Interpolation formula for the energy of a two-dimensional electron gas in the lowest Landau level

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The energy per particle of a two-dimensional electron system in the lowest Landau level is studied making use of the particle-hole symmetry. An accurate interpolation formula for the dependence of the energy on the filling factor is given; the formula is compared with the results of finite-size calculations performed in the spherical geometry.

One of the main problems in the theory of the fractional quantum Hall effect (FQHE) is the determination of the ground-state energy E per particle as a function of the filling factor ν. According to current-day theories, cusps in $E(\nu)$ are needed in order to explain the plateaus of the Hall conductivity. Furthermore, if we denote by $E_T(\nu)$ the energy per particle of the lowest translationally invariant state, and by $E_{\rm WC}(\nu)$ (WC denotes Wigner crystal) the energy per particle of the lowest state having crystalline order, it is generally believed[1,2] that there exists a liquid-solid transition below a certain value ν_c of ν, i.e., $E_{\rm WC}(\nu)<E_T(\nu)$ for $\nu<\nu_c$.

A crude empirical formula for the energy $E(\nu)$ has been suggested by Laughlin.[1] Assuming e^2/a_0 to be the energy unit, where $a_0=(\hbar c/eB)^{1/2}$ is the magnetic length, Laughlin's formula reads

$$E(\nu)=0.814\sqrt{\nu}(0.23\nu^{0.64}-1)\ . \tag{1}$$

Levesque, Weis, and MacDonald[3] have proposed a formula that gives a better interpolation of the energy of the Laughlin states:

$$E(\nu)=-0.782\,133\sqrt{\nu}(1-0.211\nu^{0.74}+0.012\nu^{1.7})\ . \tag{2}$$

However, none of these formulas satisfies the relation

$$\nu[E(\nu)-E(1)]=(1-\nu)[E(1-\nu)-E(1)] \tag{3}$$

which follows from the particle-hole symmetry of the first Landau level.[4] Furthermore, these formulas give poor results for $\nu=\frac{1}{2}$. Formula (1) gives $E(\frac{1}{2})=-0.49$ and formula (2) gives $E(\frac{1}{2})=-0.485$, while a recent result is[5] $E(\frac{1}{2})=-0.469\pm0.005$.

The proof of the relation (3) holds for all popular geometries that have been used in the theory of the FQHE, viz., the square with periodic boundary conditions, the sphere, or the disk in the limit of a large number of electrons.

In our numerical calculations we have used the spherical geometry.[6,7] Let N denote the number of electrons, and let $2S+1$ denote the number of available single-particle states on the sphere. It can be proved that for the case of a completely filled Landau level ($N=2S+1$) the energy per particle is given by

$$E_S(1)=-\frac{2^{2S-1}}{\sqrt{S}}\frac{(2S+1)!}{(4S+1)!!}\ . \tag{4}$$

In the case of the spherical geometry the particle-hole symmetry relation (3) holds with $\nu=N/(2S+1)$, and $E(1)$ substituted by $E_S(1)$. Of course taking the thermodynamic limit of relation (4) we recover the well-known result

$$E(1)=\lim_{S\to\infty}E_S(1)=-(\pi/8)^{1/2}\ . \tag{5}$$

From (3) it follows that the function

$$G(\nu)=\nu[E(\nu)-E(1)]-E(1)\nu(\nu-1)=\nu[E(\nu)-\nu E(1)] \tag{6}$$

is symmetric with respect to the exchange $\nu\leftrightarrow1-\nu$; therefore it is reasonable to try an expansion of the form

$$G(\nu)=\sum_{k=0}^{\infty}\alpha_k[\nu(1-\nu)]^{k/2}\ . \tag{7}$$

We have

$$\begin{aligned}E(\nu)&=G(\nu)/\nu+E(1)\nu\\&=E(1)\nu+\alpha_0/\nu+\alpha_1\nu^{-1/2}(1-\nu)^{1/2}+\alpha_2(1-\nu)\\&\quad+\alpha_3\nu^{1/2}(1-\nu)^{3/2}+\cdots\ .\end{aligned} \tag{8}$$

On the other hand, we know that for small ν the system behaves like a Wigner crystal; thus for small ν the energy approaches the expression[8] $E(\nu)\sim-0.782\,133\nu^{1/2}$. We set $\alpha_0=\alpha_1=\alpha_2=0$ and $\alpha_3=-0.782\,133$, and we obtain the following zero-order approximation to the energy per particle:

$$E_0(\nu)=-(\pi/8)^{1/2}\nu-0.782\,133\nu^{1/2}(1-\nu)^{3/2}\ , \tag{9}$$

which does not contain any free parameter. This formula has the correct behavior near $\nu=0$ and $\nu=1$. However, it does not produce accurate results when ν lies in the middle of the range $0<\nu<1$. In order to improve the approximation, we add two terms of the expansion (8), i.e., we consider the correction:

$$E_1(\nu)=\alpha_4\nu(1-\nu)^2+\alpha_5\nu^{3/2}(1-\nu)^{5/2} . \tag{10}$$

The expression

$$E(\nu)=E_0(\nu)+E_1(\nu) \tag{11}$$

with suitably chosen values for α_4 and α_5 fits with good accuracy all data known on FQHE states. For instance, let us consider the following set of very accurate results:[3,9,10]

$$\nu=\tfrac{1}{3},\quad E=-0.409\,73$$
$$\nu=\tfrac{1}{5},\quad E=-0.3277 \tag{12}$$
$$\nu=\tfrac{2}{5},\quad E=-0.431 .$$

We make a least-squares fit of these three data in order to determine α_4 and α_5. The results are

$$\alpha_4=0.55 \text{ and } \alpha_5=-0.463 . \tag{13}$$

We can compare the energy curve $E_0(\nu)+E_1(\nu)$ so obtained with our finite-size calculations on the sphere (N electrons on a sphere of radius R). Since the polynomial part of the Laughlin and Halperin wave functions [whose energies are given by (12)] are translational invariant, and since translational invariance in the plane maps to rotational invariance on the sphere,[7] we limit ourselves to a set of states on the sphere with a total angular momentum of zero. This set of states and the corresponding energies are shown in Table I. In Table I we denote by E_c the value of the energy multiplied by the size correction factor[10] $f=(\rho_N/\rho_\infty)^{-1/2}$, where $\rho_N=N/(4\pi R^2)=N/(4\pi S)$ is the areal density and

$$\rho_\infty=(1/2\pi)[N/(2S+1)]=\nu/2\pi .$$

To give an idea of the accuracy of the finite-size calculations on the sphere, we recall that the value of $E(\frac{1}{3})$, computed by putting ten electrons on the sphere, is $-0.410\,628$ (see Ref. 7); this result is to be compared with the value (12), obtained by means of Monte Carlo calculations.[9]

TABLE I. Ground-state energy values for N electrons on the sphere.

N	$2S+1$	ν	E_c
6	36	0.142 857[a]	−0.2807
6	26	0.230 769	−0.3517
8	32	0.25	−0.3619
6	22	0.272 727	−0.3782
9	25	0.36	−0.4268
8	22	0.363 636	−0.4290
6	14	0.428 571	−0.4474
10	22	0.454 545	−0.4587
8	17	0.470 588	−0.4659
12	24	0.5	−0.472

[a]The case $N=6$, $2S+1=36$ corresponds to a Laughlin state with $\nu=\frac{1}{7}$. Consequently, we have computed the size-correction factor using the value $\rho_\infty=(1/2\pi)\frac{1}{7}$.

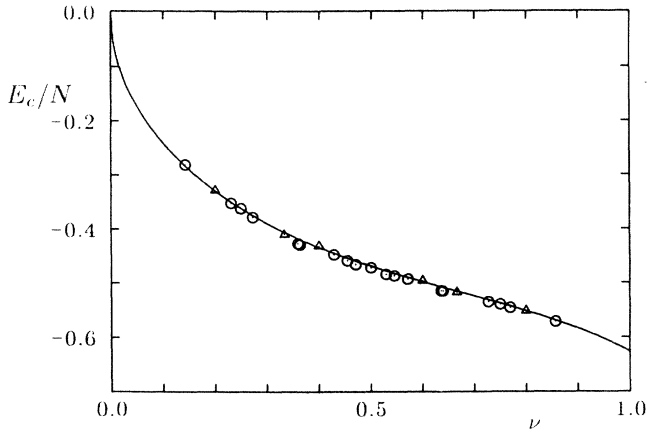

FIG. 1. Ground-state energy per particle vs ν. The curve corresponds to the interpolation formula $E=E_0+E_1$, with $\alpha_4=0.55$ and $\alpha_5=-0.463$. Circles are for $\nu=\frac{1}{3}$, $\frac{1}{5}$, and $\frac{2}{5}$ [see Eq. (12)] and triangles for the values of ν given in Table I.

In Fig. 1 it is shown that the energy curve (11) with $\alpha_4=0.55$ and $\alpha_5=-0.463$ and the data of Table I are in good agreement. The particle-hole symmetry relation (3) is used in order to obtain the whole set of data, i.e., also those for $\nu>\frac{1}{2}$.

Of course, we can improve our least-squares fit by taking into account all the data (i.e., the three values for $\nu=\frac{1}{3}$, $\frac{1}{5}$, and $\frac{2}{5}$ *and* those of Table I). The resulting values of α_4 and α_5 are

$$\alpha_4=0.683 \text{ and } \alpha_5=-0.806 . \tag{14}$$

Figure 2 shows the energy curve (11) corresponding to these values of α_4 and α_5, together with the data of the numerical calculations.

So far we have used only energy values corresponding

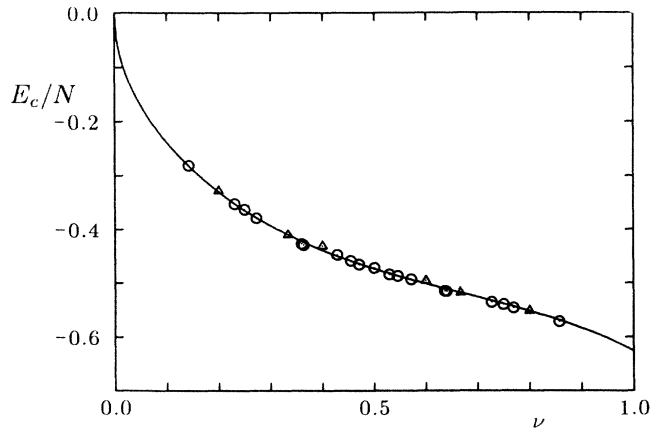

FIG. 2. Same as Fig. 1 with $\alpha_4=0.683$ and $\alpha_5=-0.806$.

to translationally invariant states. On the other hand, it is known that, at least in the Hartree-Fock approximation, the following formula,[3]

$$E_{\mathrm{CDW}} = -0.782\,133\sqrt{\nu}(1-0.372\nu-0.013\nu^2)\ , \quad (15)$$

reproduces quite well the energies of a charge-density wave with hexagonal symmetry for $\nu < \frac{1}{3}$. In Ref. 3 it was found that curves (2) and (15) cross at $\nu \sim 0.1$; this result holds unchanged using our interpolation formula in the place of Eq. (2). However, this result should not be taken as a true prediction of the ν value corresponding to the liquid-solid transition; indeed our zero-order approximation (9) behaves like a Wigner crystal for small ν, and the expression (15) does not take into account the correlation energy. Our formula $E = E_0 + E_1$ must be considered as a sort of "backbone energy" of the FQHE states. It might be of some interest to know that the energies of all the translational invariant states that we have considered fulfill our formula, and no macroscopic cusp appears.

Of course, more detailed questions about the existence of small cusps cannot be answered at this level of approximation.

Finally, we notice that there is a slight inconsistency in the derivation of our formula, since all the data employed in order to determine the coefficients α_4 and α_5 are obtained from variational wave functions corresponding to different finite-size systems; therefore there is not a single value of $E(1)$ which fulfills the particle-hole relation (3) for all these systems. However, it can be seen that the consequent numerical error is negligible. For instance, in the case of the spherical geometry, we can use $E(1) = -(\pi/8)^{1/2}$ in the place of $E_S(1)$ due to the rapid convergence of the sequence (4). Similar arguments hold for the Monte Carlo calculations by Morf *et al.*[9,10]

[1]R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[2]S. M. Girvin, in *The Quantum Hall Effect,* edited by R. E. Prange and S. M. Girvin (Springer-Verlag, Berlin, 1986).
[3]D. Levesque, J. J. Weis, and A. H. MacDonald, Phys. Rev. B **30**, 1056 (1984).
[4]E. Tosatti, M. Parrinello, and C. D. Chen, in *Focal Points in Theoretical Condensed Matter Physics,* edited by F. Bassani, F. Fumi, and M. P. Tosi (Academic, New York, 1985), p. 453.
[5]G. Fano, F. Ortolani, and E. Tosatti, Nuovo Cimento **9D**, 1337 (1987).
[6]F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).
[7]G. Fano, F. Ortolani, and E. Colombo, Phys. Rev. B **34**, 2670 (1986).
[8]L. Bonsdall and A. A. Maradudin, Phys. Rev. B **15**, 1959 (1977).
[9]R. Morf and B. J. Halperin, Z. Phys. B **68**, 391 (1987).
[10]R. Morf, N. d'Ambrumenil, and B. I. Halperin, Phys. Rev. B **34**, 3037 (1986).