}(s)=\frac{q}{\epsilon}\frac{1}{(e^2+f^2)(g^2+h^2)}\left[[c(eg-fh)+d(fg+eh)]\int_{-\infty}^{\infty}e^{-\sigma t}A(t)\cos(\omega t)dt \right.$$
$$\left. +[d(eg-fh)-c(fg+eh)]\int_{-\infty}^{\infty}e^{-\sigma t}A(t)\sin(\omega t)dt\right] . \tag{33}$$

A similar formula for $\hat{Z}_{2i}(s)$ can also be derived involving (29b).

Avoiding these integral calculations as in (33) is possible with a somewhat different approach to finding an impedance. The alternate approach performs a finite Fourier transformation in time over a sampling time interval. This approach is discussed in the next section.

## IV. IMPEDANCE BASED ON A FINITE FOURIER TRANSFORM

Questions of the behavior of the physical variables at infinite times can be completely avoided by using finite Fourier transform methods instead of Laplace transform methods enlisted in the last section. A transform pair is defined as

$$f(t)=\sum_{m=-\infty}^{\infty} F_m e^{jm\omega_T t}, \quad t_0<t\le t_0+T \tag{34a}$$

$$F_m=\frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-jm\omega_T t}dt , \tag{34b}$$

where

$$\omega_T=\frac{2\pi}{T} . \tag{35}$$

Notice that the physical variables are only described in the time interval starting at $t_0$ with width $T$. The sampling frequency is given by $\omega_T$ in (35).

Transform operations in the governing equation (12) require again that transforms of higher-order derivatives exist. It is easy to show that such is the case for an arbitrary physical variable $f(t)$. Take the derivatives of both sides of (34a), causing

$$g(t)=\frac{df(t)}{dt}=\sum_{m=-\infty}^{\infty}(jm\omega_T)\tilde{f}(m)e^{jm\omega_T t}$$

$$=\sum_{m=-\infty}^{\infty}\tilde{g}(m)e^{jm\omega_T t} . \tag{36a}$$

Because of the similarity of (34b) and (36a)

$$G_m=\tilde{g}(m)=\frac{1}{T}\int_{t_0}^{t_0+T} g(t)e^{-jm\omega_T t}dt , \tag{36b}$$

or

$$(jm\omega_T)^a\tilde{f}(m)=\frac{1}{T}\int_{t_0}^{t_0+T}\frac{d^a f(t)}{dt^a}e^{-jm\omega_T t}dt , \tag{37}$$

where the process has been extended to the $a$th derivative of $f(t)$.

Applying the transform operators in (34b) and (37) to the time-domain governing equation (12), produces the frequency domain relationship

$$\left[\epsilon(jm\omega_T)^2+\frac{\epsilon}{\tau_p}jm\omega_T+\frac{q^2n_d}{m^*}\right]\tilde{V}_d(m)$$

$$=\frac{q}{T}\int_{t_0}^{t_0+T}A(t)e^{-jm\omega_T t}dt$$

$$-l\left[jm\omega_T+\frac{1}{\tau_p}\right]\tilde{J}(m) . \tag{38}$$

Equation (38) could have been obtained from (21) by the assignments $\sigma\to 0$, $s\to j\omega$, and $\omega\to m\omega_T$.

Consider the development of a comprehensive spectral impedance definition through investigation of the spectral series expansions for terminal voltage and current density using (34a)

$$V_d(t)=\sum_{m=-\infty}^{\infty}\tilde{V}_d(m)e^{jm\omega_T t}$$

$$=\tilde{V}_d(0)+\sum_{m=1}^{\infty}\mathrm{Re}[\tilde{V}_d(m)e^{jm\omega_T t}] , \tag{39a}$$

$$J(t)=\sum_{p=-\infty}^{\infty}\mathrm{Re}[\tilde{J}(p)e^{jp\omega_T t}] .$$

$$=\tilde{J}(0)+\sum_{p=1}^{\infty}\mathrm{Re}[\tilde{J}(p)e^{jp\omega_T t}] . \tag{39b}$$

One-sided spectral representations of (39) hold due to the fact that $V_d$ and $J$ are real physical observables implying $\tilde{V}_d(-m)=\tilde{V}_d^*(m)$ and $\tilde{J}(-p)=\tilde{J}^*(p)$. It is therefore natural to define the $mp$th spectral impedance, constructed by measuring the $m$th spectral component of terminal voltage due to the $p$th spectral component of terminal test current density, as

$$\tilde{Z}_{mp}=-\frac{\tilde{V}_d(m)}{\tilde{J}(p)} . \tag{40}$$

Clearly, $\tilde{Z}_{mp}$ has little meaning if the $p$th spectral component of current density is zero. The $m$th-order harmonic impedance due to the $m$th-order harmonic current component is denoted by $\tilde{Z}_m=\tilde{Z}_{mm}$. In this case for $m=1$, the fundamental (or first-order) harmonic impedance $Z_1$ follows.

Recalling the frequency domain governing Eq. (38) and utilizing (40) to provide the same test and measurement harmonic frequency components for impedance,

$$\tilde{Z}_m=\frac{l}{\epsilon}\frac{\left[jm\omega_T+\frac{1}{\tau_p}\right]}{\left[-m^2\omega_T^2+j\frac{m\omega_T}{\tau_p}+\omega_p^2\right]}$$

$$-\frac{q}{\epsilon}\frac{\frac{1}{T}\int_{t_0}^{t_0+T}A(t)e^{-jm\omega_T t}dt}{\left[-m^2\omega_T^2+j\frac{m\omega_T}{\tau_p}+\omega_p^2\right]\tilde{J}(m)} . \tag{41}$$

The only change needed to convert (41) into a $\tilde{Z}_{mp}$ formula would be to multiply the first term by $\tilde{J}(m)/\tilde{J}(p)$ and replace $\tilde{J}(m)$ by $\tilde{J}(p)$ in the second-term denominator. Real and imaginary parts of the first term $\tilde{Z}_1(m)$ are

$$\mathrm{Re}[\tilde{Z}_1(m)]=\frac{l}{\epsilon}\frac{\omega_p^2/\tau_p}{(\omega_p^2-m^2\omega_T^2)^2+(m\omega_T/\tau_p)^2} , \tag{42a}$$

$$\mathrm{Im}[\tilde{Z}_1(m)]=\frac{l}{\epsilon}\frac{\omega_p^2-m^2\omega_T^2-1/\tau_p^2}{(\omega_p^2-m^2\omega_T^2)^2+(m\omega_T/\tau_p)^2} . \tag{42b}$$

Apparently, (42a) shows that $\tilde{Z}_1(m)$ cannot possess negative resistance, while the character of the susceptance is controlled by the (42b) numerator.

$\tilde{Z}_2(m)$, the nonlinear second term of (41), can be set down in the felicitous construction

$$\tilde{Z}_2(m)=-\frac{q}{\epsilon T}\int_{t_0}^{t_0+T} A(t)D(t)dt \ , \tag{43}$$

$$D(t)=\frac{e^{-jm\omega_T t}}{\left[-m^2\omega_T^2+j\frac{m\omega_T}{\tau_p}+\omega_p^2\right]\tilde{J}(m)} \ . \tag{44}$$

Real and imaginary components of $\tilde{Z}_2(m)$ are

$$\tilde{Z}_{2r}(m)=\mathrm{Re}[\tilde{Z}_2(m)]=-\frac{q}{\epsilon T}\int_{t_0}^{t_0+T} A(t)\mathrm{Re}[D(t)]dt \ , \tag{45a}$$

$$\tilde{Z}_{2i}(m)=\mathrm{Im}[\tilde{Z}_2(m)]=-\frac{q}{\epsilon T}\int_{t_0}^{t_0+T} A(t)\mathrm{Im}[D(t)]dt \ . \tag{45b}$$

Suppose the terminal test current exists for all time and is

$$J(t)=J_0+J_1\sin(\omega_0 t) \ . \tag{46}$$

The transform of (46) is

$$\tilde{J}(m)=J_0\delta_{m,0}+\frac{J_1}{2jT}\frac{1}{(\omega_0-m\omega_T)}(e^{j\omega_0 T}-1)e^{j(\omega_0-m\omega_T)t_0}+\frac{J_1}{2jT}\frac{1}{(\omega_0+m\omega_T)}(e^{-j\omega_0 T}-1)e^{-j(\omega_0+m\omega_T)t_0} \ . \tag{47}$$

When the driving frequency $\omega_0$ is not an integer multiple of the sampling frequency $\omega_T$, $\omega_0\neq p\omega_T$, all sampling spectral components $\tilde{J}(m)$ are nonzero. Moreover, these components have the very undesirable property of depending upon the initial sampling time $t_0$. Consequently, allow only $\omega_0=p\omega_T$, $p$ an integer. Then merely a few spectral components occur,

$$\tilde{J}(m)=\begin{cases} J_0, & m=0 \\ J_1/2j, & m=p \\ -J_1/2j, & m=-p \\ 0, & m=\text{otherwise} \ . \end{cases} \tag{48}$$

In what comes below, just the $p=1$ case is discussed. Potential Gibbs oscillation problems due to nonperiodicity in the nonlinear response functions $V_d(t)$ and $A(t)$ can be avoided by employing a continuous periodic $J(t)$, as in (46), provided the system responds in a continuous periodic fashion.

The real and imaginary parts of the integrand factor $D(t)$ are delineated by

$$\mathrm{Re}[D(t)]=\frac{1}{(e^2+f^2)(c^2+d^2)}[(ce-df)\cos(m\omega_T t)-(cf+de)\sin(m\omega_T t)] \ , \tag{49a}$$

$$\mathrm{Im}[D(t)]=\frac{-1}{(e^2+f^2)(c^2+d^2)}[(cf+de)\cos(m\omega_T t)+(ce-df)\sin(m\omega_T t)] \ , \tag{49b}$$

using (48) in the form

$$\tilde{J}(m)=c(m)+jd(m) \ , \tag{50}$$

and the other redefined variables

$$e(m)=\omega_p^2-(m\omega_T)^2 \ , \tag{51a}$$

$$f(m)=m\omega_T/\tau_p \ . \tag{51b}$$

Insertion of (49a) into (45a) produces the $\tilde{Z}_{2r}(m)$ impedance component for the $m$th harmonic:

$$\tilde{Z}_{2r}(m)=-\frac{q}{\epsilon T}\frac{1}{(e^2+f^2)(c^2+d^2)}\left[(ce-df)\int_{t_0}^{t_0+T} A(t)\cos(m\omega_T t)dt-(cf+de)\int_{t_0}^{t_0+T} A(t)\sin(m\omega_T t)dt\right] \ . \tag{52}$$

$\tilde{Z}_{2i}(m)$ can be expressed by a comparable formula utilizing (49b).

Despite the nonlinear quality of $A(t)$, it is possible to use a linear two-sided spectral description for $A(t)$ as in (34a) because of the finite sampling interval $T$. Such a description is employed below since it allows the analytic evaluation of the integrals contained in (52). Hence,

$$A(t)=\sum_{y=-\infty}^{\infty} A_y e^{jy\omega_T t}, \quad t_0<t\leq t_0+T \ . \tag{53}$$

A more expedient form of (53) for integral reduction is the one-sided spectral picture

$$A(t)=a_0+\sum_{y=1}^{\infty}[a_y\cos(y\omega_T t)+b_y\sin(y\omega_T t)] \ , \tag{54}$$

where $a_0=A_{0r}$, $a_y=2A_{yr}$, $b_y=-2A_{yi}$ ($r$ and $i$ subscripts indicating real and imaginary components). Placing (54) into (52),

$$\tilde{Z}_{2r}(m) = -\frac{q}{\epsilon T}\frac{1}{(e^2+f^2)(c^2+d^2)} \times \left[ (ce-df)\left[a_0 \int_{t_0}^{t_0+T} \cos(m\omega_T t)dt + (1-\delta_{m,0})a_m \int_{t_0}^{t_0+T} \cos^2(m\omega_T t)dt\right] - (1-\delta_{m,0})(cf+de)b_m \int_{t_0}^{t_0+T} \sin^2(m\omega_T t)dt \right] \quad (55)$$

comes about from the sinusoidal function orthogonality on the interval $T$. For the *dc* harmonic component, that is where $m=0$, $c(0)=J_0$, and $d(0)=0$, and (55) reduces to

$$\tilde{Z}_{2r}(0) = -\frac{qa_0}{\epsilon\omega_p^2 J_0} \quad (56)$$

drawing on (51).

For the first-harmonic component, $m=1$, $c(1)=0$, and $d(1)=-J_1/2$, and (55) manifests itself as

$$\tilde{Z}_{2r}(1) = -\frac{qa_1}{\epsilon J_1}\frac{1}{(\omega_p^2-\omega_T^2)^2+(\omega_T/\tau_p)^2} \times \left[\frac{\omega_T}{\tau_p} + \frac{b_1}{a_1}(\omega_p^2-\omega_T^2)\right] . \quad (57)$$

When

$$\omega_p \gg \omega_T,\ \sqrt{\omega_T/\tau_p}\ , \quad (58a)$$

$$|b_1| \geq |a_1|\ , \quad (58b)$$

(57) vastly simplifies to

$$\tilde{Z}_{2r}(1) \approx -\frac{qb_1}{\epsilon J_1 \omega_p^2} . \quad (59)$$

Estimates of $a_0$, and $a_1$ and $b_1/a_1$ directly provide, through (56) and (57), the *dc* and first-harmonic component impedances. If the energy difference quantity $A(t)$ has its one-sided spectral components available (at least up to the first harmonic), say from a simulation on interval $T$, then $\tilde{Z}_{2r}(0)$ and $\tilde{Z}_{2r}(1)$ can be resolved exactly. Of utmost importance is the aspect of the impedances that they do not depend on the initial sampling time $t_0$.

## V. DISCUSSION

Closed-form impedance representations were found for a two-terminal semiconductor device using infinite $[\tilde{Z}(s)]$ and finite $(\tilde{Z}_{mp})$ transform techniques. The representations are general and can in principle be evaluated exactly for semiconductor parameters and bias conditions of interest. Specification of large signal driving test terminal currents enabled explicit formulas for the impedances to be found in terms of time integrals of an energy difference function $A(t)$. Through proper sampling of the large signal device physical variables including $A(t)$ and the response terminal voltage $V_d(t)$, a very transparent harmonic impedance expression was arrived at.

The study here employed a single electron gas model. This approach is reasonable for many valleyed semiconductors provided proper care is taken during the process of reduction down to the single electron gas model. Demonstrations of such reductions are available for GaAs and InP, for example.[11,12] The use of Fourier transformation techniques to obtain impedance information from transient simulations based on one[7,12,13] and two[7] electron gas models in III-V materials are covered in recent literature. Impedance formulas similar to those derived here for electrons should also be obtainable for a single hole gas model.


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