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Percolation and Cosserat elasticity: Exact results on a deterministic fractal

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The elastic problem in two-dimensional percolation is investigated on a hierarchical lattice, in the context of Cosserat elasticity (or granular elasticity). For this regular fractal, the exponent of elastic moduli (T/ν) is slightly smaller than the expected value ($t/\nu+2$) deduced from the conductivity exponent (t/ν). The correction is related to the distribution of a new geometric parameter (the "eccentricity"). We also discuss the inequalities $1+d\nu<T<t+2\nu$, the anisotropy of the elastic response, and the case of scalar elasticity.

The critical behavior of elasticity in lattice percolation has motivated various studies.[1–11] By now, four microscopic models of elasticity have been considered,[1–3,6] each of them describing a different kind of real system. In this article, we are concerned with the two-dimensional form of what Feng[6] has named the "granular model." In this case, the elastic energy W of the network depends on the displacement $\mathbf{u}_i=\mathbf{R}_i'-\mathbf{R}_i$ of each site i, and on its in-plane rotation θ_i:

$$2W=\sum_{\langle ij\rangle}p_{ij}\{k_\parallel(\mathbf{u}_{ij})_\parallel^2+k_\perp[(\mathbf{u}_{ij})_\perp-\tfrac{1}{2}(\theta_i+\theta_j)(\hat{\mathbf{z}}\times\mathbf{R}_{ij})]^2+k_{\theta\theta}(\theta_{ij})^2\}\,, \quad (1)$$

where $\hat{\mathbf{z}}$ is the unit vector normal to the plane and $p_{ij}=1$ or 0 with respective probabilities p and $1-p$ (note the convention $\theta_{ij}=\theta_i-\theta_j$ which also holds for $\mathbf{u}$ and $\mathbf{R}$). This expression can be understood[6,7] as the potential energy of a random network of rigid disks with elastic junctions (depleted granular medium). An equivalent picture[8–10] is also a percolating network of elastic bars rigidly joined at nodes ("fibrous" medium), such as the honeycomb studied in the experiment of Allain *et al.*[11] In this case, the three bond stiffnesses $k_\parallel$, $k_\perp$, and $k_{\theta\theta}$ are, respectively, associated with an elongation, a shear,[12] and a pure bending[13] of the bars. Because of the presence of angular elasticity, this model is conceptually close to the "bond-bending" model,[3–5] and both of them are expected to be equivalent with respect to the critical properties.[6] In particular, when the three bond stiffnesses are nonzero, the backbone of the infinite clusters, and also the percolation threshold p_c, are identical to those defined from geometric connectivity.[7]

Contrary to the central-forces model,[2] the present one does not lead to the usual, symmetric theory of elasticity when considered from a macroscopic viewpoint. For length scales larger than the correlation length $\xi\sim(p-p_c)^{-\nu}$, such a lattice constitutes what is called in mechanics a "Cosserat continuum."[14] In particular, the elastic energy density involves both the symmetric part of the strain $u_{\alpha\beta}=\frac{1}{2}(\partial_\alpha u_\beta+\partial_\beta u_\alpha)$, and the antisymmetric part $\omega_{xy}=\frac{1}{2}(\partial_x u_y-\partial_y u_x)$:

$$2w=\lambda u_{\alpha\alpha}u_{\beta\beta}+2\mu u_{\alpha\beta}u_{\alpha\beta}+2\mu'(\omega_{xy}-\theta)^2+\sigma'(\nabla\theta)^2\,, \quad (2)$$

where λ, μ, μ', and σ' are the generalized Lamé coefficients which describe an isotropic medium. The Cosserat theory belongs to a class of unusual theories of asymmetric elasticity,[14] which are expected to take into account the influence of a microstructure in a material. This microstructure may be a grain in a granular material, or a crystallite in a polycrystal, and the classical theory is just an approximation, which is enough for most problems of practical interest. A novelty introduced by these theories was the appearance of couple stresses[14] in the equilibrium equations, in addition to the usual stresses. In our case, this corresponds to the propagation of bending moments[13] in the chains of the backbone.

In this article, we study the elastic properties of a fractal model of the backbone, for this kind of elasticity. Its predictions are in perfect agreement with the previous theoretical works:[3,6,9] the critical exponent T of the usual elastic moduli $\lambda\sim\mu\sim(p-p_c)^T$ is larger than the conductivity exponent t of a factor of order 2ν. The new result is that this factor is in fact slightly smaller than 2ν, and that the correction involves the distribution of a new geometric parameter, the "eccentricity" of the chains relative to their end points. Thus, our estimate $T\cong3.87$ is just bounded between those which are, respectively, deduced from the red bonds' contribution[3,6] $1+2\nu\cong3.67$, and from the "scalar analogy"[9] between bending and conducting $t+2\nu\cong3.97$. These two bounds will also be derived again in the context of Cosserat elasticity. A more detailed derivation of some of these results is also available in a separate publication,[10] but is focused on a less realistic representation of the backbone.

The recursive rule of construction of our model is suggested in Fig. 1, as an inflation process. After n iterations, the Euclidian distance between the end points A and B is equal to $L(n)=(1+\sqrt{2})^nL(0)$. This object is just the hierarchical model of the backbone on which de Arcangelis *et al.*[15] have studied the voltage moments, but with some additional geometric specifications. Following this reference, we can calculate some typical exponents, such as the fractal dimension of the red bonds d_{red}, the one of the backbone d_{BB}, the exponent ζ_R of the electric resistance ($R\sim L^{\zeta_R}$), and also ζ_{SAW} which describes the scaling of the average number of steps in the set of all self-avoiding walks: $N_{\text{SAW}}\sim L^{\zeta_{\text{SAW}}}$. They are compared

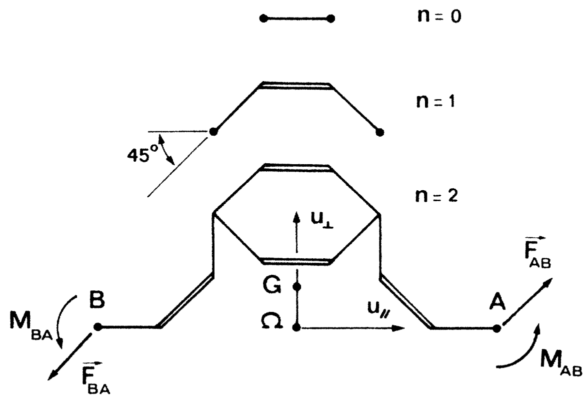

FIG. 1. Two successive stages of iteration of the hierarchical model under study, and the stresses which can be exerted on its end points. Note that static equilibrium implies $\mathbf{F}_{AB}+\mathbf{F}_{BA}=0$ and $M_{AB}+M_{BA}+LF_{AB}^{\perp}=0$.

in Table I with the known percolation values in two dimensions. The agreement is rough but comparable with that obtained by Mandelbrot and Given[16] on a Koch curve. Note that in our case, the distribution of the red bonds is more realistic, since their gyration radii $\langle u_{\parallel}^2\rangle^{1/2}$ and $\langle u_{\perp}^2\rangle^{1/2}$ relative to the point Ω of Fig. 1 are both nonzero and therefore scale like L. This property of percolation clusters has been numerically checked by Kantor,[17] and is very important in their elastic behavior.[3,6]

We now define another geometric parameter as the "eccentricity" $\mathbf{\Omega G}$, where G is the gravity center of the red bonds, and Ω is the center of the end points A and B (see Fig. 1). This parameter will be very important in the loops' behavior. It scales like L, just as the gyration radii, but vanishes when it is averaged over the two possible configurations of the chain between A and B. It seems rather justified to expect a similar property for percolation clusters, but with a continuous distribution of eccentricity instead of a binary one.

We now need a way to describe the elasticity of our structure between its two end points. It is first important to realize that the bond interaction in Eq. (1) is not the most general expression of a rotationally invariant elastic energy. In the general case, the three bond stiffnesses must be replaced[10] with a 3×3 symmetrical matrix ${}^3\overline{\overline{K}}$, which can be called the "stiffness matrix." Thus we can write the elastic energy stored up between A and B as $2U_{AB}={}^3\mathbf{v}_{AB}\cdot{}^3\overline{\overline{K}}\cdot{}^3\mathbf{v}_{AB}$ with ${}^3\mathbf{v}_{AB}=(\mathbf{v}_{AB},\theta_{AB})$ and $\mathbf{v}_{AB}=\mathbf{u}_{AB}-\frac{1}{2}(\theta_A+\theta_B)(\hat{\mathbf{z}}\times\mathbf{R}_{AB})$. If both the axes $(\Omega u_{\parallel})$ and $(\Omega u_{\perp})$ were axes of symmetry, this quadratic form would become diagonal as in Eq. (1). It is convenient to define conjugated stresses ${}^3\mathbf{F}_{AB}=(\mathbf{F}_{AB},M'_{AB})$ of ${}^3\mathbf{v}_{AB}$. These stresses can also be understood as the decomposition of the external stresses of Fig. 1, which is suggested in Figs. 2(a) and 2(b). $\mathbf{F}_{AB}$ is simply the force which is transmitted from A to B, and M'_{AB} is the part of the pure bending of the moments $M'_{AB}=\frac{1}{2}(M_{AB}-M_{BA})$. The "deformability matrix" ${}^3\overline{\overline{S}}={}^3\overline{\overline{K}}^{-1}$ plays the role of the electric resistance in a generalized Ohm's law:

$$\begin{bmatrix}\mathbf{v}_{AB}\\ \theta_{AB}\end{bmatrix}=\begin{bmatrix}\overline{\overline{S}} & \mathbf{S}_\theta\\ \mathbf{S}_\theta & S_{\theta\theta}\end{bmatrix}\begin{bmatrix}\mathbf{F}_{AB}\\ M'_{AB}\end{bmatrix}. \qquad (3)$$

The scalar $S_{\theta\theta}$ and the tensor $\overline{\overline{S}}$ can be called, respectively, the "bending" and "stretching" deformabilities. $\mathbf{S}_\theta$ is a possible coupling vector which will be related later to the eccentricity. One can describe our structure with ${}^3\overline{\overline{K}}$ as well as with ${}^3\overline{\overline{S}}$, but it is easier to renormalize ${}^3\overline{\overline{S}}$. In particular, one can derive series formulas for ${}^3\overline{\overline{S}}$, but with some cautions since M'_{AB} is nonconservative as in the theory of beams,[13] and $\mathbf{v}_{AB}$ is nonadditive. This gives,[10] for a series combination of N elastic elements $(A_{i-1}A_i)$, between the two terminals $A_0=A$ and $A_N=B$:

$$\begin{aligned} S_{\theta\theta}&=\sum_{i=1}^{N}S_{\theta\theta}^{(i)}\,,\\ \mathbf{S}_\theta&=\sum_{i=1}^{N}[\mathbf{S}_\theta^{(i)}-S_{\theta\theta}^{(i)}(\hat{\mathbf{z}}\times\mathbf{r}_i)]\,,\\ \overline{\overline{S}}&=\sum_{i=1}^{N}[\overline{\overline{S}}^{(i)}-\mathbf{S}_\theta^{(i)}\otimes(\hat{\mathbf{z}}\times\mathbf{r}_i)-(\hat{\mathbf{z}}\times\mathbf{r}_i)\otimes\mathbf{S}_\theta^{(i)}\\ &\qquad+S_{\theta\theta}^{(i)}(\hat{\mathbf{z}}\times\mathbf{r}_i)\otimes(\hat{\mathbf{z}}\times\mathbf{r}_i)]\,, \end{aligned} \qquad (4)$$

where $\otimes$ designates the dyadic product between two vectors, and where $\mathbf{r}_i$ is the position of the center of $(A_{i-1}A_i)$ relative to Ω, center of (AB): $\mathbf{r}_i=\frac{1}{2}(\mathbf{\Omega A}_{i-1}+\mathbf{\Omega A}_i)$. We can deduce from these equations some qualitative indications concerning the scaling behavior of the deformabilities. In the first equation, the bending deformability is additive, just like the electric resistance R. This suggests that the scaling exponents of these two quantities

TABLE I. Comparison of some typical exponents of our model with their equivalents in 2D percolation.

	$d_{\rm red}$	ζ_R	$d_{\rm SAW}$	d_{BB}
Percolation	$\frac{1}{\nu}=0.75^{\rm a}$	$\frac{t}{\nu}=0.97^{\rm a}$	1.3ᵃ	1.6ᵃ
Model	0.786	1.040	1.247	1.572

ᵃFrom Refs. 15 and 16, and references therein.

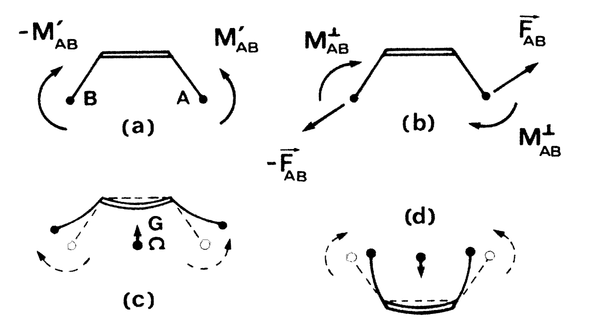

FIG. 2. The external stresses can be analyzed as the superposition of a bending (a), and of a stretching (b) with $M_{AB}^{\perp}=-\frac{1}{2}LF_{AB}^{\perp}$ (rotational equilibrium). Because of the eccentricity $\mathbf{\Omega G}$, the bending will induced an elongation of one side of the loops, and a contraction of the other [(c) and (d)].

($R \sim L^{\zeta R}$ and $S_{\theta\theta} \sim L^{\zeta}$) could be rather close ($\zeta \simeq \zeta_R$). In fact, because of the existence of loops, we can only apply this argument to the series combination of the red bonds, which underestimates $S_{\theta\theta}$ and R. Therefore we can only say that both exponents have the same lower bound: $\zeta > d_{\rm red}$ and $\zeta_R > d_{\rm red}$.

Let us now consider the second equation. We understand a bit more about the relationship between the coupling vector $\mathbf{S}_\theta$ and the eccentricity, for which we can now give a better definition: $\mathbf{\Omega} G = \sum S_{\theta\theta}^{(i)} \mathbf{r}_i / \sum S_{\theta\theta}^{(i)}$ where the sums are performed both over the red bonds and over the loops. In particular, because of the eccentricity contribution in $\mathbf{S}_\theta$, the bending of our structure [Fig. 2(a)] should involve an elongation [Fig. 2(c)], in addition to the expected end-point rotations. It is important to realize that this effect will induce opposed elongations in loops [see Figs. 2(c) and 2(d)]. This implies than an "electric"[9] distribution of moments (with no forces) flowing from A to B is not the correct distribution of stresses in the case of Fig. 2(a), since it would induce incompatible displacements. The correct distribution of stresses minimizes the elastic energy $\frac{1}{2} S_{\theta\theta}(M'_{AB})^2$, and therefore $S_{\theta\theta}$ is smaller than its "electric" estimate deduced from a complete analogy between $S_{\theta\theta}$ and R. This argument is equivalent to that recently applied by Roux[9] to the macrolinks of the backbone, and leads to the inequality $\zeta < \zeta_R$. The new idea proposed here is that this inequality is just a consequence of the "eccentricity" of the pattern.

Finally, we note that the dependence of the stretching deformability upon the gyration radii discussed by Kantor and Webman[3] and by Feng,[6] is contained in the bending terms of $\bar{\bar{S}}$. Because of their r^2 dependence, these terms are expected[3,6,9] to dominate the influence of the others. A similar idea can be applied to $\mathbf{S}_\theta$, which suggests the following scaling laws: $\bar{\bar{S}} \sim L\mathbf{S}_\theta \sim L^2 S_{\theta\theta} \sim L^{\zeta+2}$ with, as we have seen, $d_{\rm red} < \zeta < \zeta_R$. These results generalize the conclusions of Refs. 3, 6, and 9. Their field of relevance is of course not restricted to our hierarchical model, but this one now allows for a direct check of these qualitative arguments.

We note first that the stiffness matrix ${}^3\bar{\bar{K}} = {}^3\bar{\bar{S}}^{-1}$ is additive in a parallel combination, since ${}^3\mathbf{v}_{AB}$ is the same for all the chains. Also, because of the symmetries of our fractal, its stretching matrix $\bar{\bar{S}}$ is diagonal in the framework $(u_\parallel, u_\perp)$ of Fig. 1, and its $\mathbf{S}_\theta$ vector is reduced to an $S_{\theta\parallel}$ component. We finally get the recursion relation between two scales L and $L' = L(1+\sqrt{2})$:

$$
\begin{aligned}
S'_{\theta\theta} &= \tfrac{5}{2} S_{\theta\theta} - \frac{(S_{\theta\parallel})^2}{2S_\parallel}\,,\\
S'_{\theta\parallel} &= -\sqrt{2} S_{\theta\parallel} + \frac{3}{2\sqrt{2}} L S_{\theta\theta} - \frac{L}{2\sqrt{2}} \frac{(S_{\theta\parallel})^2}{S_\parallel}\,,\\
S'_\parallel &= \tfrac{3}{2} S_\parallel + S_\perp - L S_{\theta\parallel} \\
&\quad + \frac{L^2}{2} S_{\theta\theta} - (S_{\theta\parallel})^2 \left(\frac{1}{2S_{\theta\theta}} - \frac{L^2}{4S_\parallel} \right),\\
S'_\perp &= S_\parallel + \tfrac{3}{2} S_\perp - (1+\sqrt{2}) L S_{\theta\parallel} + \left(\frac{3}{4} + \frac{1}{\sqrt{2}} \right) L^2 S_{\theta\theta}\,.
\end{aligned} \tag{5}
$$

These recursion relations must be initiated with the bond deformabilities $S_\parallel(0) = k_\parallel^{-1}$, $S_\perp(0) = k_\perp^{-1}$, $S_{\theta\parallel}(0) = 0$, and $S_{\theta\theta}(0) = k_{\theta\theta}^{-1}$. Asymptotically, a fixed point is reached, the properties of which do not depend on the selected initial values (universality):

$$
\begin{aligned}
&\frac{S_\parallel}{L^2 S_{\theta\theta}} \cong 0.0286\,, \quad \frac{S_\perp}{L^2 S_{\theta\theta}} \cong 0.103\,,\\
&\frac{S_{\theta\parallel}}{L S_{\theta\theta}} \cong 0.127\,, \quad L^2 S_{\theta\theta} \to +\infty\,.
\end{aligned} \tag{6}
$$

One can deduce from (5) and (6) the asymptotic scaling laws. For large L,

$$S_{\theta\theta} \sim L^{\zeta}\,, \quad S_{\theta\parallel} \sim L^{\zeta+1}\,, \quad S_\parallel \sim S_\perp \sim L^{\zeta+2} \tag{7}$$

with $\zeta \cong 0.903$. These results agree with the above qualitative discussion. One can check in Table I the inequalities $d_{\rm red} < \zeta < \zeta_R$, where we observe that ζ is appreciably reduced from ζ_R. This was to be expected from Eq. 5(a), which differs from its equivalent in the electric case ($R' = \frac{5}{2} R$) from an eccentricity correction. This correction results from the incompatibility effect discussed above. It means that conduction and bending are not equivalent in general, because of the eccentricity distribution.

In the percolation case, the reality is, of course, much more complicated. The scaling law $\bar{\bar{S}} \sim L\mathbf{S}_\theta \sim L^2 S_{\theta\theta} \sim L^{\zeta+2}$ holds as an average over a random distribution, and the incompatibility effects will also involve an $S_{\theta\perp}$ component. However, we can derive the counterpart of this scaling law in the homogeneous range ($L \gg \xi$), by adapting the usual scaling argument,[18] which relates the electric conductivity $\sigma \sim (p-p_c)^t$ with the conductance $G \sim L^{-\zeta R}$ in the fractal range. In the case of σ' (respectively, λ, μ, μ') G must be replaced by $K_{\theta\theta} \sim L^{-\zeta}$ (respectively, $\bar{\bar{K}} \sim L^{-(\zeta+2)}$), which gives

$$\sigma' \sim (p-p_c)^{t'} \text{ and } \lambda \sim \mu \sim \mu' \sim (p-p_c)^T\,, \tag{8}$$

with $t' = \nu(\zeta + d - 2)$ and $T = \nu(\zeta + d) = t' + 2\nu$ where $d = 2$ is the Euclidean dimension. Knowing that $d_{\rm red} < \zeta < \zeta_R$ and $d_{\rm red} = 1/\nu$, we obtain the inequalities $1 + (d-2)\nu < t' < t$ or equivalently $1 + d\nu < T < t + 2\nu$. In this way we retrieve, in a more general form, the two bounds for T of Refs. 3, 6, and 9. Note that σ' behaves to some extent like a conductivity for the rotations, the flow of charges being replaced with the propagation of couple stresses[14] in the material. Nevertheless, we stress the fact that, contrary to what is often conjectured,[5,9] t' differs from t, and consequently T differs from $t + 2\nu$, because of the eccentricity distribution in percolation clusters.

Now, we can get an estimate of t' and T from our value $\zeta \simeq 0.903$ and from[19] $\nu = \frac{4}{3}$, which gives

$$t' \cong 1.20 \text{ and } T \cong 3.87\,. \tag{9}$$

These values are to be compared with $t = \nu(\zeta_R + d - 2) \cong 1.3$ and with the two bounds for T, $1 + 2\nu \cong 3.67$ and $t + 2\nu \cong 3.97$. Of course our estimate must be taken with some caution since it is impossible to evaluate its degree of accuracy. Because of the roughness of the fit in Table I, it is reasonable to estimate T of order 3.8 or 3.9.

In particular, the recent numerical simulations of Zabilitzky *et al.*,[4] which give $T = 3.96 \pm 0.04$, suggest that the correction to $t + 2\nu$ could be smaller than in our hierarchical model. Further studies are needed to define the order of magnitude of this incompatibility correction.

Finally, it is noteworthy that the hierarchical model reproduces the anisotropy of the elastic response observed by Kantor:[17] The ratio of transverse to elongation deformabilities $S_\perp/S_\parallel \cong 3.61$ is larger than 1, and is consistent with his estimate 3.4 ± 0.2 deduced from the distribution of red bonds. This anisotropy disappears in the case of the Born model[1] (scalar elasticity) which can be viewed as the limiting case $k_{\theta\theta} \to +\infty$ of Eq. (1). This corresponds to an unstable fixed point of the transformation (6): $S_\perp/S_\parallel = 1$, $LS_{\theta\parallel}/S_\parallel = 0$, and $L^2 S_{\theta\theta}/S_\parallel = 0$, which leads to $S_\parallel \sim S_\perp \sim R \sim L^{t}_{\xi}$, and consequently to de Gennes' finding[1] $T = t$. The asymptotic value $S_\perp/S_\parallel = 1$ expresses the return to isotropy which is observed in numerical simulations:[7] Near threshold, the x and y coordinates split into two independent scalar problems, which yields $K/\mu = 1$ for the asymptotic ratio of bulk to shear moduli. Instead of this value, the anisotropy discussed above implies that this ratio is larger than 1 for a Cosserat material at threshold, which agrees with the numerical data.[4,7]

Extensions of this work to higher dimensions are now in progress. It is clear that the incompatibility effect will also occur, and thus, intuitively, I expect an incompatibility correction $t + 2\nu - T > 0$ for any dimension $d < 6$.

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