Electrical breakdown in a fuse network with random, continuously distributed breaking strengths

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We investigate the breakdown properties of a random resistor-fuse network in which each network element behaves as a linear resistor if the voltage drop is less than a threshold value, but then "burns out" and changes irreversibly to an insulator for larger voltages. We consider a fully occupied network in which each resistor has the same resistance (in the linear regime), and with the threshold voltage drop uniformly distributed over the range $v_- = 1 - w/2$ to $v_+ = 1 + w/2$ $(0 < w \leq 2)$. The breakdown properties of this model depend crucially on w, and also on L, the linear dimension of the network. For sufficiently small w, "brittle" fracture occurs, in which catastrophic breaking is triggered by the failure of a vanishingly small fraction of bonds in the network. In this regime, the average voltage drop per unit length required to break the network, $\langle v_b \rangle$, varies as $v_- + O(1/L^2)$, and $L \to \infty$, and the distribution of breakdown voltages decays exponentially in v_b. By probabilistic arguments, we also establish the existence of a transition between this brittle regime and a "ductile" regime at a critical value of $w = w_c(L)$, which approaches 2, as $L \to \infty$. This suggests that the fuse network fails by brittle fracture in the thermodynamic limit, except in the extreme case where the distribution of bond strengths includes the value zero. The ductile regime, $w > w_c(L)$, is characterized by crack growth which is driven by increases in the external potential, before the network reaches the breaking point. For this case, numerical simulations indicate that the average breaking potential decreases as $1/(\ln L)^y$, with $y \leq 0.8$, and that the distribution of breakdown voltages has a double experimental form. Numerical simulations are also performed to provide a geometrical description of the details of the breaking process as a function of w.

I. INTRODUCTION

Breakdown phenomena have been the object of considerable study, as the mechanical failure of materials is an issue of basic importance in engineering and technology (see, e.g., Refs. 1–10 for a sampling of recent work). Although there has been a great wealth of experimental data and modeling at the phenomenological level, much of it has been oriented towards specific applications, rather than for the purpose of identifying possible universal principles underlying breaking. Recently, the electric breakdown of random fuse networks has been investigated,[11–14] in which breaking originates from the simple microscopic process of the failure of a single bond when the voltage drop across it exceeds a threshold value. If the voltage drop exceeds the threshold, the element "blows" and changes irreversibly into an insulator, i.e., the bond behaves as a series combination of a resistor and a fuse. The failure of a random network consisting of these elements, as the external potential is raised, is meant to mimic the mechanical failure of a random elastic network under the condition of increasing uniform tension. The model is appealingly simple, and from investigations of models of this type, one may hope to develop general insights about failure in real materials. It is also noteworthy that the breakdown in a fuse network involves the high-voltage tail of the voltage distribution of a random resistor network,[15–18] and this connection may be worth developing.

In addition to defining the current-voltage response of each network element, the kinetics of the breaking process must also be defined in order to specify the model. In this work, we shall assume that the breaking of a single bond is "slow" in that the characteristic time required to break an "overstressed" bond is much larger than the time for the currents in the network to reequilibrate after a bond breaks. We also assume that the bond-breaking time is much smaller if the overstress is larger, so that only one bond, the most overstressed one, is broken at any stage. These basic assumptions of rapid current redistribution and single-bond breaking form the basis of our numerical simulations. For this model, interesting questions to study are the detailed nature of the breaking process itself, the average value of the external potential

required to break the network, and the distribution of these breakdown potentials.

For a fully occupied network consisting of nearly identical fuses, the behavior of the system is trivial, because the failure of one bond leads to the formation of a linear crack that breaks the entire network. More interesting is the case of disordered networks, and perhaps the simplest case is that of percolation disorder, in which lattice bonds are randomly occupied with probability p by identical elements.[11,12] Near the percolation threshold, singly-connected bonds[19] are most susceptible to failure, and the burning out of a singly-connected bond will cause the entire network to fail. Above the percolation threshold, the behavior is considerably more subtle. Duxbury *et al.*[12] have recently argued that, for weak disorder, the breaking process is dominated by the largest "crack" in the initial state of the system. At the tip of this largest crack, local current flow is enhanced by a factor which is proportional to the square root of the crack length (in two dimensions). Furthermore, in a system of linear dimension L, the length of the largest crack is proportional to $\ln L$, and this leads to the average breaking potential of the network vanishing asymptotically as $1/(\ln L)^y$, with $\frac{1}{2} < y < 1$. This is qualitatively consistent with the numerical work of Ref. 12, although the exponent y was not estimated from the data.

While percolation disorder may be appropriate for accounting for the breaking of materials such as unwoven textiles, the general situation where bond conductances and/or bond-breaking strengths are drawn from continuous distributions may be more appropriate for discussing breaking processes in disordered solids. Moreover, a completely occupied network, with the properties of each element continuously distributed, is the situation that results if a percolating network, which is above the percolation threshold, were coarse grained. Consequently, such a continuous model may provide a better representation of the thermodynamic limit than percolation models. Another aspect of considering continuous, rather than percolation distributions, is that the microscopic-crack size distribution can be monitored during the breaking process, and this may be useful for providing detailed geometric information.

For simplicity, we will restrict ourselves to the case where each bond has unit conductance, while the failure-inducing voltage drop, or current, i.e., the "strength" of each bond, is continuously distributed over some range. The breaking properties of the network depend on the shape of this distribution, especially at the low-strength limit. However, we will focus on the uniform distribution, in which the threshold voltage is uniformly distributed between $v_- = 1 - w/2$ to $v_+ = 1 + w/2$, as this case appears to be rather typical. Notice that the average threshold voltage is equal to unity, and the width of the bond-breaking distribution, w, lies between 0 and 2.

In analogy with fuse networks with percolation disorder, the value of the external potential drop required to break the network, i.e., the macroscopic breaking strength, decreases as L increases,[11,12] but at a rate that depends on w (Fig. 1). For sufficiently small w, a regime of "brittle" behavior occurs, in which one of the first few

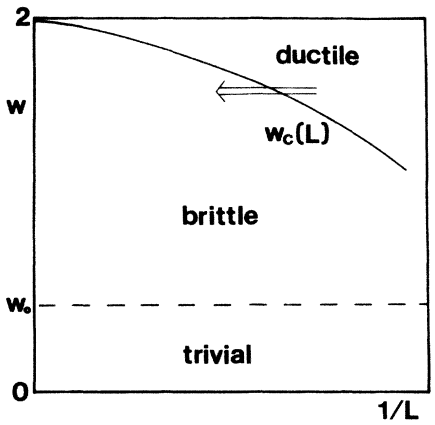


FIG. 1. Phase diagram for the random fuse network in the parameter space spanned by $1/L$ and w. For $0 < w < w_c(L)$, the network is brittle, with trivial behavior occurring in the limit $w < w_0$, while for $w > w_c(L)$, the network is ductile. The arrow schematically indicates the expected crossover from ductile to brittle behavior when a sequence of systems with fixed w and increasing values of L are considered.

bonds broken nucleates a crack which propagates across the system. Breaking is therefore governed by the weakest, or one of the weakest, bonds in the initial distribution. For larger w, breakdown of the network is more gradual than in brittle fracture, as there is a large range over which individual bond breakings are driven by increases in the external potential. We term this regime of behavior as "ductile." From a probabilistic argument, ductility is expected if the number of voltage-driven bond breakings is more than of order L, in an $L \times L$ network. (For a d-dimensional network of linear dimension L, this critical value generalizes to a number that is of order $L^{d/2}$.) The behavior of the breaking voltage in the ductile regimes seems to parallel that of the fuse network with percolation disorder. Numerical simulations suggest that the average breaking potential per unit length, $\langle v_b \rangle$, decays as $1/(\ln L)^y$ as $L \to \infty$, with y estimated to be less than 0.8. However, $\langle v_b \rangle$ can never be less than v_-, and this leads to an eventual crossover to brittle behavior as L increases (Fig. 1), except for the extreme case $w = 2$.

The outline of the paper is as follows: In Sec. II, we study the failure of the fuse network in the strongly brittle regime, where breaking occurs by single-crack growth. In Sec. III, we use the approximation that a crack of length 2 is immediately unstable to catastrophic failure, together with general probabilistic arguments, to estimate the location of the transition between brittleness and ductility as a function of w. In Sec. IV, we employ a dilute-crack approximation and the Lifshitz-type argument of Duxbury *et al.*[12] to estimate the average breaking strength of the network in the ductile regime. The approximations used in Secs. III and IV are rather different in nature, but we expect that they are reasonable for the early and later stages of the breaking process, respectively. In Sec. V, we present numerical results for the average breaking potential and its distribution, as well as the total number of broken bonds at the fracture point, to

help characterize brittle versus ductile fracture, geometrically. Finally, in Sec. VI, we briefly discuss generalizations to other types of disorder and to three-dimensional systems. We also raise some questions which may be suitable for future work.

II. FRACTURE BY A SINGLE CRACK

To understand the breaking of a random fuse network, first consider what happens when a single bond is broken. The degree of stability of this initial one-bond crack is useful for determining the subsequent breaking process. For a sufficiently large system of linear dimension L, the first bond will break when the external voltage drop per unit length is $v_- + O(1/L^2)$ [see Eq. (6), below]. For the present discussion, however, we simply take this minimum voltage drop to be v_-. After the breaking of this weakest bond, the voltage drops across the bonds in this new network can be calculated exactly.[12,20] One finds that the largest enhancement of the local voltage drops occurs across vertical bonds which are horizontally adjacent to the initial crack; these are the bonds which are most likely to break without any additional increase of the external potential drop.

More generally, if the voltage drop across the system is V and the corresponding voltage drop, v_i, across bond i is greater than v_-, then the probability that this bond will break is equal to

$$p_i = \frac{v_i - v_-}{w} . \tag{1}$$

Then the probability p_s, that no additional bonds break at external potential V, after the first bond has been broken, equals

$$p_s = \prod_i{}' (1-p_i) . \tag{2}$$

Here the prime on the product denotes that it is taken over all bonds with $p_i > 0$, that is, only those bonds where the defect has caused an enhancement of the local voltage drop above v_-. The dependence of this stability probability on w is shown in Fig. 2 for a sufficiently large network where finite-size effects are negligible. For w less than a value w_0, the initial single-bond crack is always unstable to further cracking. This defines a "trivial" regime of behavior, where the breaking of the first bond immediately leads to a catastrophic failure of the network by the growth of a straight-line crack.

An estimate for the range of w for which trivial behavior occurs can be found as follows. If the weakest (vertical) bond in the system fails when the voltage drop across it is $v_- = 1 - w/2$, then the voltage drop across the two vertical bonds immediately adjacent to the initial broken bond is αv_-, where the enhancement factor α is weakly dependent on the system size, the location of the broken bond, and on the boundary conditions; however, in the thermodynamic limit, $\alpha = 4/\pi$.[12,20] If αv_- exceeds the breaking strength of the strongest possible bond in the system, v_+, then the initial bond failure necessarily leads to the formation of a linear crack that breaks the network. Using $\alpha = 4/\pi$, this criterion leads to a critical

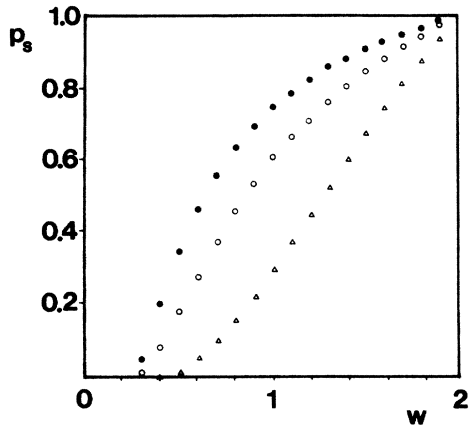


FIG. 2. Plot of the stability probability p_s, i.e., the probability that no additional bonds break when the external potential is set to the value which is sufficient to cause the weakest bond in the network to break. The plots are based on data from a 91×91 network, in which the bond at the center is removed. For comparison, we show this probability when stability of only the two edge bonds are tested (●), when all the bonds in the same horizontal row as the initial crack are tested (○), and when all the bonds in the lattice are tested (△).

value of $w = w_0$, whose numerical value, according to the approximations employed, is given by

$$2\left[\frac{4-\pi}{4+\pi}\right] \simeq 0.2404 . \tag{3}$$

This represents a lower bound for the true value of w_0.

Since the breaking mechanism is dependent only on the properties of the weakest bond in the entire network, the strength of the network and its distribution can be calculated from extreme-value statistics.[21] If the external voltage drop per unit length is v, then the probability that any one bond breaks is simply $p = (v - v_-)/w$. Here we are tacitly assuming that v is less than v_+, so that $p < 1$. Therefore, the probability that none of the L^2 vertical bonds in an $L \times L$ system break when the potential is v, or less, is simply

$$\mathcal{F}(v) = (1-p)^{L^2} . \tag{4a}$$

This quantity can be reinterpreted[21] as the probability that the weakest bond in the system breaks when the voltage drop is equal to or greater than the value v. Consequently, the probability that the weakest bond breaks, when the voltage drop across it is between v and $v + dv$, is given by

$$\mathcal{F}'(v)dv = \frac{L^2}{w}(1-p)^{L^2-1}dv . \tag{4b}$$

Notice that $\mathcal{F}'(v)$ is normalized in the sense that $\int_{v_-}^{v_+} \mathcal{F}'(v)dv = 1$. Thus the distribution of network breaking strengths decays exponentially in the voltage. Furthermore, from (4), we can straightforwardly calculate the mean voltage drop per unit length, at which the weakest bond breaks,

$$\langle v_b \rangle = \int_{v_-}^{v_+} v \mathcal{F}'(v) dv$$
$$= v_- + \frac{w}{L^2+1} . \quad (5)$$

This result merely reflects the fact that when the breaking strengths of the L^2 vertical bonds are uniformly distributed within the range w, then the typical difference between two adjacent breaking strengths will be of order w/L^2. Consequently, the weakest bond strength will typically be larger than v_- by this same amount.

III. UNSTABLE-CRACK APPROXIMATION

For $w > w_0$, there is a nonzero probability that the initial crack is stable to further breaking when the external potential is at the value required to break the first bond. Consequently, the potential must be increased in order to break additional bonds. To study how the network fails in this case, we develop a very crude probabilistic argument which is based on the approximation that a crack of length greater than 1 is immediately unstable to catastrophic failure. From this argument, we also predict that there is a transition between brittleness and ductility which occurs at a critical value, $w = w_c(L)$, which depends on the linear dimension of the system. For $w < w_c(L)$, a network fails by brittle fracture, while for $w > w_c(L)$, the breakdown process is more gradual. Interestingly, our argument gives $w_c(L) \to 2$ as $L \to \infty$, and this prediction seems to be independent of detailed approximations. This suggests, therefore, that the random fuse network with continuously distributed bond-breaking strengths fails only by brittle fracture in the thermodynamic limit, except in the extreme case where the distribution of bond-breaking strengths extends to zero, i.e., $v_- = 0$ (or $w = 2$).

First, we determine an approximate criterion for the onset of brittle fracture. This criterion is based on the picture in which the initial phase of breaking involves only the failure of the sequence of weakest available bonds in the network. Accordingly, these initial single-bond cracks will be spatially uncorrelated. For a uniform distribution of bond-breaking strengths, it then follows from Eq. (5) that the average breaking strength of the nth weakest bond in the network varies as

$$\langle v_{\text{weakest}}(n) \rangle \simeq v_- + nw/L^2 , \quad (6a)$$

so that the voltage drop at the edge of each crack will be $\langle v_{\text{weakest}}(n) \rangle \alpha$. This result is verified to a good approximation in our numerical simulations. As each single-bond crack forms, the number of bonds at the edges of all the cracks increases by two. If these edge bonds are statistically independent, then after n single-bond cracks are created, the average breaking strength of the weakest of the ensemble of $2n$ edge bonds is [cf. Eq. (5)]

$$\langle v_{\text{edge}}(n) \rangle = v_- + \frac{w}{2n+1} . \quad (6b)$$

For a network which is initially stable to breaking, $\langle v_{\text{weakest}}(1) \rangle \alpha < \langle v_{\text{edge}}(1) \rangle$. However, as the weakest available bonds are sequentially broken, $\langle v_{\text{weakest}}(n) \rangle \alpha$ increases in n, while $\langle v_{\text{edge}}(n) \rangle$ decreases. Eventually a point is reached where

$$\langle v_{\text{weakest}}(n) \rangle \alpha = \langle v_{\text{edge}}(n) \rangle , \quad (7)$$

and, at this stage, the unstable growth of an existing crack is as likely to occur as forming a new crack. We define this point as the onset of brittle fracture. The number of broken bonds at this point, $n_c(w)$, is found to have two possible asymptotic forms as $L \to \infty$. In the general case where $w_0 < w$ and $2 - w \gg O(1/L)$, then from Eq. (7), the value of $n_c(w)$ tends to a *finite* value as $L \to \infty$, which is given by

$$n_c(w < 2) \simeq \frac{1}{2} \left[\frac{1}{w_0} - \frac{1}{w} \right] \frac{w}{v_-} , \quad (8a)$$

with correction terms which are of order $1/L^2$. On the other hand, when $2 - w \ll O(1/L)$, i.e., $w \to 2$, then

$$n_c(w = 2) \simeq L/\sqrt{2\alpha} , \quad (8b)$$

with correction terms of order unity. Thus $n_c(w)$ increases with w, for $w > w_0$, and then saturates at a value given by Eq. (8a) when $2 - w$ becomes greater than $O(1/L)$. Therefore, as $L \to \infty$, the random fuse networks fails by brittle fracture for all values of $w < 2$, since the number of single-bond cracks needed before the system undergoes brittle fracture, is finite.

However, the assumption of spatial independence in the initial sequence of single-bond cracks eventually breaks down in a finite-size network, and we now determine whether this occurs before or after brittle fracture has been reached. That is, in the process of breaking the weakest available bond, it may eventually happen that the next bond to be broken is adjacent to a previously broken bond. If this happens *before* brittle fracture has occurred, then further crack growth will not necessarily be catastrophic, but rather, will typically be driven by additional increases in the external potential. This change in behavior is the criterion we use for determining the location of the brittle-ductile transition.

We therefore reexamine the assumption of spatial independence in the sequence of initial broken bonds, by computing the probability that the nth weakest vertical bond will be spatially independent of all the other cracks already in the system. The notion of "independence" is not sharply defined in the present context, but one can formulate an intuitively plausible criterion based on the pattern of bond voltage drops in a network with two closely separated single-bond cracks. Even though the voltage drops around one crack are substantially perturbed by the neighboring crack, we have numerically verified that the stability probability of the two-crack system is very nearly equal to the square of the stability probability for a single-crack system, for a variety of locations of the two neighboring cracks. Thus, in terms of the susceptibility for additional bond breaking, two cracks can be very close together and still be independent.

Therefore, we use the geometrical criterion that two bonds are independent if they do not coincide or if they are not horizontally adjacent. Consequently, each broken bond removes three possible locations in the lattice at which the next broken bond is independent of its predecessors. For a system of linear dimension L, the probability that the nth broken bond is independent is then given by

$$\mathcal{P}_n = 1 \left[\frac{L^2-3}{L^2-1} \right] \left[\frac{L^2-6}{L^2-2} \right] \left[\frac{L^2-9}{L^2-3} \right] \cdots \left[\frac{L^2-3(n-1)}{L^2-(n-1)} \right] . \tag{9a}$$

(This probability is closely analogous to the familiar problem of the probability of "birthday coincidences" in a finite population.) We next estimate the value of n for which $\mathcal{P}_n$ decays to approximately $1/e$. Rewriting (9a) as

$$\mathcal{P}_n = \frac{[1-(3/L^2)][1-(6/L^2)][1-(9/L^2)] \cdots \{1-[3(n-1)/L^2]\}}{[1-(1/L^2)][1-(2/L^2)][1-(3/L^2)] \cdots \{1-[(n-1)/L^2]\}} , \tag{9b}$$

and then making the approximation $(1-k/L^2) \sim e^{-k/L^2}$, the product can be expressed as the exponential of a sum, from which we obtain

$$\mathcal{P}_n \simeq e^{-n(n-1)/L^2} . \tag{10}$$

Thus we deduce a characteristic value

$$n_{\text{indep}} \sim L , \tag{11}$$

beyond which the independent assumption fails.

If brittle fracture has not yet occurred by the time that n_{indep} bonds are broken, i.e., $n_{\text{indep}} < n_c(w)$, then crack growth is as likely to occur as new crack initiation when the external potential is increased, and the failure of the network will be more gradual than brittle fracture. Conversely, for $n_{\text{indep}} > n_c(w)$, the system is brittle, as the sequence of weakest broken bonds are still spatially independent when failure at the edge of a newly formed crack occurs. The brittle-ductile transition is therefore defined by n_{indep} and $n_c(w)$ being of the same order. This then leads to a critical value of w given by

$$w_c(L) \sim 2 \left[1 - O\left(\frac{1}{L} \right) \right] , \quad L \to \infty . \tag{12}$$

This result actually represents an upper bound to the true value of $w_c(L)$, owing to the nature of the unstable crack approximation. Since single-crack growth does not necessarily end in catastrophic failure, a point in the brittle region of the phase diagram may actually be ductile, and correspondingly the true value of $w_c(L)$ is lowered. Nevertheless, attempts to develop more refined arguments all lead to the feature that $w_c(L) \to 2$, as $L \to \infty$. Thus we predict that the random fuse network is brittle in the thermodynamic limit, except in the special case where $w=2$.

IV. DILUTE-CRACK APPROXIMATION

We now discuss the failure of the random fuse network by using an approximation in which cracks of length greater than unity may be stable, but also in which cracks are sufficiently dilute so that interaction effects are ignored.[22] This is very similar to the approach introduced by Duxbury *et al.*[12] for the case of percolation disorder. We further assume that cracks are linear in shape, and that a crack grows by advancing by one lattice spacing, rather than by a more complicated kinetic process. This means that the probability of finding a crack of a given size will be independent of its history. The crack length distribution can now be calculated in terms of a one-dimensional correlated percolation problem, in which the degree of correlation involves the length-dependent amplifications of the current at the crack tip. For a given value of w and V, there will be a critical size, n_c, for which the tip current is sufficient to lead to unstable crack growth. The breaking potential of the network is then given by the value at which the expected number of cracks of size n_c, or larger, is of order unity.

When an external potential $V = Lv$ is applied, the probability that any one bond breaks, in the absence of correlations, is $p_0 = (v - v_-)/w$. Once this has occurred, then the probability, p_1, that the next horizontally adjacent bond breaks is larger than p_0, owing to the local current enhancement. Thus $p_1 = (\alpha_1 v - v_-)/w$, where for the enhancement factor α_1, we use the value $4/\pi$ of the infinite lattice. More generally, the probability of breaking the kth bond in the crack is

$$p_k = \frac{v_k - v_-}{w} , \tag{13}$$

where $v_k \equiv v\alpha_k$ is the voltage drop at the tip of a crack of length k, and α_k is the corresponding amplification factor. Asymptotically v_k varies as,[12]

$$v_k \sim v(1 + f\sqrt{k}) , \tag{14}$$

with f numerically estimated to be 0.55. However, numerical data for v_k show that this asymptotic form is not evident until $k \gtrsim 30$ (Fig. 3). As a result, calculations based on Eq. (14) may have limited utility in accounting for the behavior of the breaking potential for the range of network sizes accessible in our simulations, as crossover effects may play a relatively important role.

Thus, the probability that there exists a (one-dimensional) crack of size n is equal to

$$P(n) = p_0 p_1 p_2 \cdots p_n (1 - p_{n+1})^2 . \tag{15}$$

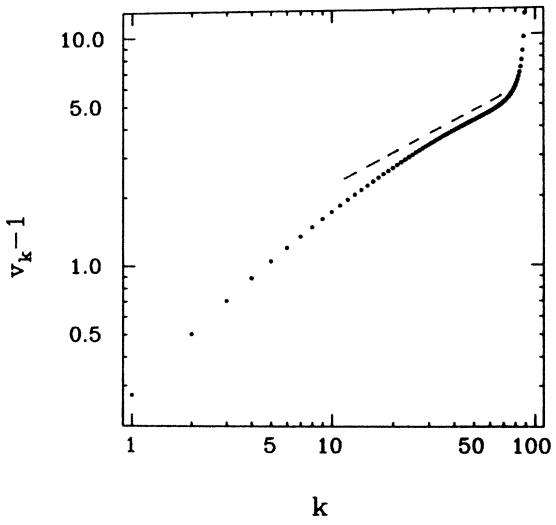


FIG. 3. Double logarithmic plot of $v_k - 1$ vs k, to illustrate the range of validity of the asymptotic form, $v_k \sim 1 + f\sqrt{k}$. For the data shown, the vertical position of the crack was halfway between the bus bars on a 91×91 network. There are periodic boundary conditions in the transverse direction, and the voltage drop across the network is equal to $V = 91$. The dashed line corresponds to the behavior where $v_k - 1$ increases as $\sqrt{k}$, and the rapid increase in $v_k - 1$ for $k \gtrsim 70$ is due to finite-size effects.

In keeping with our assumption that a crack fails *only* at the tip, Eq. (15) involves the boundary factor $(1 - p_{n+1})^2$, associated with the two perimeter bonds. However, as illustrated in Fig. 2, this assumption yields only a fair approximation for the stability of the network to additional breaking after the failure of a single bond. Let us now estimate how $P(n)$ depends on n in the dilute-crack approximation. If all the bond-breaking probabilities were equal (uncorrelated percolation), then the crack-length distribution would be a pure exponential, and the breaking strength of the network would decay as $1/\sqrt{\ln L}$, by following the Lifshitz-type argument given by Duxbury *et al.*[12] For the fuse network with a continuous distribution of bond-breaking strengths, however, the crack-length distribution will be modified by enhancement effects. Using the asymptotic form for v_k in p_k, we have

$$P(n) \simeq \prod_{k=1}^{n} \frac{v_k - v_-}{w} \equiv \prod_k (p_0 + q\sqrt{k}) , \tag{16}$$

where $p_0 = (v - v_-)/w$ and $q = vf/w$. By considering $\ln P(n)$, the resulting sum can then be approximated by an integral for n large, giving

$$\ln P(n) \sim \int_0^n \ln(p_0 + q\sqrt{k})\, dk , = \left[n - \frac{p_0^2}{q^2} \right] \ln(p_0 + q\sqrt{n}) - \left[\frac{n}{2} - \frac{p_0}{q}\sqrt{n} \right] + \frac{p_0^2}{q^2} \ln p_0 . \tag{17}$$

We are interested in the behavior of $\ln P(n)$ for the situation where the voltage at the tip of the crack is close to v_+. This is equivalent to the condition

$$p_0 + q\sqrt{n} \sim 1 - \delta \tag{18a}$$

or

$$\delta \simeq \frac{v_+ - v\alpha_n}{w} . \tag{18b}$$

Now expanding $\ln(1-\delta) \simeq -\delta$, and dropping all lower-order terms in Eq. (17), we find

$$\ln P(n) \sim -n \left[\frac{v_+}{w} + \frac{1}{2} \right] + n^{3/2} \frac{fv}{w} . \tag{19}$$

Correspondingly, the total number of cracks of size n in a system of linear dimension L scales as

$$N(n) = L^2 P(n) . \tag{20}$$

By taking this number to be unity, one finds a characteristic length, n_{longest}, for which there will be *one* such crack in a system of linear dimension L, which is determined by

$$n_{\text{longest}} \left[\frac{v_+}{w} + \frac{1}{2} \right] - n_{\text{longest}}^{3/2} \frac{fv}{w} = 2 \ln L . \tag{21}$$

For $w \simeq 2$, the coefficient of $n^{3/2}$ is typically quite small. Therefore, even for reasonably large values of n, $\ln P(n)$ decreases almost linearly in n, leading to $n_{\text{longest}} \sim \ln L$. For very large n, the $n^{3/2}$ term causes $P(n)$ to decay slower than a pure exponential, and eventually reach a minimum at a value $n = n_0$, corresponding to $p_{n=n_0} = 1$. Longer cracks are necessarily unstable, as the crack-length distribution ostensibly increases with n. However, by comparing the full solution to Eq. (21) with that obtained by keeping only the linear term, one finds that the two solutions differ only by a numerical factor which is less than or equal to $\frac{4}{3}$, as illustrated in Fig. 4.

Thus using $n_{\text{longest}} \sim \ln L$, we estimate the breaking potential $\langle v_b \rangle$ from the condition

$$\langle v_b \rangle (1 + f\sqrt{n_{\text{longest}}}) \sim v_+ , \tag{22a}$$

i.e., the voltage drop at the tip of the longest crack just equals v_+. This then yields

$$\langle v_b \rangle \sim \frac{v_+}{\sqrt{\ln L}} . \tag{22b}$$

However, (22b) must break down when $\langle v_b \rangle$ is smaller than v_-, corresponding to L being large enough that there is no solution to Eq. (21). In this case, the network is brittle, as an external potential V which is infinitesimally greater than Lv_- immediately leads to the creation of a longest crack that is already beyond the stability limit. Only in the case $w = 2$, does Eq. (21) have a solution for all values of L, so that the network is always ductile.

We now attempt to understand the influence of the interactions between cracks on the breakdown of the ran-

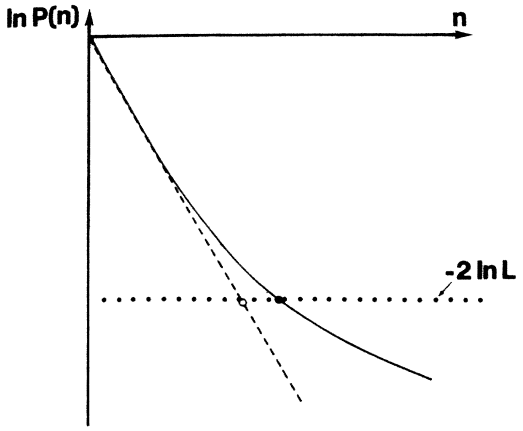


FIG. 4. Schematic plot of $\ln P(n)$ from Eq. (19) vs n, in order to illustrate the nature of the solution to (21). The dashed line is the result when only the linear term in $\ln P(n)$ is retained. The corresponding approximate solution (○) and the full solution (●) to (21) are shown. They differ only by a numerical factor which is less than $\frac{4}{3}$.

dom fuse network. Related questions have been considered previously,[22] and it has been found that either screening or enhancement effects can occur, depending on the relative orientations and positions of nearby cracks. We focus here on enhancement effects, as these are more likely to modify the size and voltage dependence of the breakdown voltage. Our approach is again probabilistic; while it is very crude, it does not depend strongly on microscopic details and may therefore have general applicability.

Consider two colinear cracks which, for simplicity, are separated by a distance d, and are of equal lengths n. As discussed in Refs. 22, the degree of correlation between the cracks is characterized by the ratio n/d. For $n/d<1$, interactions play almost no role, and the breakdown of this two-crack system is close to that of two independent single cracks. In the opposite case, there is a strong enhancement of the two-crack failure probability. We now ask whether there is an appreciable probability that a network of linear dimension L contains a configuration of two closely separated colinear cracks, in which the sum of the lengths of the two cracks is greater than n_{longest}. If such a configuration exists, then the region between the two cracks will likely break down without raising the external potential beyond the point necessary to create the initial two-crack configuration. This two-crack configuration would then play the role of an effective single critical crack of length n_{eff}, which is greater than n_{longest}, and the independence assumption of the dilute-crack approximation would fail.

To determine whether n_{eff} is much greater than n_{longest}, we employ an exponential crack-length distribution, $P(n)\sim e^{-an}$, from which the number of cracks of length $n_{\text{longest}}/2$ or greater is proportional to L. If these L "half"-cracks are randomly distributed, then from Sec. III, there is a sufficient number of them to make it likely that two of them are nearby. Furthermore, if these cracks are on the same row, then the local current enhancement will typically lead to coalescence of the two cracks. By construction, the length of the newly formed crack is of order n_{longest}. Thus the L dependence of n_{eff} is the same as that of n_{longest} found in the dilute-crack approximation. The same result follows when considering k cracks of length n_{longest}/k in the same row. This argument therefore suggests that the L dependence of the breaking potential should not be affected strongly by interactions.

V. NUMERICAL RESULTS

In order to test the predictions of the previous sections, and to provide a more quantitative account of the breaking process, we have performed numerical simulations on $L\times L$ random fuse networks on the square lattice, with L in the range 10–80. Opposite edges of the lattice are connected to bus bars across which there is a potential drop unity (for calculational convenience), and periodic boundary conditions in the transverse direction are imposed. Each bond is a resistor-fuse combination which has unit resistance (in the linear regime), but in which the breaking voltage for the ith bond, $v_b(i)$, is uniformly distributed in the range $v_-=1-w/2$ to $v_+=1+w/2$, with $0<w\leq 2$. The potential drops v_i across each bond i are calculated by the conjugate gradient method.[23] We then break the bond for which the difference $\Delta_i=v_b(i)-v_i$ is the smallest, i.e., the resistance of the bond is set to infinity, and the procedure of solving for the voltages and breaking bonds, one at a time, is repeated until the network breaks. The breaking potential of the network (normalized per unit length) v_b is the value $v_b(i)$ corresponding to the largest Δ_i. For large-system sizes, and for $w\simeq 2$, many bonds need to be broken before the system breaks down, and the computation time required to reach this point can become quite large. For example, for $L=80$ and $w=2$, a single configuration required approximately 2 h of CPU time on an IBM 3090, beginning with the pure lattice and ending with the broken network after approximately 1300 bonds were broken.

To illustrate the detailed evolution of the breaking process, we plot the value of the external potential as each bond is broken (Fig. 5), in which each data point is coded according to whether the broken bond forms a new crack (vertical line), adds to an existing crack (horizontal line), or joins together two already existing cracks (open circles).[24] For a very brittle network, the breaking process is represented by a horizontal sequence beginning with a single vertical line, and then followed by a sequence of horizontal lines. As w increases, the initial stages of breaking become closer to a percolation-like process where independent single-bond cracks are formed. Correspondingly, the evolution plot should begin with a sequence of vertical lines that lie on a line of slope w/L^2, according to Eq. (6b). For the case $L=50$ and $w\gtrsim 1.3$, this initial slope agrees with (6b) to an accuracy of about 10%.

The breakdown of the network by brittle fracture is signaled by the occurrence of catastrophic crack growth after a relatively small number of cracks has formed [Fig. 5(b)]. In the ductile regime, there is a substantial range of increasing external potential over which gradual crack

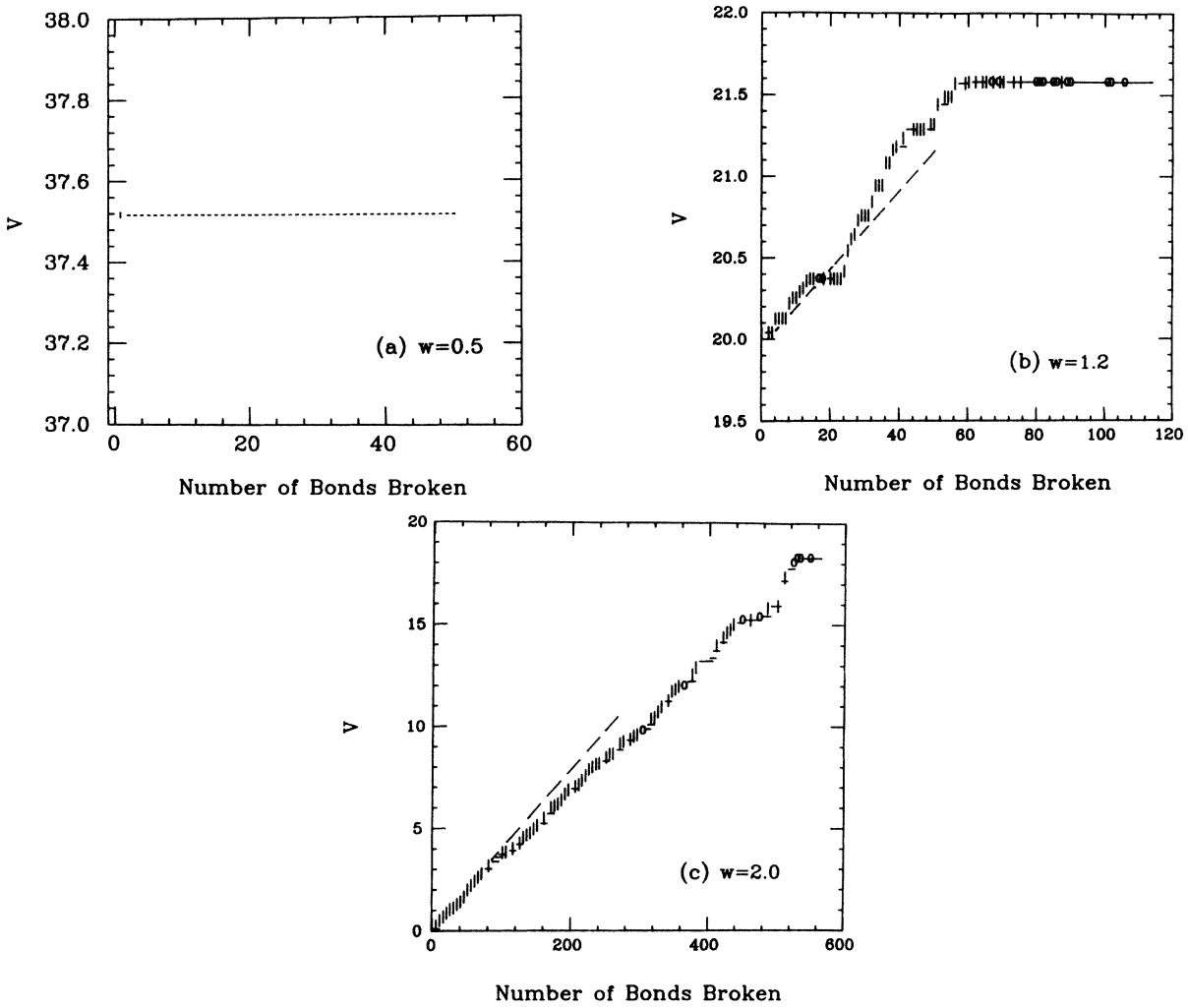


FIG. 5. A plot of the (unnormalized) external voltage drop v vs the number of bonds broken, to illustrate the breaking of a 50×50 network for (a) $w=0.5$, (b) 1.2, and (c) 2.0. The data points are coded according to whether the newly broken bond creates a new crack (vertical line), adds to an existing crack (horizontal line), or joins cracks (open circle). For clarity, only every fifth point in the sequence is shown in (c). The dashed lines in (b) and (c) indicate the initial slope expected from the independent bond-breaking picture [cf. Eq. (6a)]. In (c) notice that both crack growth and crack coalescence occur as the external potential is being increased.

growth occurs [Fig. 5(c)]. This type of behavior is generally observed when w is, typically, equal to or greater than 1.5. In the evolution plot, we see that the initial percolation regime of new crack formation gradually becomes interspersed by crack growth and crack coalescence, and these events ultimately lead to network failure.

Next we discuss the behavior of the average breaking potential of the network, $\langle v_b \rangle$. When $w < w_0$, Eq. (5) gives $\langle v_b \rangle \sim v_- + O(1/L^2)$, and for $w > w_0$, but still in the brittle regime, $\langle v_b \rangle$ is still expected to show this same L dependence. This is checked in Fig. 6(a), where $\langle v_b \rangle - v_-$ is plotted versus L on a double logarithmic scale for the representative case of $w=1$. Asymptotically, this data appear to lie on a straight line of slope -2, consistent with our theoretical expectations. This behavior is found to persist for w as large as 1.2, for the range of L accessible in the simulations, except that for $w=1.2$, the asymptotic behavior sets in at larger L.

For $w=2$, we first test whether the decay of $\langle v_b \rangle$ is consistent with power-law behavior in L by plotting $\langle v_b \rangle$ versus L on a double logarithmic scale [Fig. 6(b)]. This plot suggests that $\langle v_b \rangle$ decays slower than any power of $1/L$, although we cannot definitively exclude the possibility of a power-law decay with a very small exponent value. A plot of $\langle v_b \rangle$ versus $\ln L$ on a double logarithmic scale is more linear [Fig. 6(c)], and this suggests that $\langle v_b \rangle$ does vary as $1/(\ln L)^y$. However, there is a slight, but systematic, curvature in the data and by successively deleting the smaller-L points, the slope of a linear fit changes from 1.05 (when all the data is fit), to $\simeq 0.8$, suggesting that $y \lesssim 0.8$. This behavior appears to follow, qualitatively, the finite-size crossover found in the dependence of v_k on k (Fig. 3). Thus, it is conceivable that $\langle v_b \rangle$ varies asymptotically as $1/\sqrt{\ln L}$, as predicted by Duxbury *et al.*[12] for the case of percolation disorder, but

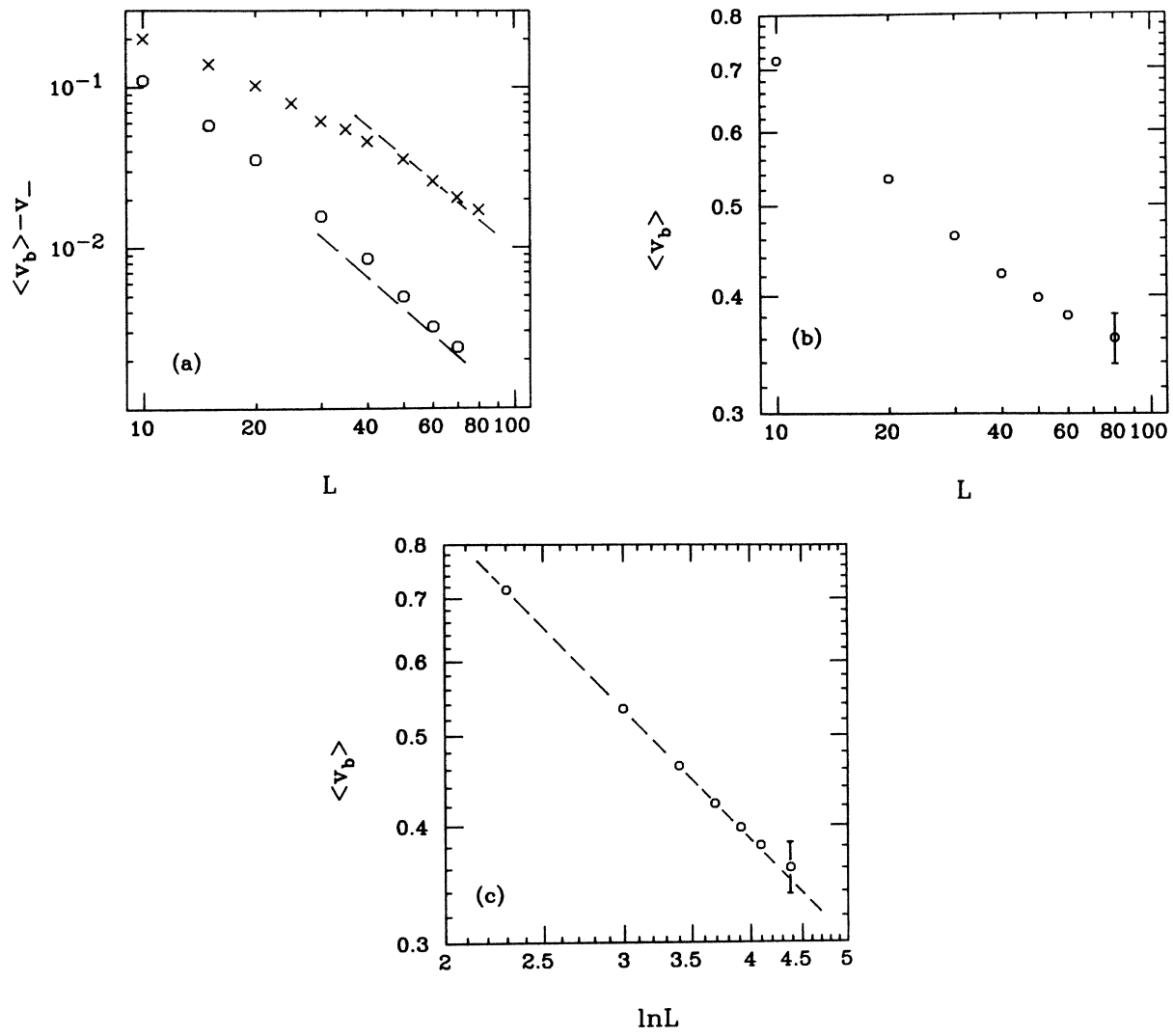


FIG. 6. Plot of $\langle v_b \rangle$ vs L for representative values of w. In (a), $\langle v_b \rangle - v_-$ vs L is plotted on a double logarithmic scale for $w=1.0$ ($\circ$) and $w=1.2$ ($\times$). The dashed line corresponds to the behavior predicted in Eq. (5). This data is based on averaging 100 realizations (for $L=80$) to 1000 realizations (for $L=20$). In (b) and (c), $\langle v_b \rangle$ is plotted vs L and vs ln$L$, respectively, on a double logarithmic scale for the case $w=2$. The dashed line in (c) has slope -1.05, but note the systematic curvature in the data. For $w=2$, the data is based on averaging 12 realizations for $L=80$, 400 realizations for $L=60$, and gradually increasing to 1000 realizations for $L=20$. Statistical errors are equal to or smaller than the size of the data points, except for the one point shown.

that crossover and finite-size effects mask this result for values of L accessible in our simulations.

We have also studied the distribution of $\langle v_b \rangle$, as this is a relatively sensitive probe of the underlying breaking mechanism, which is often used as a tool for reliability analysis. It is conventional to consider the probability that a network has failed at a voltage v_b, or less, $F(v_b)$. In Fig. 7, we plot this distribution for two examples whose behavior is representative of brittle and ductile behavior. In the brittle regime, $1-F(v_b)$ decays exponentially in v_b [cf. Eq. (4)], and our data is in excellent agreement with this prediction. In the ductile regime, two natural choices to consider for $F(v_b)$ are the Weibull form,

$$F(v_b)=1-\exp(-av_b^m)\ , \tag{23a}$$

which is often invoked in many investigations of mechanical failure, and the double exponential form

$$F(v_b)=1-\exp[-a\exp(-bL^2v_b^{-m})]\ , \tag{23b}$$

which has been argued to be appropriate in accounting for the distribution of breaking strengths for the random fuse network with percolation disorder.[12] In this case, the underlying source of the double exponential form is the exponential form of the microscopic crack-length distribution.

In Fig. 7(b), we first show the result of attempting to fit the distribution of breakdown strengths to a Weibull form, for the case $L=50$ and $w=2$. There is a relatively poor fit, with $\chi^2=11$, and the corresponding parameter values $\ln a=-37.8$ and $m=12.4$ from the best fit are rather anomalous. The double exponential form with $a=28.5$, $b=6.36$, and $m=2.8$ provides a much better

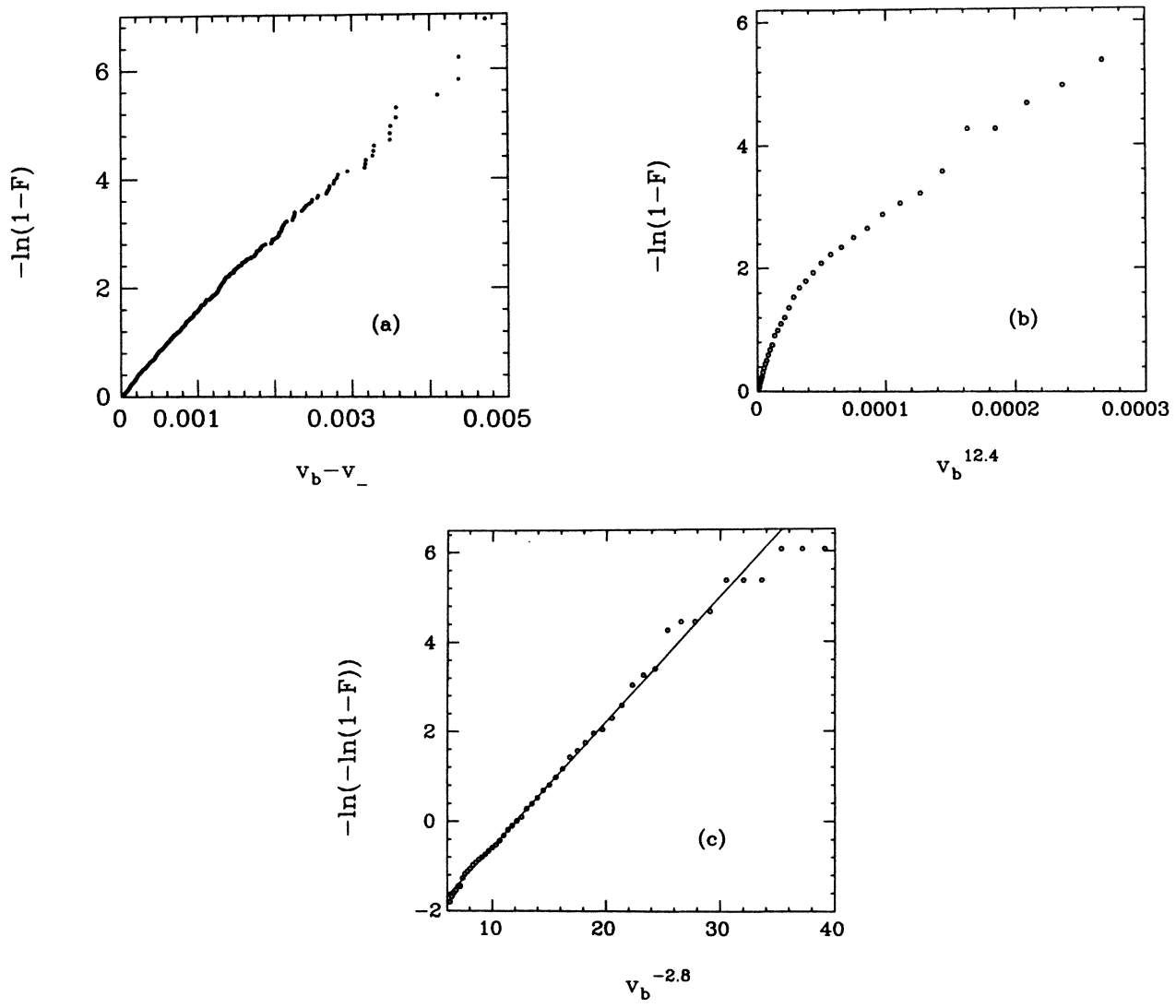
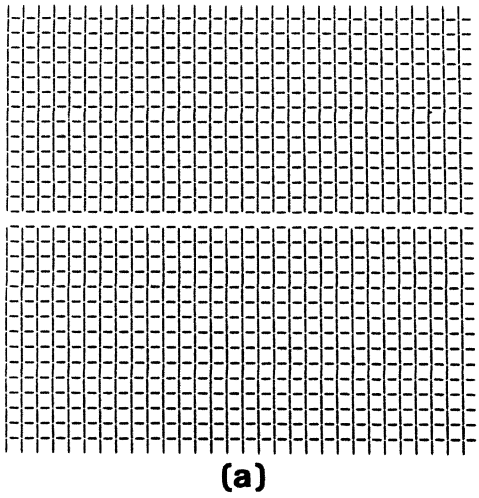


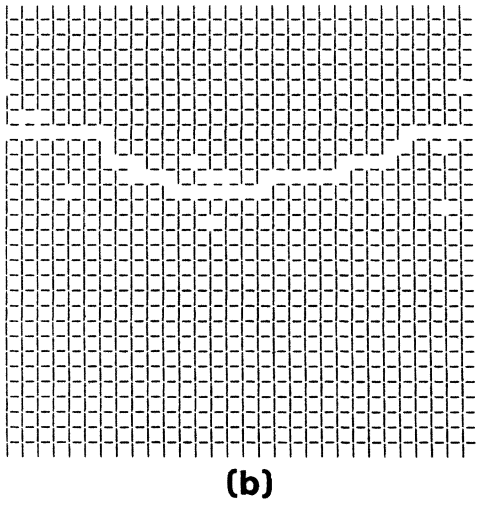
FIG. 7. The distribution of breaking strengths $F(v_b)$. In (a), $-\ln[1-F(v_b)]$ is plotted vs $v_b - v_-$ for the case $w=0.5$, for 1000 configurations of a 30×30 network, to illustrate that $1-F(v_b)$ decays exponentially in v_b. In (b) and (c), data based on 750 configurations on a 50×50 network for the case $w=2$ is shown. In (b), $-\ln[1-F(v_b)]$ is plotted vs v_b^m, where for m we use the value $m=12.4$, obtained by attempting to fit to the Weibull form. In (c), $\ln(-\ln[1-F(v_b)])$ is plotted vs v_b^m, where we now use the value $m=2.8$ from fitting to the double exponential form. The best fit is indicated by the straight line.

quantitative account of the data. Notice that if the distribution of crack lengths was purely exponential, and if the asymptotic relation that the voltage at the tip of a crack of length k increases as $\sqrt{k}$ is used, then we would expect the exponent m to be equal to 2. The discrepancy with the simulations suggests that the crack-length distribution may not be a pure exponential, as was found within the dilute-crack approximation. Furthermore, interactions between cracks might be expected to cause further modifications in the crack-length distribution.

While measurements of $\langle v_b \rangle$ and its distribution are basic quantities which characterize breaking, it is also useful to study properties which are sensitive to geometrical aspects of the breaking process. One such property is suggested by looking at "snapshots" of the network at the breaking point, for values of w which are representative of the brittle and ductile regimes (Fig. 8). In Fig. 8(a), fracture is caused by single-crack growth, while in 8(b) there is a small degree of microcracking, consistent with our qualitative picture of brittle behavior. Finally in 8(c), the critical crack has become less straight, and there is considerable "damage" to the network, in addition to the critical crack.

These pictures suggest that it is useful to study the w dependence of the average number of bonds that have failed, $\langle N_b \rangle$, when the breaking point of the network is reached. For convenience we consider the scaled quantity, $x \equiv \langle N_b \rangle / L - 1$, which appears to be a sensitive way of discerning between brittle and ductile behavior. In the trivial regime, x is identically equal to zero, as the system fails by the formation of a single linear crack which spans the network. In the brittle regime, only a *finite* number





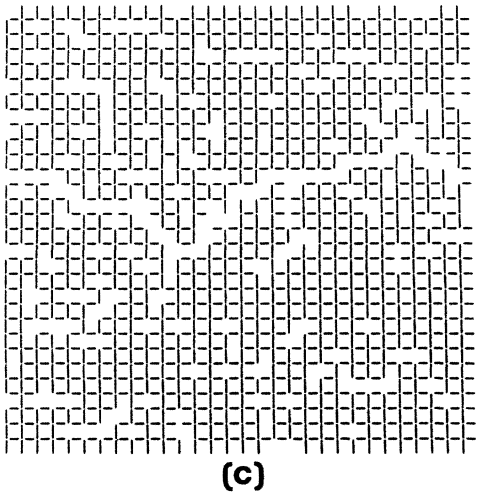


FIG. 8. Snapshots of a 30×30 network at the breaking point for the cases (a) $w=0.5$, (b) $w=1.2$, and (c) $w=2.0$.

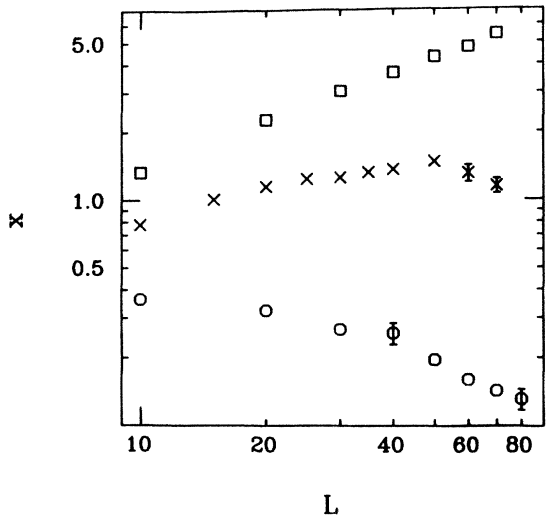


FIG. 9. Plot of $x \equiv \langle N_b \rangle / L - 1$ vs L on a double logarithmic scale, for the representative values $w=0.9$ (○), 1.2 (×), and 1.5 (□). The statistical error bars on the data points are all smaller than those shown explicitly. For $w=0.9$ and 1.2, the data asymptotically lie on a straight line of slope -1, indicating that x vanishes as $1/L$. When $w=1.2$, the data is nonmonotonic, indicative of the crossover from brittle to ductile behavior as L increases. From the figure, we see that the value of w at the crossover point, $w_c(L)$ increases with L. For $w=1.5$, note that x is growing approximately as $L^{0.6}$. Typical error bars are shown.

of bonds need to break before there is catastrophic formation of a linear crack which spans the network. Consequently, $\langle N_b \rangle \sim L + \text{const}$ or $x \sim \text{const}/L$, as $L \to \infty$. We have verified this behavior for w in the range 0.7–1.0 (Fig. 9), and this plot provides a useful check of the assumptions used in our theoretical treatment of brittle fracture. Thus in both the trivial and brittle regimes, the order parameter vanishes, asymptotically.

For larger values of w ($w \gtrsim 1.5$), x is an increasing function of L for all values of L attainable in the simulations, and this is indicative of ductile behavior. Moreover, x appears to be growing as a noninteger power of L in this regime, with a characteristic exponent that seems to be approaching 1 for values of w approaching 2. Furthermore, a nonmonotonicity in x may occur, and this provides a sensitive way of observing the crossover from ductile to brittle behavior as a function of L, for intermediate values of w (cf. Fig. 1). For example, when $w \simeq 1.2$ and L sufficiently small, the network seems to be in the ductile regime of the phase diagram, and x is an increasing function of L. However, for larger values of L, the network is predicted to cross over to brittle behavior, as predicted by Eq. (12), and x will correspondingly vanish as $L \to \infty$ (Fig. 9).

VI. DISCUSSION

We have investigated the breaking of a two-dimensional fuse network, with random, continuously distributed breaking strengths, as a function of the applied external potential. From a general probabilistic ap-

proach, we can distinguish between the brittle and the ductile regimes by the number of bonds broken at the fracture point. In the brittle regime, the average breaking strength of the network, and its distribution, can be found from elementary aspects of extreme value statistics. These theoretical predictions are in excellent agreement with simulation results. In addition, our theoretical approach quantitatively accounts for the number of broken bonds at the fracture point.

In the ductile regime, we used a dilute-crack approximation together with a Lifshitz-type argument to estimate the average breaking strength of the network. This approach is based on approximations whose accuracy is not easily assessed, however. Interactions between cracks might be anticipated to play a role in the breakdown process. Furthermore, the behavior in the ductile regime may be influenced by a variety of crossover effects. These unresolved issues are reflected in our numerical results for the ductile regime. The data does suggest that $\langle v_b \rangle$ decays as $1/(\ln L)^y$, with $y \lesssim 0.8$, and that the distribution of breakdown voltages is of a double exponential form, with a characteristic exponent $m \simeq 2.8$. In comparison, the approximations of dilute *uncorrelated* cracks and a Lifshitz-type argument give $y=0.5$ and $m=2$. Further work is needed to understand whether these apparent differences arise from interaction and enhancement effects, or whether the differences are merely a manifestation of the fact that in our simulations we cannot treat large enough systems.

For the future, three-dimensional fuse networks should also be considered. Much of the analysis of Secs. II and III can be readily extended to arbitrary spatial dimension d, from which one can identify two competing influences which govern the transition between brittle and ductile fracture as a function of d. First, for larger d, the local enhancement of current flow around a small defect is less pronounced. (For a single-bond crack on the simple-cubic lattice, the voltage drop across the nearest-neighbor bonds is approximately equal to 1.0926..., compared to the corresponding value of $4/\pi \simeq 1.2732...$ in two dimensions.) On the other hand, a single-bond crack is horizontally adjacent to $2(d-1)$ "edge" bonds. Thus, while local current enhancement diminishes with increasing d, there are many more possible ways, geometrically, by which the initial crack could propagate. From numerical simulations, the w dependence of the stability probability (cf. Sec. II A), for an $L=51$ simple-cubic network, when a single vertical bond at the center is removed, is quite similar to that of the square lattice. Thus the competition between lessened current enhancement and more routes for crack propagation nearly balances out, at the level of a single bond breaking. Moreover, the transition between the brittle and ductile regimes is predicted to occur when $O(L^{d/2})$ bonds have broken, and the corresponding transition value of w is $w_c(L) \sim 2[1-O(L^{-d/2})]$.

Another interesting aspect of three-dimensional breaking is that the fracture interface will be considerably more complex than in two dimensions. In two dimensions, it is clear how two nearby cracks which are noncolinear can join by the formation of a "kink." However, in three dimensions, the manner in which two nearby planar cracks join will be more complicated. It is also interesting to inquire how linear cracks, which occur in the initial stage of a breaking process, will evolve into the planar cracks, which are experimentally known to be the basic ingredients of breaking in three dimensions.

A second general question to consider is the influence of the form of the bond breaking and bond conductance distributions on breaking. To account for more general bond breaking strengths, consider the power-law distribution

$$P(v) \sim (v - v_-)^k, \quad v_- \le v \le v_+ , \tag{24}$$

and with $v_- > 0$. By following the reasoning in Sec. II, the breaking strength of the weakest bond in a finite sample of L^2 bonds, whose strengths are distributed according to Eq. (24), is given by

$$\langle v_{\text{weakest}} \rangle \simeq v_- + \frac{\text{const}}{(L^2)^{\mu}} , \tag{25}$$

where $\mu = 1/(k+1)$. Thus for a system with relatively few weak bonds, i.e., the case $k > 1$, the weakest bond in the system is further from the theoretical minimum than in the uniform distribution. In addition, the strength of the nth weakest vertical bond in an $L \times L$ network is, in analogy with Eqs. (6),

$$\langle v_{\text{weakest}}(n) \rangle = v_- + bw \left[\frac{n}{L^2} \right]^{\mu} , \tag{26a}$$

and the expected strength of the weakest of the $2n$ bonds at the edges of the n single-bond cracks is

$$\langle v_{\text{edge}}(n) \rangle = v_- + \frac{aw}{2n^{\mu}} , \tag{26b}$$

with a and b constants of order unity. From these results, it follows that the nature of the brittle-ductile transition is qualitatively similar to that found for the uniform distribution, except that crossover effects may be expected to play a relatively more important role. However, in the extreme case where $v_- = 0$, the breaking of the network might be expected to be more sensitive to the low-voltage tail of the distribution.

An additional question to consider is the kinetics of the breaking process. We have considered the general situation where the stress relaxation time is less than the bond-breaking time. The opposite case where all the "overstressed" bonds break at a single time step may be worth pursuing. This "rapid breaking" rule is also much less demanding computationally, and simulations could be extended to larger system sizes to help clarify some of the slow crossover effects discussed above. The rapid breaking rule also gives rise to interesting geometric patterns. From small-scale simulations, we observe features which are similar to the crazing that is sometimes observed at the tip of a crack in various materials. These features are also reminiscent of the patterns observed in dielectric breakdown phenomena.[25] This also leads to the related question in the time dependence of the breaking process, and we hope to develop models by which the breakdown time can be calculated.

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