Directed percolation in 3+1 dimensions

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Directed percolation in four dimensions is of direct physical relevance to the world with three space and one time dimension. We present a comprehensive analysis of recently extended series for the moments of the cluster-size distribution and for the percolation probability in a "field" on the hypercubic lattice. We find a critical threshold, $p_c=0.3025\pm0.0010$, and dominant critical exponents $\gamma=1.21\pm0.05$, for the mean cluster size; $\beta=0.82\pm0.03$ and $1/\delta=0.45\pm0.02$ for the percolation probability in the thermal and field directions respectively; and a gap exponent of $\Delta=2.03\pm0.06$. We find a thermal-correction exponent $\Delta_1=0.55\pm0.15$ and a field correction of $\Omega=0.3\pm0.1$. We also calculate some universal critical-amplitude ratios.

I. INTRODUCTION

Directed percolation in three spatial and one time dimension ($d=3+1$, or four dimensions in total) is of direct physical relevance to the real world. Applications include the spread of disease through any three-dimensional container of stored produce, flow in a three-dimensional crack network in a porous rock, and certain questions in galactic evolution. Notwithstanding these possibilities, relatively little high-precision numerical work was undertaken for this problem until quite recently. Short series expansions[1] and ϵ-expansion calculations[2] within Reggeon field theory (RFT) led to the early exponent estimates which are given in Table I together with some very recent[3] RFT results. RFT is believed to be[2] in the same universality class as directed percolation (DP), and since the upper critical dimension for this theory is $(4+1)=5$ dimensions, the ϵ-expansion exponents in $(3+1)=4$ dimensions could be expected to be reliable. Critical thresholds, however, cannot be obtained from RFT and it is most desirable to be able to determine high-precision exponents directly within the percolation model in order to check the universality question. It is also of interest to evaluate correction to scaling exponents in order to determine whether the correspondence between DP and RFT extends to the irrelevant operators.

In order to obtain thresholds and dominant and correction exponents directly for the $d=4$ DP problem, the generation and analysis of extended low-density power-series expansions would appear to be the approach of choice. Such series generations have recently[4] been carried out for the moments of the DP cluster distribution for the hypercubic $d=4$ site problem and this generation will be discussed more fully in Sec. II.

The moments, $m^{i+j}(p)$ of the distribution of the number of connected clusters per site with s sites and perimeter t are believed to behave as

$$m^{i+j}(p)\sim |p-p_c|^{-\gamma-(i+j-1)(\gamma+\beta)} \times(1+B|p-p_c|^{\Delta_1}+\cdots) . \quad (1)$$

The sites are occupied with probability p, and i and j $(i+j>0)$ define the moment weighting with respect to the number of sites in the cluster and to the cluster perimeters, respectively. The $m^{i+j}(p)$ will also be denoted as $\langle s^i t^j\rangle$. The exponent sum $\gamma+\beta$ is usually known as the gap exponent and is sometimes denoted by Δ. The results of some preliminary threshold and gap exponent estimates from these series[4] are presented in Table I. The gap estimates were obtained by division of the series to give series with a dominant critical exponent of Δ. Series for the full perimeter polynomials for the $d=4$ hypercubic site problem up to order $s=13$ are also now available.[5,6] These have been used by Carvalho and Duarte[7] to deduce the scaling function in general dimension and to give a p_c estimate for the hypercubic site problem. This estimates is given in Table I together with some results from two order $s=12$ calculations.[6,8] In addition to the above studies, knowledge of these polynomials enables us to deduce the full percolation probability $P(p,\lambda)$, where $\lambda=e^{-H}$, and H is analogous to a magnetic field.[9] This percolation probability is assumed to have the critical behavior

TABLE I. Critical-exponent and temperature estimates for four-dimensional hypercubic-site percolation.

	Ref.	p_c	γ	β	Δ	$1/\delta$	Δ_1	Ω
			RFT ϵ expansion					
	2		1.21425	0.79410	2.01		0.3	
	3			0.82	2.03			
			DP series					
Mean cluster size ($s=8$)	1		1.230±0.005					
Moments ($s=13$)	4	0.303±0.0015		$0.81^{+0.02}_{-0.04}$	$2.02^{+0.04}_{-0.02}$			
Cluster no. ($s=13$)	6	0.30±0.02						
			This work DP series					
Overall		$0.3025^{+0.001}_{-0.001}$	1.21±0.05	0.82±0.03	2.03±0.06	0.45±0.02	0.55±0.15	0.3±0.1
Mean cluster size		0.303±0.001						
Mean perimeter		0.302±0.001						
Divided series					2.01±0.06			
Different p_c		0.303	1.230	0.820	2.05	0.43	0.55	0.33
		0.3025	1.205	0.825	2.04	0.45	0.55	0.4
		0.302	1.200	0.800	2.00	0.46	0.55	0.4
Scaling relation $\Omega=\Delta_1/\beta\delta$								0.3±0.1

$$P(p_c,\lambda)\sim(1-\lambda)^{1/\delta}[1+b(1-\lambda)^{\Omega}+\cdots] . \quad (2)$$

In a recent calculation[9] we showed that in $d=2$ and $d=3$ the "field" correction, Ω, obeyed the scaling relation[10]

$$\Omega=\Delta_1/\beta\delta , \quad (3)$$

although as discussed in Ref. 9 this relation has not yet been shown numerically to be satisfied in any other system.

In the present paper a more sophisticated analysis of the moments and the first analysis of $P(p_c,\lambda)$ is presented. This analysis is described in detail in Sec. III and addresses the question of correction exponents as well as giving more precise threshold and dominant exponent values, and a check on Eq. (3). Some universal amplitude ratios are calculated in Sec. IV.

The results are found to be in excellent agreement with the RFT values when these are available, and a discussion of their implications and of some potential applications is given in Sec. V. Our results are summarized in Table I.

II. SERIES DEVELOPMENT

Four-dimensional directed percolation is no longer usefully approached by a transfer-matrix technique[1] as happens in two and three dimensions. In this fully-directed site-percolation model the recursion decomposition described in detail by Duarte[5] was specifically implemented in four dimensions, and not as a special case of an exhaustive enumeration of d-dimensional strong embedding.

For an equivalent number of configurations, our recursion relations take longer in four than in five and six dimensions. The first neighborhood decomposition of an explicit size (s) and perimeter (t) counting, can be written by inspection

$$g_{st}=4g_{s-1,t-3}+\binom{4}{2}g_{st}^{(2)}+\binom{4}{3}h_{st}^{(3)}+j_{st}^{(4)} , \quad (4)$$

where g_{st} is the number of clusters with size s, and perimeter t, $g_{st}^{(2)}$ is the number of clusters with a planar kernel consisting of the root site and a compact source of size 2, $h_{st}^{(3)}$ is the number of clusters with a four-dimensional kernel (root site plus three of its neighbors) and $j_{st}^{(4)}$ is the number of clusters with the first four-dimensional compact source as a kernel.

The $g_{st}^{(2)}$ can still be further decomposed, using an adaptation of Eq. (C2) in Ref. 5. These economies mean that only 16% of the total number of configurations must be counted at each size.

From the complete perimeter polynomials, low-density expansions of the moments $m^{i+j}(p)$ follow

$$m^{i+j}(p)=\langle s^i t^j\rangle=\sum_{s,t} s^i t^j g_{st} p^s (1-p)^t . \quad (5)$$

All possible orders and combinations of i and j are of course possible. In Table II we publish the first set of moments up to $i+j<4$. Notice that instead of s or t the bond content could also be used for such generalized moments.

III. ANALYSIS OF THE SERIES

We consider the moments, $m^{i+j}(p)$, for several choices of i and j and $P(p_c,\lambda)$, as $\lambda\rightarrow 1$, series for the hypercubic site DP problem. We have analyzed the series both with the Roskies transform, as presented in Ref. 10 on p. 409 for isotropic percolation (see also Ref. 11), and with a graphical version of the method of Adler *et al.*[12] Both methods give plots of different Padé approximants to the dominant exponent as a function of the correction exponent for different choices of p_c in the moment series, and in the series for $P(p_c,\lambda)$, as $\lambda\rightarrow 1$. At the correct values of dominant and correction exponent these approximants are expected to converge.

In addition to these analyses of individual series, we have used a method (see, for example, Ref. 13) recently revived by Meir.[14] This method involves term-by-term division of two series with the same critical threshold and then study of the divided series. This divided series should have critical behavior of a threshold at 1 and a dominant critical exponent equal to the difference between the exponents of the two original series plus 1. The division is expected to introduce an analytic correction to scaling (i.e., $\Delta_1=1$). If this correction has a large enough amplitude it could provide a nice convergence region for the evaluation of the dominant exponent. It is to be hoped that the amplitude of the introduced analytic correction is sufficient to swamp the nonanalytic correction of the individual series which is still present. This method avoids the problems associated with uncertainties in p_c and is ideal for the moment series.

The various moments and the percolation probability have been studied with p_c choices in the range 0.300–0.305. The moments with $j\neq 0$ do not have an initial constant term and therefore their derivatives were studied. The analysis for the $\langle s\rangle$ series at $p_c=0.303$ is given in Fig. 1(a) and that for the $\langle t\rangle$ series at $p_c=0.302$ is given in Fig. 1(b). We chose to illustrate in these figures those threshold choices that gave the best convergence. The region of best convergence for the mean size, $j=0$, series fell at slightly higher p_c and γ estimates than that for the perimeter series. From these and the analyses of the other higher moments we conclude that $p_c=0.3025\pm0.001$, and $\Delta_1=0.55\pm0.15$ and give the first of our gap exponent estimates $\Delta=2.04^{+0.02}_{-0.04}$. Estimates

TABLE II. Low-density moment, $m^{i+j}=\sum_s a(i,j,s)p^s$, expansions for the four-dimensional hypercubic site problem.

i	j	s	$a(i,j,s)$	$a(i-1,j+1,s)$
1	0	0	1	0
1	0	1	4	4
1	0	2	16	12
1	0	3	58	42
1	0	4	208	150
1	0	5	724	516
1	0	6	2524	1800
1	0	7	8618	6094
1	0	8	29 682	21 064
1	0	9	100 264	70 582
1	0	10	342 958	242 694
1	0	11	1 150 145	807 187
1	0	12	3 919 488	2 769 343
1	0	13	13 060 059	9 140 571
2	0	0	1	0
2	0	1	12	4
2	0	2	92	40
2	0	3	550	274
2	0	4	2916	1558
2	0	5	14 156	7936
2	0	6	65 170	37 690
2	0	7	286 290	169 844
2	0	8	1 222 106	737 622
2	0	9	5 063 308	3 105 700
2	0	10	20 613 702	12 780 496
2	0	11	82 164 267	51 521 363
2	0	12	323 821 242	204 505 371
2	0	13	1 255 419 717	799 710 993
3	0	0	1	0
3	0	1	28	4
3	0	2	376	96
3	0	3	3466	1166
3	0	4	26 032	10 186
3	0	5	169 948	73 560
3	0	6	1 009 072	467 870
3	0	7	5 562 890	2 719 688
3	0	8	29 037 450	14 765 926
3	0	9	144 723 196	76 051 620
3	0	10	696 069 418	375 228 580
3	0	11	3 243 361 817	1 788 071 917
3	0	12	14 738800 032	8 271 733 273
3	0	13	65 436 320 865	37 331 320 941

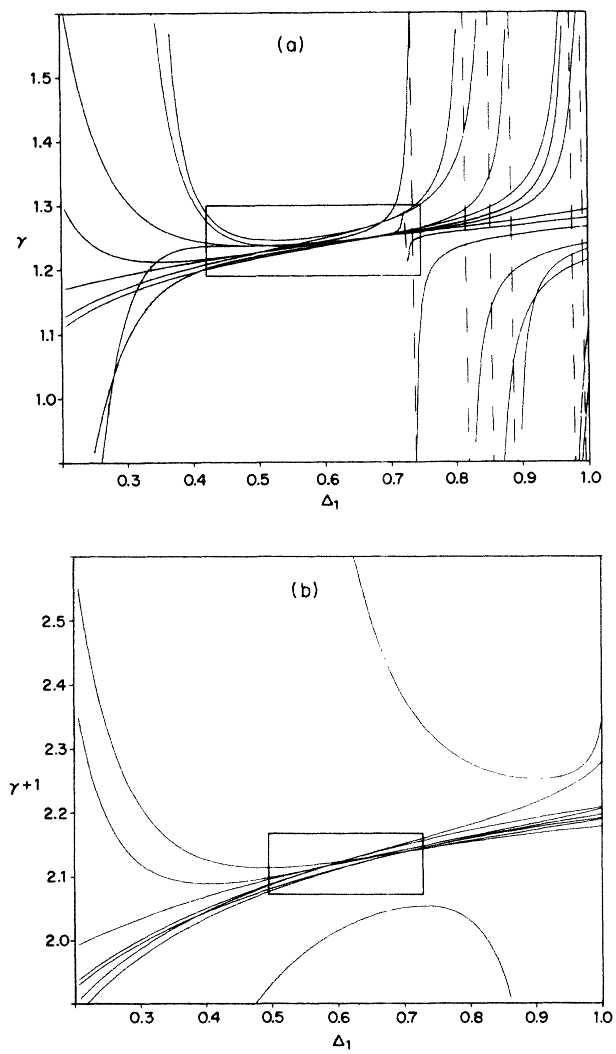


FIG. 1. Graphs of different Padé approximants to γ as functions of Δ_1 from (a) the mean cluster size series, $\langle s\rangle$, at $p_c=0.303$ and (b) the first derivative of the mean perimeter series, $\langle t\rangle$, at $p_c=0.302$, using the Roskies transform method of Ref. 10.

of the exponents favored for the different extremes of the p_c range are given at the bottom of Table I. The β estimates are deduced from the difference between the gap exponent and γ both for each p_c choice and for the overall values.

The analysis of two different combinations of the divided series is given in Fig. 2. In Fig. 2(a) we illustrate the analysis of the series obtained from the ratio of the $\langle s^2 \rangle$ and $\langle s \rangle$ series, and in Fig. 2(b) we show that of the $\langle s^3 \rangle$ and $\langle s \rangle$ series. We see that the former is best converged over a fairly wide range of Δ_1 choices, but that this range is clearly centered at a higher value than either of those in Fig. 1, but not necessarily quite at $\Delta_1 = 1.0$. Thus, the analytic correction may not completely swamp the nonanalytic correction of the individual moments. Nevertheless, we read off the Δ values at $\Delta_1 = 1.0$, noting that since the graphs are relatively flat, this choice is not crucial. We find $\Delta = 2.01 \pm 0.06$ from these and other divided series plots which leads us to an overall Δ estimate of 2.03 ± 0.06. It should be noted that the direct gap exponent estimate is somewhat lower than the individual moment one. This could lead us to the conclusion that we should favor a central p_c choice of, say, 0.302 25. The overall choice of 0.3025 is based on tightness of convergence right up to 0.303 for the moment series but would be only marginally preferable on this basis.

A graph of Padé approximants to $1/\delta$ from the analysis of the percolation probability series at $p_c = 0.302\,25$ and $\lambda = 1$ is given in Fig. 3. We see here optimal convergence for $\Omega \sim 0.4$ and note that Ω decreases as p_c increases. The δ and Ω estimates for different p_c choices are given in Table I.

We now discuss the validation or otherwise the scaling relation given in Eq. (3). From the overall exponent estimates we calculate that $\Delta_1/\beta\delta = 0.3 \pm 0.1$. This is in excellent agreement with the direct evaluation of Ω and does seem to to support the choice of the higher central p_c estimate since the higher p_c values give lower Ω values.

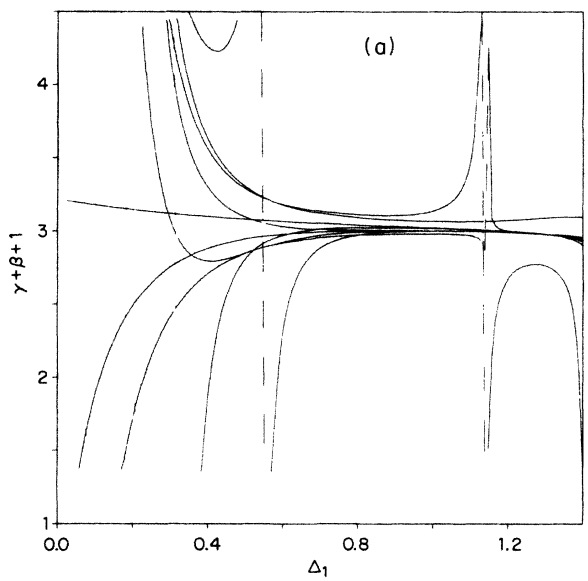
IV. AMPLITUDE RATIOS

Amplitude ratios, though less studied than exponents, also serve as universal quantities which may be used to characterize the universality class.[15] In our case the quantities

$$R_{kl/mn} = \langle s^k \rangle \langle s^l \rangle / \langle s^m \rangle \langle s^n \rangle$$

are universal and are also equal to

$$\langle s^{k-1} t \rangle \langle s^{l-1} t \rangle / \langle s^{m-1} t \rangle \langle s^{n-1} t \rangle \ .$$



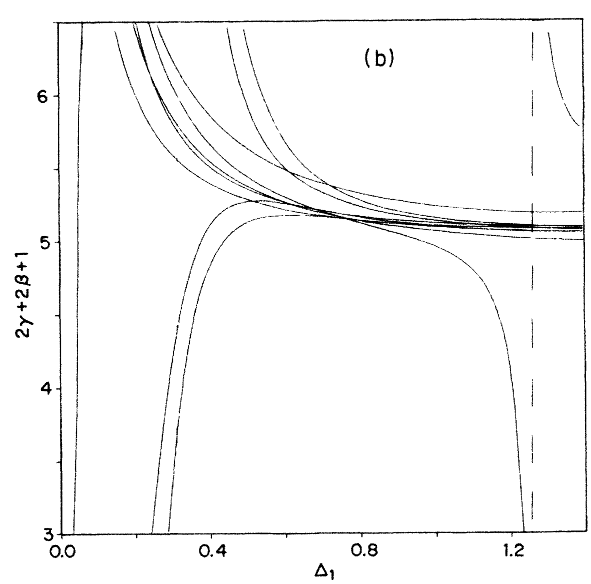


FIG. 2. Graphs of different Padé approximants to $\Delta = \gamma + \beta$ as functions of Δ_1 from (a) the term-by-term divided $\langle s^2 \rangle / \langle s \rangle$ ratios, (b) the term-by-term divided $\langle s^3 \rangle / \langle s \rangle$ ratios, using the Roskies transform (Ref. 10) on the divided series.

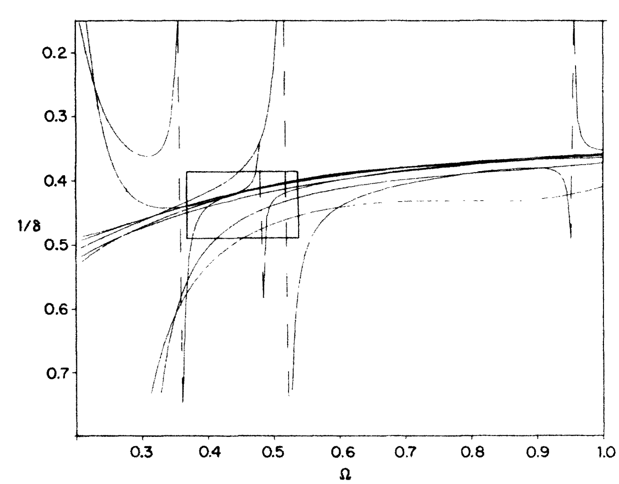


FIG. 3. Graphs of different Padé approximants to $1/\delta$ as a function of Ω from the percolation probability $P(p_c, \lambda)$ with $p_c = 0.302\,25$ at $\lambda = 1$ using the Roskies transform (Ref. 10) method.

In order to estimate the amplitude ratios we used a newly developed method.[14] This method, which involves multiplying and dividing the coefficients of the corresponding series, leads to estimates of the universal quantities

$$S_{kl/mn}=\frac{R_{kl/mn}\Gamma(\gamma_m)\Gamma(\gamma_n)}{\Gamma(\gamma_k)\Gamma(\gamma_l)} . \tag{6}$$

Here $\gamma_i=\gamma+(i-1)(\gamma+\beta)$ and Γ is the usual Γ function. The $S_{kl/mn}$ are unbiased by the value of the critical threshold or by the values of the critical exponents. Application of this method to the ratio $\langle s^2\rangle\langle s^4\rangle/\langle s^3\rangle\langle s^3\rangle$ leads to the estimate $S_{24/33}=0.480\pm0.005$. The ratio $\langle st\rangle\langle s^3t\rangle/\langle s^2t\rangle\langle s^2t\rangle$ leads to the same estimate, where the mean field value in five dimensions is $S_{24/33}=\frac{1}{2}$. Although S is a universal quantity, one may use the estimates obtained in this paper for the exponents in order to express the ratios in terms of R, which leads to $R_{24/33}=2.47$ (compared to a mean field value of 3). The error in R is very sensitive to the error estimates for the exponents, due to the γ function involved, and we do not quote it here.

Although the field theory for this model has been extensively studied,[2] there is no ϵ expansion for the amplitude ratios. In a future work we hope to report on an ϵ expansion for the amplitude ratios in DP in the context of field theory as well as extension of our amplitude ratio series analysis to other dimensions.

V. DISCUSSION

A glance at Table I confirms that our dominant exponent values are very close to those of RFT. However, the measured DP correction exponent Δ_1 is rather higher than the RFT one. The quoted RFT value is calculated via the relation $\Delta_1=\lambda\nu$ from the λ and ν estimates.[2] The ε-expansion RFT λ and ν values are somewhat smaller than the direct loop RFT calculations in the lower dimensions, and might likewise be too low in $3+1$ dimensions. However, we cannot exclude the possibility that the correction we are observing is due to a different irrelevant operator to that seen in the RFT. This is especially likely in view of the fact that this is a consistent trend. In $2+1$ dimensions the observed[9] Δ_1 of 0.75 ± 0.10 is somewhat higher than one of the RFT estimates of 0.62 ± 0.02, although it is consistent with the other, 0.74 ± 0.02. If the irrelevant operator from the RFT calculation is distinct from the one that we are observing in $3+1$- and $2+1$-dimensional DP, then it may have a small amplitude in higher dimensional DP. We do not have an available resummed Ω value calculated within the RFT, but if we calculate Ω via Eq. (3) from the RFT values with our $1/\delta$, then we find a value of less than 0.2 which is inconsistent with our direct evaluations. Thus, we may conclude that we do not observe the same first irrelevant operator for $3+1$-dimensional DP as that calculated for the three-dimensional RFT.

We conclude with a brief description of how our results relate to a potential application of our model. DP in two space and one time dimension has proven to be very useful for modeling galaxies that are almost flat.[16] There are also some galaxies of interest that have a three-dimensional nature and the development in time of such systems could be modeled by a $3+1$-dimensional DP model. One possible approach to the study of such a system would be a $3+1$-dimensional Monte Carlo simulation for a family of site DP models that could include the one described above as a special case. Thus, our determination of p_c would provide an important baseline check for the programs written for such a project.

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