Energy flow during the soliton-antisoliton interaction in extended Klein-Gordon systems

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The energy flow during the soliton-antisoliton interaction in an extended Klein-Gordon system is investigated by means of a state-plane technique. The energy flow is divided into traveling-wave and standing-wave components, and the energy-transfer mechanism between them is discussed. Numerical examples are given for the extended sine-Gordon system, and are compared with the pure sine-Gordon system. Finally, the effect of local distortions produced during the interaction on the energy flow is clarified.

The solutions in dissipative, externally driven, nonlinear Klein-Gordon systems, which are referred to as extended Klein-Gordon systems, play a crucial role for many phenomena appearing in condensed-matter physics. It was shown by Tateno that the exact behavior of the soliton-antisoliton interaction in one-dimensional extended Klein-Gordon systems can be treated by using a state-plane technique.[1,2] Using this technique we can transform the systems directly into expressions of their energy balance. Moreover, the solution for ϕ_x and ϕ_t, where the subscripts represent derivatives, can be divided into the traveling-wave and the standing-wave components. Thus, there is the possibility of investigating the energy exchange mechanism between them during the interaction. In this paper, we derive the expressions on such an energy exchange mechanism using the state-plane technique, and then apply them to the extended sine-Gordon system in comparison with the pure sine-Gordon system, clarifying the effect on the energy flow of local distortions appearing during the interaction.

We treat the systems described by

$$\phi_{xx}-\phi_{tt}-F(\phi)=G\phi_t-J_B \ , \tag{1}$$

where G is the dissipation coefficient, and J_B the uniformly applied external driven force. Every quantity of notations in Eq. (1) is normalized by a certain unit quantity. The exact behavior of solutions to Eq. (1) can be determined numerically under the condition that the solution approaches asymptotically the stationary solitary-wave one as $|x|$ goes to infinity.[1,2] In the state-plane technique, we consider that ϕ is determined by designating x and t, and then state variables $\Pi(x,t)$ concerned with Eq. (1) such as ϕ_x,ϕ_t, and so on are determined by designating ϕ. In order that $\Pi(x,t)$ may be a state variable, that is, in a functional relation with ϕ, the following condition should be sustained:

$$\frac{\partial(\Pi,\phi)}{\partial(x,t)}=0 \ , \tag{2}$$

that is,

$$\frac{\partial\Pi}{\partial\phi}=\frac{\Pi_x}{\phi_x}=\frac{\Pi_t}{\phi_t} \ . \tag{3}$$

From (3) we obtain

$$\Pi_x=\phi_x\frac{\partial\Pi}{\partial\phi},\ \Pi_t=\phi_t\frac{\partial\Pi}{\partial\phi} \ . \tag{4}$$

We rewrite Eq. (1) using Eq. (4) in the form of the derivative of ϕ, and then integrate the result with respect to ϕ. As a result, we obtain the energy balance relation as

$$\tfrac{1}{2}\phi_x^2-\tfrac{1}{2}\phi_t^2=\int[F(\phi)+G\phi_t-J_B]d\phi \ . \tag{5}$$

It is convenient to use the transmission line equivalent to Eq. (1) as shown in Fig. 1 (Ref. 3), where G represents the shunt conductance per unit length, $F(\phi)$ the nonlinear nondissipative shunt element per unit length, and the inductance and capacitance per unit length are normalized to unity, respectively. Then, $-\phi_x$ represents the current along the line, and ϕ_t the voltage between the line.

The power flow $p(x,t)$ of the transmission line is given by

$$p(x,t)=\frac{\partial}{\partial t}\int_{-\infty}^{x}(\tfrac{1}{2}\phi_{x'}^2+\tfrac{1}{2}\phi_t^2)dx' + \int_{-\infty}^{x}\phi_t[G\phi_t+F(\phi)-J_B]dx' \ . \tag{6}$$

The energy flow is defined by integration of $p(x,t)$ with respect to t from $t=-\infty$ to t. The energy density $\mathcal{H}(x,t)$ is obtained by differentiating the energy flow with respect to x:

$$\mathcal{H}(x,t)=\tfrac{1}{2}\phi_x^2+\tfrac{1}{2}\phi_t^2+\int_{\phi_0}^{\phi}[G\phi_t+F(\phi)-J_B]d\phi \ , \tag{7}$$

where ϕ_0 is ϕ at $t=-\infty$. We can rewrite Eq. (7) using Eq. (5) as

$$\mathcal{H}(x,t)=\phi_x^2 \ . \tag{8}$$

Thus, it is seen that the energy density at x is twice the energy density supplied to the inductance element in Fig. 1.

We can choose the solution between $\phi=\phi_{0,2n-2}$ and $\phi_{0,2n+2}$ when we treat two-soliton problems, where n is an integer denoting the position of the singular point at $|\xi|=|x-ut|$ being infinity. The energy $\varepsilon^{(c)}(x,t)$ from $x'=-\infty$ up to x is then given from Eq. (8) by

$$\varepsilon^{(c)}(x,t)=\int_{-\infty}^{x}\mathcal{H}(x',t)dx'=\int_{\phi_{0,2n}}^{\phi}\phi_x d\phi \ . \tag{9}$$

If we take directionality into consideration, the net energy $\varepsilon^{(c)}(\infty,t)$ becomes always zero as expected, because of $\phi=\phi_{0,2n}$ at $x=\pm\infty$.[2]

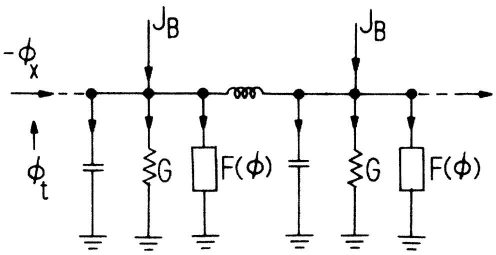

FIG. 1. Equivalent transmission line to extended Klein-Gordon systems.

We can set ϕ_t and ϕ_x as

$$\phi_t = V(x,t)g(t), \quad \phi_x = -V(x,t)h(x)/u \ , \tag{10}$$

where $V(x,t)$ is a state variable denoting the traveling-wave component of ϕ_t, and $g(t)$ and $h(x)$ are the other variables associated with nonlinear coordinates $T(x,t)$ and $X(x,t)$ defined by[1,2]

$$T(x,t) = \int^t g(t')dt' + T_0(x) \ , \tag{11}$$

$$X(x,t) = \int^x h(x')dx' + X_0(t) \ , \tag{12}$$

where $T_0(x)$ and $X_0(t)$ are arbitrary functions of x and t associated with singularities.[1,2] We differentiate Eq. (9) with respect to t, regarding x as a function of t with the relation $dx/dt = -\phi_t/\phi_x$. The result shows the power flow, which is another expression of Eq. (6):

$$p(x,t) = -\phi_x\phi_t \ . \tag{13}$$

Thus, the net energy flow $\varepsilon^{(n)}$ at a position x_0 is expressed by

$$\varepsilon^{(n)} = \int_{-\infty}^{+\infty} p(x_0,t)dt = -\int_{\phi_{0,2n-2}}^{\phi_{0,2n+2}} \phi_x d\phi \ . \tag{14}$$

ϕ_x is also expressed by $\phi_x = (\phi_x)^{(t)} + (\phi_x)^{(r)}$, where the superscripts (t) and (r) denote the traveling-wave component and the standing-wave component, respectively, $(\phi_x)^{(t)} = -(\phi_t)^{(t)}/u$, $(\phi_t)^{(t)} = V(x,t)$, $(\phi_x)^{(r)} = [h(x) - 1](\phi_x)^{(t)}$, and $\Xi(x,t) = X(x,t) - uT(x,t)$.

$V(\Xi)$ satisfies the following equation[1,2] which is another form of Eq. (1):

$$\frac{\partial V}{\partial \phi} = \frac{u^2}{1-u^2}\,\frac{F'(\Xi) + GV(\Xi) - J_B}{V(\Xi)} \ , \tag{15}$$

where

$$F'(\Xi) = F(\phi) - (\phi_x)_x^{(r)} + (\phi_t)_t^{(r)} + (\Xi_x - 1)V_\Xi/u + (\Xi_t + u)V + G(\phi_t)^{(r)} \ .$$

$$(\phi_t)^{(r)} = [g(t) - 1]V(\Xi) \ .$$

If we take account of $V(\Xi) = 0$ at $\phi = \phi_{0,2n\pm 2}$, we obtain the traveling-wave component of $\varepsilon^{(n)}$, $\varepsilon^{(t)}$, from Eq. (15) as[1,2]

$$\varepsilon^{(t)} = -\int_{\phi_{0,2n-2}}^{\phi_{0,2n+2}} (\phi_x)^{(t)} d\phi = \frac{2J_B(\phi_{0,2n} - \phi_{0,2n-2})}{uG} \ , \tag{16}$$

where

$$\int_{\phi_{0,2n-2}}^{\phi_{0,2n+2}} F'(\Xi)d\phi = \int_{\phi_{0,2n-2}}^{\phi_{0,2n+2}} F(\phi)d\phi = 0 \ . \tag{17}$$

In Eq. (16), if $u < 0$, u is replaced by $-u$, that is, the directionality of the energy flow has not been taken into consideration. It is seen from Eq. (16) that $\varepsilon^{(t)}$ is independent of x. Equation (16) also indicates that the energy dissipation of the traveling-wave component due to G is always compensated by the external energy supply by J_B throughout ϕ from $\phi_{0,2n-2}$ to $\phi_{0,2n+2}$. Thus, the traveling-wave component is always sustained. From Eq. (15) the net energy loss $\varepsilon^{(1)}$ is written as

$$\varepsilon^{(1)} = \varepsilon^{(t)} + \varepsilon^{(r)} = \frac{1}{u}\int_{\phi_{0,2n-2}}^{\phi_{0,2n+2}} \phi_t d\phi \ , \tag{18}$$

where $\phi_t = (\phi_t)^{(t)} + (\phi_t)^{(r)}$ and $(\phi_t)^{(r)} = [g(t) - 1]V(\Xi)$. Since the term $G(\phi_t)^{(r)}$ in $F'(\Xi)$ describes the dissipation due to the standing-wave component, one can express $\varepsilon^{(r)}$ from Eq. (17) as

$$\varepsilon^{(r)} = \frac{1}{u}\int_{\phi_{0,2n-2}}^{\phi_{0,2n+2}} (\phi_t)^{(r)} d\phi = -\frac{1}{uG}\int_{\phi_{0,2n-2}}^{\phi_{0,2n+2}} F''(\Xi)d\phi \ , \tag{19}$$

where

$$F''(\Xi) = -(\phi_x)_x^{(r)} + (\phi_t)_t^{(r)} + (\Xi_x - 1)V_\Xi/u + (\phi_t + u)V_\Xi \ .$$

Equation (19) indicates that the net energy dissipation of the standing-wave component due to G is just compensated by the energy supply presented by the integration of $-F''(\Xi)$ with respect to ϕ from $\phi_{0,2n-2}$ to $\phi_{0,2n+2}$. Thus, $\varepsilon^{(r)}$ constructs the standing-wave component. Since $\varepsilon^{(r)}$ is associated with the opposite wave to the traveling wave, the net energy flow $\varepsilon^{(n)}$ is represented by

$$\varepsilon^{(n)} = \varepsilon^{(t)} - \varepsilon^{(r)} \ . \tag{20}$$

By comparing the term on $(\phi_x)^{(r)}$ in Eq. (14) with $\varepsilon^{(r)}$ in Eq. (20), we obtain the average value of $g(t)$, $\langle g(t)\rangle$ related to $h(x)$ as follows:

$$\langle g(t)\rangle = \int_{\phi_{0,2n-2}}^{\phi_{0,2n+2}} g(t)V(\Xi)d\phi \Big/ \int_{\phi_{0,2n-2}}^{\phi_{0,2n+2}} V(\Xi)d\phi = 2 - h(x) \ , \tag{21}$$

which also represents the ratio of $\varepsilon^{(1)}$ to $\varepsilon^{(t)}$. With decreasing $|x|$ from infinity, $\langle g(t)\rangle$ is increased monotonically from 1, because $h(x)$ is monotonically decreased from 1, and as $|x|$ approaches 0, $\langle g(t)\rangle$ approaches 2, because of $h(0) = 0$.

The relation between ϕ and ϕ_t is calculated from an ordinary differential equation equivalent to Eq. (1).[1,2] As an example, we treat the extended sine-Gordon system, i.e., $F(\phi) = \sin\phi$. Numerical analysis is made for $J_B = 0.1$, 0.4, and 0.6, respectively, where G is fixed at 0.018. For comparison with the result for $J_B = 0.4$, the pure sine-Gordon system, i.e., $J_B = G = 0$, is first treated for $u = 0.99845$, which is in agreement with the result for $J_B = 0.4$. The integration of ϕ_t with respect to ϕ is made to obtain the net dissipation energy, and then its traveling-wave component ε is calculated from Eq. (10) as marked by black triangles in Fig. 2. As expected, we see that ε does not depend upon the value of $|x|$. On the analogy of Eq. (16) that $\varepsilon^{(t)}$ is twice the value for the corresponding stationary traveling wave,[4] we can set

$$\varepsilon = 16/(1-u^2)^{1/2} , \qquad (22)$$

which is twice the energy of a stationary soliton in the pure sine-Gordon system,[5] and is shown by the broken line. The traveling-wave component for $J_B = 0.4$ is depicted by the solid line with black circles, and is in agreement with the result calculated from Eq. (16), i.e.,

$$\varepsilon^{(t)} = 4\pi J_B/uG . \qquad (23)$$

$\varepsilon^{(t)}$ is divided into two, i.e., $\varepsilon_-^{(t)}$ in $t<0$ and $\varepsilon_+^{(t)}$ in $t>0$, where their centers collide at $(x,t)=(0,0)$. Then, we treat a situation that in $t<0$ they are approaching each other, and in $t>0$ they are going away from each other. At the position where $|x|$ is larger than about 0.3, $\varepsilon_-^{(t)}$ is regarded as almost equal to $\varepsilon_+^{(t)}$ so that they are in the stationary state, where the singularity at $|t| \to \infty$ is a saddle point. With decreasing $|x|$, the interaction between the soliton and the antisoliton is stronger. As a result, $\varepsilon_-^{(t)}$ is decreased and $\varepsilon_+^{(t)}$ is increased, where the effect of the local distortion is significant. However, it disappears when $|x|$ is smaller than the value corresponding to boundary of a saddle point and a node which are singularity at $|x| \to \infty$. In any case, the following relation is always preserved:

$$\varepsilon^{(t)} = \varepsilon_-^{(t)} + \varepsilon_+^{(t)} = \text{const} .$$

As $|x|$ decreases more, the value of $\varepsilon_\pm^{(t)}$ becomes constant. Next, notice the standing-wave component $\varepsilon^{(r)}$, which is the sum of two components, i.e., $\varepsilon_-^{(r)}$ in $t<0$ and $\varepsilon_+^{(r)}$ in $t>0$. $\varepsilon^{(r)}$ increases from zero with decreasing $|x|$ and becomes equal to $\varepsilon^{(t)}$ at $x=0$, where the net energy flow $\varepsilon^{(n)}$ disappears. The condition $\varepsilon_-^{(r)} > \varepsilon_+^{(r)}$ is always preserved so that a decrease of $\varepsilon_-^{(t)}$ brings an increase of $\varepsilon_-^{(r)}$ and the increase of $\varepsilon_+^{(t)}$ brings the decrease of $\varepsilon_+^{(r)}$.

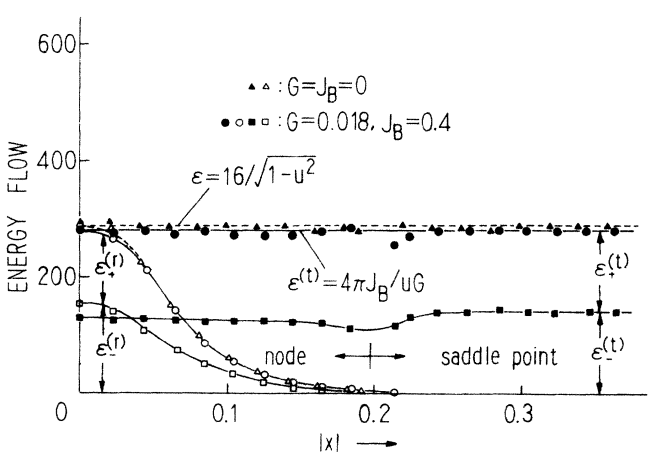

FIG. 2. Comparison of energy flows between the extended sine-Gordon system for $G=0.018$ and $J_B=0.4$ and the pure sine-Gordon system, where $u=0.99845$ for both cases. Every quantity of notations in the figure is normalized by a certain unit quantity.

The results for $J_B=0.1$ and 0.6 are depicted in Figs. 3(a) and 3(b), respectively. For $J_B=0.1$, the values of $\varepsilon^{(t)}$ are closer to the value of ε than for $J_B=0.4$, and the values of $\varepsilon_+^{(t)}$ are almost in agreement with the ones of $\varepsilon_-^{(t)}$, irrespective of the value of $|x|$. For $J_B=0.6$, the difference between the values of $\varepsilon_+^{(t)}$ and $\varepsilon_-^{(t)}$ becomes more pronounced. Naturally, the difference between the values of $\varepsilon_-^{(r)}$ and $\varepsilon_+^{(r)}$ becomes also pronounced.

The local distortion takes a form of the wedge-shaped distortion or the thorn-shaped distortion depending upon whether it appears before or after the collision of the center.[1,2] In the case of Fig. 3(a), the local distortion becomes considerably weakened compared with the case for Fig. 2, because of the much smaller value of J_B than for Fig. 2. Thus, $\varepsilon_+^{(t)}$ is almost equal to $\varepsilon_-^{(t)}$. In Fig. 3(b), the distortion becomes stronger than for the case of Fig. 2, because of the larger value of J_B. From these examples, it is understood that the occurrence of the wedge-shaped distortion in $t<0$ acts to weaken the traveling-wave component, and that the thorn-shaped distortion $t>0$ acts to strengthen it by the same amount lost in $t<0$ to keep $\varepsilon^{(t)}$ constant. The degree of them depends upon the magnitude of their distortion.

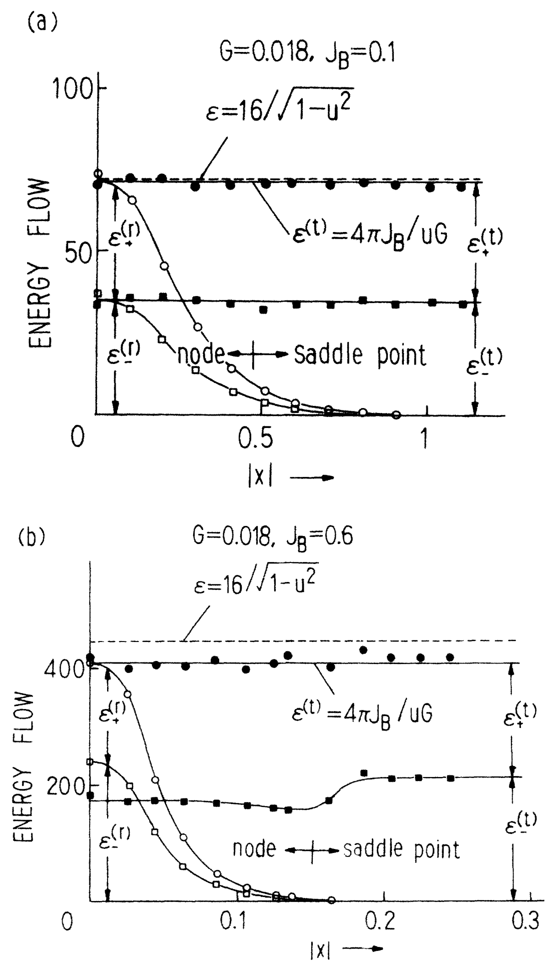

FIG. 3. Energy flow in the extended sine-Gordon system, where $G=0.018$. (a) $J_B=0.1$ and (b) $J_B=0.6$.

We set Eq. (22) equal to Eq. (23). Then, we obtain

$$u = \pm 1/[1+(4G/\pi J_B)^2]^{1/2} , \quad (24)$$

which is in agreement with the result by Mclaughlin and Scott[6] obtained by regarding the right side of Eq. (1) as a perturbation. Thus, the difference between $\varepsilon^{(t)}$ and ε denotes a measure of inaccuracy for application of Eq. (24).

Consider the soliton-antisoliton interaction in more detail. As the soliton and the antisoliton approach each other, the attractive force[5] becomes stronger. This is because the phase velocity $w(x,t) = ug(t)/h(x)$ is larger toward the wave front of each wave as they approach each other, where $g(t)$ is a decreasing function of $|t|$ and $h(x)$ an increasing function of $|x|$. Thus, distortion is produced in each wave. In the pure sine-Gordon system, the distortion is smoothly distributed throughout the wave on account of no moving singularity. On the other hand, in the extended sine-Gordon system, the excess stress that acts so as to divide the wave into two is concentrated at the position satisfying $h(x)^2/u^2 - g^2(t) = 0$. Thus, other parts of the wave can endure the stress. As a result, the wedge-shaped distortion grows there. We may also regard this process as a kind of local decaying process of the wave. Thus, $\varepsilon_-^{(t)}$ is decreased with decreasing $|x|$. In this situation, the singularity at $t = -\infty$ is still a saddle point, so that the solution around the singular point cannot be changed appreciably in this process. Thus, the wave can preserve its identity as a soliton or an antisoliton as a whole against the stress by producing such a local distortion. When $|x|$ is smaller than a certain value, the singularity at $t = -\infty$ is changed from a saddle point to a node. This means that the soliton or antisoliton cannot preserve its identity under the increasing stress, so that the waveform can be changed freely according to the stress. Thus, the local distortion disappears, and the stress is distributed smoothly over the wave. In $t > 0$, the wave is decelerated this time on account of the attractive force. When $|x|$ reaches a certain value with increasing $|x|$, the singularity at $t = +\infty$ is changed from the node to a saddle point, and then a thorn-shaped distortion appears. Thus, the identity as a soliton or an antisoliton is again recovered because the wave is able to overcome the decreasing stress with increasing $|x|$. We may regard this process as a kind of a local generalization process of the wave. Thus, $\varepsilon_+^{(t)}$ is decreased as the local distortion is weakened with increasing $|x|$. As $|x|$ increases sufficiently, the distortion almost disappears so that the attractive force disappears, and $\varepsilon_+^{(t)}$ becomes just half the value of $\varepsilon^{(t)}$.

We have investigated the situation of energy flow during the soliton-antisoliton interaction in extended Klein-Gordon systems, by dividing the energy flow into traveling-wave and standing-wave components. It is concluded that (i) the energy density of the systems is expressed by ϕ_x^2, (ii) the sum of these energy flows is obtained by integrating ϕ_t with respect to ϕ and then dividing by u, (iii) the average value of $g(t)$ over $V(\Xi)$ at a given value of x is equal to the ratio of the sum of the above two energy components to the traveling-wave component. Numerical examples have been given for the extended sine-Gordon system. It is also concluded for the system that (iv) the traveling-wave component of the energy flow is invariable irrespective of the value of $|x|$, and that (v) with decreasing $|x|$ the wedge-shaped distortion acts to decrease the traveling-wave component in $t < 0$ and the thorn-shaped distortion acts to increase it in $t > 0$.

[1]H. Tateno, Phys. Rev. B **36**, 2170 (1987).

[2]H. Tateno, J. Math. Phys. (to be published).

[3]A. C. Scott, *Active and Nonlinear Wave Propagation in Electronics* (Willey-Interscience, New York, 1970).

[4]S. Sakaki and H. Tateno, Jpn. J. Appl. Phys. **22**, 1374 (1983).

[5]J. Rubenstein, J. Math. Phys. **11**, 258 (1970).

[6]D. W. Mclaughlin and A. C. Scott, Phys. Rev. A **18**, 1652 (1978).