Comments

Comments are short papers which comment on papers of other authors previously published in the **Physical Review**. *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Comment on the conductivity exponent in continuum percolation

J. Machta
Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01003
(Received 20 April 1987)

The field theory introduced by Lubensky and Tremblay [Phys. Rev. B **34**, 3408 (1986)] for continuum percolation is reanalyzed. Dynamical exponents are found which agree with those found by Straley [J. Phys. C **15**, 2343 (1982)] and Machta *et al.* [Phys. Rev. B 33, 4818 (1986)] using a nodes-link-blobs approach.

Several investigators[1-4] have recently studied the dynamical exponents for percolation networks where the bonds of the network are chosen from a broad distribution. An example of this kind of problem is a diluted resistor network with a broad distribution of resistances. Suppose that a fraction p of bonds have finite resistance and that these resistors are chosen from a distribution having a power-law tail such that

$$\text{prob}(R>X)\sim X^{-\alpha},\ X\to\infty\ . \tag{1}$$

Here R is any of the finite resistors in the network and $0<\alpha<1$ characterizes the tail of the distribution.[5] The recent interest in this problem stems from the observation[1] that the dynamical properties of continuum percolation systems are governed by distributions satisfying Eq. (1).

It is generally agreed[1-4,6] that as $\alpha\to 1$ the conductivity exponent t approaches its universal value t_0. When α is sufficiently small, t is nonuniversal and behaves like

$$t=t(\alpha)=(d-2)\nu+1/\alpha\ , \tag{2}$$

where ν is the correlation-length exponent. The crossover from t_0 to $t(\alpha)$ is the subject of controversy in the literature. On the one hand, investigations[2,6] based upon the nodes-links-blobs picture of the incipient infinite cluster yield

$$t=\max[t_0,t(\alpha)]\ , \tag{3}$$

so that there is a crossover from universal to nonuniversal behavior when $\alpha=\alpha_c$ such that $t_0=t(\alpha_c)$. The nodes-links-blobs analysis thus predicts that the broad distribution of bonds is irrelevant for $\alpha_c<\alpha<1$. On the other hand, a field-theoretic ϵ expansion due to Lubensky and Tremblay[4] (LT) predicts that $t>t_0$ for all $\alpha<1$ and that there is a crossover to $t=t(\alpha)$ at a value of α smaller than α_c.

In this Comment, I reanalyze the renormalization-group flows obtained by Lubensky and Tremblay and argue that the field theory is in agreement with the nodes-links-blobs results, Eq. (3). The starting point is the one-loop recursion relations for the parameters v and w given in LT as Eqs. (29)-(31):

$$\frac{dv}{dl}=\frac{v}{\nu}\ , \tag{4}$$

$$\frac{dw}{dl}=\frac{w}{\nu}-\tfrac{1}{4}u_3^2E\ , \tag{5}$$

where

$$E=\int_0^\infty\frac{dp}{p}\,\frac{4\alpha^2v^2(p^2)^{2\alpha-1}+4w^2p^2-8\alpha vwp^{2\alpha}}{(1+vp^{2\alpha}-wp^2)^4}\ . \tag{6}$$

ν is the correlation-length exponent, and u_3 is the coupling constant whose fixed-point value to leading order in $\epsilon=6-d$ is $(u_3^*)^2=2\epsilon/7$. The parameters v and w describe the distribution function of the resistances. In particular, v describes the amplitude of the power-law tail of this distribution and must be non-negative. If there is no power-law tail v vanishes and $-w$ is the mean of the distribution, in which case w must be negative. On the other hand, when $v>0$, w may take either sign and has no straightforward interpretation.

Equation (6) leads to nonlinear couplings between v and w. To simplify the calculation, LT introduce a new field $g=wv^{-1/\alpha}$ and carry out a small g expansion of the recursion relations. The present approach is based instead upon a large-g expansion of the recursion relations. The appropriateness of this expansion will be discussed at the end of the Comment.

I define $h=(-g)^{-\alpha}=(-w)^{-\alpha}v$, change variables of integration to $z=wp^2$, and expand Eq. (6) in powers of h

holding w fixed. To linear order in h the result is

$$E=2w\int_C dz\frac{1}{(1-z)^4}-4whe^{i\pi\alpha}\int_C dz\left[\frac{2z^\alpha}{(1-z)^5}+\frac{\alpha z^{\alpha-1}}{(1-z)^4}\right]. \quad (7)$$

If $w>0$ the contour of integration C is along the positive real axis and if $w<0$ the contour is along the negative real axis. If $w>0$ the integrals in Eq. (7) do not exist because of the pole at $z=1$. However, LT argue that this is the result of truncating the k expansion of the quadratic coupling constant in the field theory and that the inclusion of higher-order terms would move the pole off the axis. Assuming the pole is moved to $1+i\eta$ the integrals can be done and are independent of the sign of w,

$$E=-\tfrac{2}{3}w-wh\Gamma(1+\alpha)\Gamma(4-\alpha)\ . \quad (8)$$

Combining Eqs. (4), (5), and (8) one obtains recursion relations: for h and w

$$\frac{d\ln w}{dl}=\frac{\phi_0}{\nu}+\tfrac{1}{4}u_3^2h\Gamma(1+\alpha)\Gamma(4-\alpha) \quad (9)$$

and

$$\frac{d\ln h}{dl}=\frac{1}{\nu}(1-\phi_0\alpha)-\frac{\alpha}{4}u_3^2h\Gamma(1+\alpha)\Gamma(4-\alpha)\ , \quad (10)$$

where $\phi_0=1+\nu(u_3^*)^2/6$.

There are two nontrivial fixed points of these recursion relations. The first is at $w^*=h^*=0$. This is the universal fixed point and is stable for $\phi_0>1/\alpha$. The second, α-dependent, fixed point is at $w^*=0$ and $h^*=[4(1/\alpha-\phi_0)]/[\nu(u_3^*)^2\Gamma(1+\alpha)\Gamma(4-\alpha)]$ and is stable for $\phi_0<1/\alpha$. These fixed points have the same stability and eigenvalues as the pair of fixed points found in the renormalization-group treatment of a hierarchical nodes-links-blobs model.[2] In Ref. 2, the parameters describing the distribution of resistors are A and B with $B=-w$ and $(A/B)^\alpha=h$ to linear order in w and h.

The stable fixed point governs the critical behavior of the conductivity and the exponent t is obtained from the largest eigenvalue of the linearized flow near the stable fixed point. The method is described in Sec. IV B of LT. The result is that $t=t_0=(d-2)\nu+\phi_0$ when $\alpha>\alpha_c$ and $t=t(\alpha)$ when $\alpha<\alpha_c$ in agreement with Eq. (3) and the nodes-links-blobs analysis.

Near the exchange in stability of the two fixed points h^* is small for both fixed points justifying the use of the small h expansion in investigating the crossover from universal to α-dependent behavior. Physically, this is the statement that near the crossover point the tail of the distribution of resistors renormalizes to a small value. To obtain the dependence of t on α in the whole range, $0<\alpha<1$, I assume that away from the crossover region the qualitative features of the renormalization-group flow persist. Specifically, I suppose that there are two fixed points; the universal $(w^*=h^*=0)$ fixed point stable for $\alpha>\alpha_c$ and a finite h^* fixed point stable for $\alpha<\alpha_c$. The eigenvalues of the finite h^* fixed point are more easily obtained using the variables v and h from which it follows that $t=t(\alpha)$ whenever this fixed point is stable. For α sufficiently less than α_c, h^* is expected to be large so that the small g analysis of LT holds and one again recovers the result $t=t(\alpha)$. However, the small g expansion cannot be used to study the regions $\alpha\approx\alpha_c$ or $1<\alpha<\alpha_c$ since g is large or infinite here. I conclude that the field theory,[4] when properly analyzed, is in agreement with the nodes-links-blobs analysis.[2,6]

I am pleased to acknowledge A.-M.S. Tremblay and T. Lubensky for useful discussions. This work was supported by National Science Foundation Grant No. DMR 83-17442.

[1]B. I. Halperin, S. Feng, and P. N. Sen, Phys. Rev. Lett. **54**, 2391 (1985); Phys. Rev. B **35**, 197 (1987).
[2]J. Machta, R. A. Guyer, and S. M. Moore, Phys. Rev. B **33**, 4818 (1986).
[3]P. N. Sen, J. N. Roberts, and B. I. Halperin, Phys. Rev. B **32**, 3306 (1985).
[4]T. C. Lubensky and A.-M. S. Tremblay, Phys. Rev. B **34**, 3408 (1986).
[5]"α" is defined as "$1-\alpha$" of Ref. 4.
[6]J. P. Straley, J. Phys. C **15**, 2333 (1982); **15**, 2343 (1982).