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Thermoelectric fluctuations in multilead devices

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The Chester-Thellung relation between transport coefficients of a degenerate Fermi gas is used to extend recent results on the long-range character of the conductivity tensor to thermoelectric and heat conductivity tensors. Consequences are discussed for combined voltage and temperature measurements in multilead devices. Among our predictions is that the temperature and temperature-induced voltage drops determined via a four-probe measurement should fluctuate dramatically, both in sign and magnitude, as a function of the chemical potential and external magnetic field. If more than four probes are used, any two measured temperature and temperature-induced voltage drops will be uncorrelated, both in sign and magnitude, at fixed values of chemical potential and magnetic field. We propose two representative experiments, which involve the application of a temperature rather than a voltage difference between the two end probes. (i) A grounded sample which is thermally isolated except at the probes. Here, large temperature fluctuations will be observed. (ii) A sample in good thermal contact with the substrate, but electrically isolated except at the probes. Here, induced voltage will show large fluctuations.

In the past several years a remarkable discovery has been made[1] that, at low temperatures, electronic transport in disordered conductors is essentially non-self-averaging. This is manifested by the sample specific fluctuations of the transport coefficients as a function of magnetic field or chemical potential and is due to quantum interference effects. Experimental observation of such fluctuations became possible with the manufacturing of ultrasmall devices such as metallic wires and rings and Si-metal-oxide-semiconductor field-effect transistors (MOSFET's).

So far, most of the effort in both theory and experiment has been made in investigation of conductance and voltage fluctuations where the system is assumed to be held at constant temperature. In Ref. 2 it was shown that when a temperature gradient is present across the sample, the resulting thermoelectric effect also exhibits quantum fluctuations, in fact, in a more striking manner than conductance fluctuations. The theory of Ref. 2, however, is correct only for two-lead rectangular geometry and is not applicable for multiprobe devices. In this paper we extend the theory developed for such devices[3,4] to include thermoelectric effects. The central equation to be derived is

$$I_i=\sum_j G_{ij}V_{ij}+N_{ij}T_{ij} \ , \tag{1}$$

where I_i is the current through the ith lead, and V_{ij} and T_{ij} are the potential and temperature differences between the ith and jth leads. G_{ij} and N_{ij} can be called conductance and thermoelectric conductance tensors, and will be defined more precisely later on. We will show that the fluctuation in N_{ij} is such that typically $\delta N/N\gg 1$, hence the temperatures or induced voltages measured at the various probes will bear no relationship whether in sign or magnitude to the average temperature or induced voltage drop across the sample. This is in contrast with voltage fluctuations under conditions of uniform temperature, where the potential drop measured at the probes basically follows the sign of the overall drop. To begin with, let us briefly review the deviations of (1) with $T_{ij}\equiv 0$. In Ref. 3, it was shown that the conductivity tensor $\sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}')$ is the key quantity to consider. From Kubo's formula, it is the velocity-velocity correlation function and can be interpreted as a real-space version of the transmission coefficient. As such, $\sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}')$ must be long range. Indeed, due to the current conservation it satisfies the condition[3]

$$\nabla_\alpha\sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}')=\nabla'_\beta\sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}')=0 \ . \tag{2}$$

It is precisely this conservation that causes the velocity-velocity correlation function to decay much slower than the mean-free path. Since the current density

$$j_\alpha(\mathbf{r})=\int\sigma_{\alpha\beta}(\mathbf{r}\mathbf{r}')E_\beta(\mathbf{r}')d\mathbf{r}' \ , \tag{3}$$

this implies the current through the ith lead is

$$\begin{aligned}I_i&=\int j_\alpha(\mathbf{r})dS^i_\alpha\\&=\int\sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}')E_\beta(\mathbf{r}')d\mathbf{r}'dS^i_\alpha \ .\end{aligned} \tag{4}$$

Now using $E_\beta(\mathbf{r})=\nabla'_\beta V(\mathbf{r})$ and integrating by parts, we obtain,[5] from (4) with the use of (2),

$$I_i=\sum_j G_{ij}V_j \ , \tag{5}$$

where V_j is the voltage at the jth lead and G_{ij} is given by

$$G_{ij}=\int dS^i_\alpha dS'^j_\beta\sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}') \ , \tag{6}$$

the integration being performed over the cross section of the leads and $d\mathbf{S}$ points *away* from the sample. Clearly, G_{ij} plays the role of a transmission coefficient[3] between the ith and jth lead.

In connection to the above, we note that any approximation to the ensemble averaged version of $\sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}')$ should also satisfy the conditions imposed by current conservation. For instance, in the lowest order in disorder one finds[3,6]

$$\langle \sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}')\rangle = \sigma_0[\delta_{\alpha\beta}\delta(\mathbf{r},\mathbf{r}') - \nabla_\alpha \nabla'_\beta d(\mathbf{r},\mathbf{r}')] \ , \tag{7}$$

where σ_0 is the Boltzman conductivity and $d(\mathbf{r},\mathbf{r}')$ is the diffusion propagator. The latter satisfies the equation $-\nabla^2 d(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$, subject to the boundary conditions $d(\mathbf{r},\mathbf{r}') = 0$ on a conducting boundary and $\nabla_n d(\mathbf{r},\mathbf{r}') = 0$ on an insulating boundary. In this form $\langle \sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}')\rangle$ explicitly satisfies the conditions given by Eq. (2).

Since $\sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}')$ is like the transmission coefficient, it can be written as the square of the sum over all Feynman paths of the amplitude to get from $\mathbf{r}$ to $\mathbf{r}'$. Quantum interference accounts for the fluctuations in the conductivity tensor[3] which can be described by the correlation function

$$\langle \delta\sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}')\delta\sigma_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}'_1)\rangle \ ,$$

where

$$\delta\sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}') = \sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}') - \langle \sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}')\rangle \ .$$

Similar to $\langle \sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}')\rangle$, it has a long-range part. Although the evaluation of the long-range parts of the average conductivity tensor and the conductivity tensor correlation function can be by-passed in the calculation of the conductance and conductance fluctuations in simple rectangular geometries,[3] they are crucial for the understanding of electronic transport in multilead devices[4] and devices with nontrivial geometries.[6]

In the absence of magnetic field the conductance tensor G_{ij} is symmetric, $G_{ij} = G_{ji}$, and due to current conservation it must satisfy the condition

$$\sum_i G_{ij} = \sum_j G_{ij} = 0 \ . \tag{8}$$

Because of this constraint, Eq. (5) can be written as

$$I_i = \sum_i G_{ij} V_{ij} \tag{5'}$$

which explicitly shows that only voltage differences (V_{ij}) are important, as must be the case. Thus, there are only $n(n-1)/2$ independent G_{ij} for the device with n leads and one needs precisely $n(n-1)/2$ independent measurements to determine the conductance (transmission) between any two pairs of leads.[4] By measuring current and voltage fluctuations, one can also easily extract the fluctuations of conductance:

$$\delta G_{ij} = \langle (G_{ij} - \langle G_{ij}\rangle)^2\rangle^{1/2} \ .$$

In a typical four-probe measurement,[7] as depicted in Fig. 1, the current is absent in the "voltage leads," $I_2 = I_3 = 0$, and the incoming current is fixed $I_1 = I$. This defines three variables which suffice to find the three independent voltage drops: V_{23}, V_{12}, V_{14}. Assuming that all linear dimensions in Fig. 1 are of the same order of magnitude, one finds that all conductances and voltage drops are of the same order of magnitude as well. The latter can be schematically presented as "$\Delta V = IG^{-1}$," where $\Delta V \approx V_{ij}$ and G is some combination of G_{ij} and $G \approx G_{ij}$. Obviously, then $\delta\Delta V = I\delta G/G^2$ and $\delta\Delta V/\Delta V = \delta G/G$. At low temperatures δG is "universal" (Ref. 1), $\delta G \approx e^2 h$. Alternatively, one could set constant voltage ΔV between any two leads, e.g., $V_{14} = \Delta V$, and then find voltage drops between all the others and the incoming

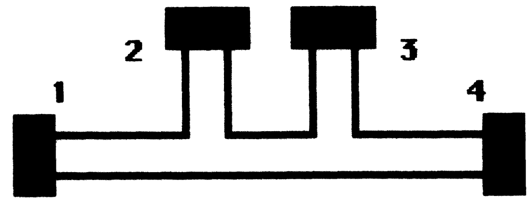

FIG. 1. A typical four-lead geometry in which leads are connected to large reservoirs, where electrons thermally equilibrate.

current I. For instance, V_{23}, V_{12}, and I can be chosen as a set of independent variables to be determined from Eq. (5'). The fluctuations $\delta\Delta V/\Delta V$, $\delta I/I$, and $\delta G/G$ are all of the same order.

Now consider the presence of temperature gradients, with $V_{ij} \equiv 0$. In this case we have

$$j_\alpha(\mathbf{r}) = \int \eta_{\alpha\beta}(\mathbf{r},\mathbf{r}')\nabla'_\alpha T(\mathbf{r}')d\mathbf{r}' \ , \tag{9}$$

which merely defines the thermoelectric conductivity $\eta_{\alpha\beta}(r,r')$. Using linear response theory, and invoking the equivalence of $T\nabla(1/T)$ to a gravitational potential gradient (Ref. 8), $\eta(\mathbf{r},\mathbf{r}')$ is simply related to the energy current-velocity correlation function. Furthermore, provided impurity scattering is elastic and the quasiparticles are noninteracting, it has been shown rigorously[9,10] that

$$\eta_{\alpha\beta}(\mathbf{r},\mathbf{r}') = -\frac{1}{eT}\int \frac{df}{d\epsilon}(\epsilon-\mu)\sigma_{\alpha\beta}(\epsilon,\mathbf{r},\mathbf{r}')d\epsilon \ , \tag{10}$$

where $\sigma_{\alpha\beta}(\epsilon)$ is the zero-temperature conductivity at chemical potential ϵ, μ is the physical chemical potential, and f the Fermi distribution. It is customary to use the Sommerfeld expansion and write (9) as

$$\eta(\mathbf{r},\mathbf{r}') = \frac{\pi^2}{3e}k_B(k_B T)\frac{\partial\sigma(\mu)}{\partial\mu} \ , \tag{10'}$$

which is valid if $\sigma(\mu)$ varies slowly on the scale of $k_B T$. While the scale of variation of $\langle\sigma(\mu)\rangle$ is μ, that of $\sigma(\mu)$ unaveraged is

$$E_c = \frac{\pi^2 D}{L_x^2} \ ,$$

where D is the diffusion constant and L_x is the sample size along the directions of the current. The fact that $\sigma(\mu,\mathbf{r},\mathbf{r}')$ is sensitive to the sample size is a reflection of its long-ranged nature.

Clearly (2) and (9) together implies that

$$\nabla_\alpha \eta_{\alpha\beta}(\mathbf{r},\mathbf{r}') = \nabla'_\beta \eta_{\alpha\beta}(\mathbf{r},\mathbf{r}') = 0 \ . \tag{11}$$

Hence following the steps leading from (2) to (5') we arrive at

$$I_i = \sum N_{ij} T_{ij} \tag{12}$$

with

$$N_{ij} = \int dS_i^\alpha dS_j^\beta \eta_{\alpha\beta}(r,r')$$

$$= \frac{-1}{eT}\int \frac{df}{d\epsilon}(\epsilon-\mu)G_{ij}(\epsilon) \tag{13}$$

$$= \frac{\pi^2}{3e}k_B(k_B T)\frac{\partial G_{ij}(\mu)}{\partial\mu}, \quad T \ll E_c \ . \tag{13'}$$

It is now consistent with the philosophy of linear response to allow for both voltage and temperature gradients by combining (5′) and (12) to give (1). Reference 2 considered the special case of two-lead rectangular systems, for which the matrix equation (13) reduces to a scalar equation. Since the energy correlation function of the conductance has been previously calculated by various authors, the fluctuation in N can be simply calculated. The results can be summarized by

$$\begin{aligned}\delta N(T)/\langle N(T)\rangle &= \frac{\mu}{E_c}\delta G(T)/\langle G\rangle, \quad T \ll E_c \\ &= \frac{\mu}{T}\delta G(T)/\langle G\rangle, \quad T \gg E_c \ , \end{aligned} \tag{14}$$

where $\delta G(t)$ is the fluctuation of the conductance at temperature T. This can be easily understood as follows. The characteristic energy scale of the conductance fluctuations is E_c or T, whichever is larger. Therefore, the fluctuation of the thermoelectric conductance will be proportional to the average value of the conductance fluctuation over this energy scale. The characteristic energy scale of the *average* conductance, on the other hand, is μ, so that the average thermoelectric conductance is proportional to the average conductance over μ. Equations (14) then follow immediately. Physical interpretation of the above argument was given in Ref. 2; it has to do with the compensation of the currents due to holes and electrons moving in the same direction under the temperature gradient. It is important, however, that such compensation occurs only on the average, hence the enhancement of the relative fluctuation of the thermoelectric conductance as compared to the conductance.

Strictly speaking, since (14) is based on (13), the former is correct only if inelastic scattering can be neglected at the temperature T. However, it was argued in Ref. 2 that the principal effect of inelastic scattering should be to destroy quantum coherence and, hence, to suppress $\delta G(T)$; and so Eq. (14) might be valid even for $T < \tau_{\text{in}}^{-1}$, the inelastic scattering rate. Thus, up to a relatively high temperature $\delta N/N \gg \delta G/G$, while, up to a lower but far from restrictive temperature $\delta N/N \gg 1$.

Returning to the multiprobe situation, the relative fluctuation in N_{ij} is enhanced over that of G_{ij} according to Eq. (14) with the subscripts ij inserted everywhere. Hence, while G_{ij} will differ little from $\langle G_{ij}\rangle$, its value if the system has a uniform resistivity, and must always be positive; N_{ij} can differ drastically from $\langle N_{ij}\rangle$ and be randomly positive or negative for different (ij) or for the same (ij) but at different magnetic field or chemical potential.

Experimentally, the fluctuations in N_{ij} manifest themselves in the form of fluctuations of the temperature or induced voltage profile along the sample, and of the induced current. Limiting ourselves to the geometry as shown in Fig. 1, the following two setups are representative.

(i) The system is grounded everywhere: $V_{ij} \equiv 0$. Leads 1 and 4 are in contact with heat baths of different temperature, with the rest of the system thermally isolated. We have the following set of equations for T_{12}, T_{23}, and T_{13} for the given T_{14} and N_{ij}'s: $I_1 = N_{1j}T_{1j}$, $0 = N_{2j}T_{2j}$, and $0 = N_{3j}T_{3j}$, where the fact that there is no current through the temperature probes has been used. The relative fluctuations of T_{12}, T_{23}, and T_{13} come out of these equations via the relative fluctuations of N_{ij}'s. Then according to Eq. (14), not only can the magnitudes of T_{12}, T_{23}, and T_{13} be much larger than $|T_{14}|$, but their signs can be opposite to that of T_{14}, too.

(ii) $T_{14} \neq 0$, but the temperature gradient is uniform along the sample due to good thermal contact with the underlying substract. Here it is the induced voltage difference between the probes that fluctuate in signs and magnitude.

In addition to studying the fluctuations of the temperature and/or voltage profile, one can of course also look at the change of any T_{ij}, V_{ij}, or the current I with a changing magnetic field or chemical potential. As noted in Ref. 2, the direction of the current will change even when T_{14} is fixed.

Finally, for completeness, one can also extend all these considerations to the heat current. The heat current density in the presence of a temperature gradient and/or electric field is

$$\begin{aligned} j_Q^\alpha = T\int \eta_{\alpha\beta}(r,r')E_\beta(r')dr' \\ +\int \kappa_{\alpha\beta}(r,r')\nabla'_\beta T(r')dr' \ . \end{aligned} \tag{15}$$

Note that the first term on the right, being a velocity-heat current correlation function, is consistent with Onsager relation. The thermal conductivity κ satisfies[9,10]

$$\kappa(r,r') = -\frac{1}{e^2T}\int \frac{\partial f}{\partial \epsilon}(\epsilon-\mu)^2\sigma(\epsilon,\mathbf{r},\mathbf{r}')d\epsilon \tag{16}$$

$$= \frac{1}{3}\left(\frac{\pi k_B}{e}\right)^2 T\sigma(\mu,\mathbf{r},\mathbf{r}'), \quad T \ll E_c \ . \tag{16′}$$

Hence, integrating (15) by parts we obtain

$$I_{Q,i} = T\sum_j N_{ij}V_{ij} + \sum_j K_{ij}T_{ij} \ , \tag{17}$$

where K_{ij} is given by (16) with σ replaced by G_{ij}. Evidently

$$\frac{\delta K}{K} = \frac{\delta G}{G} \ . \tag{18}$$

In conclusion, above we were able to show that by bringing thermoelectric conductance into play by allowing for the presence of temperature gradients in the system one can extend the range of four-probe measurements to include temperature fluctuations under application of a voltage across the sample, voltage fluctuations under application of a temperature drop across the sample, and finally temperature fluctuations under application of a temperature drop across the sample. Moreover, since the relative fluctuation of the thermoelectric conductance is larger than the relative fluctuation of conductance, one comes to the conclusion that the relative voltage and the temperature fluctuations in the settings (i) and (ii) will be more pronounced than the relative voltage fluctuations in the settings studied so far.[4,7]

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