# Effective-field renormalization-group study for the diluted Ising model

Z. Y. Li
Center of Theoretical Physics, CCAST (World Laboratory), Beijing, The People's Republic of China
and
C. Z. Yang
Department of Physics, Suzhou University, Suzhou, Jiangsu, The People's Republic of China

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**An effective-field renormalization-group method which improves the result of the mean-field renormalization-group method is proposed to study the diluted Ising system.**

## I. INTRODUCTION

Recently the mean-field renormalization-group (MFRG) method has been proposed for computing critical properties of lattice spin systems.[1] This approach is based upon a comparison of the behavior of clusters of different sizes in the presence of symmetry breaking boundary conditions which, in a mean-field way, simulate the effect of surrounding spins in the infinite system.[2] The MFRG method has been successfully applied to the study of critical properties of ordered and disordered spin systems.[3–7] In contrast with the Migdal-Kadanoff (MK) method[8,9] or the decimation techniques,[10–12] the MFRG method overestimates the interactions among the spins and consequently gives an upper bound for the critical temperature and a lower bound for the critical concentration. The critical coupling, the percolation threshold, and other critical components calculated by use of the MFRG method are different from the exact or series results. Lately a two-step renormalization-group (TSRG) method, combining the MFRG and decimation methods, has been presented and has been applied to diluted Ising system.[13,14] In the present paper, we use a new type of effective-field theory with correlation[15] instead of the mean-field theory. The theory substantially improves the standard molecular-field approximation and has been applied to a variety of interesting problems.[16–18] The effective-field results for small clusters of spins are combined with renormalization group ideas to propose a new approximate scheme. The effective-field renormalization-group (EFRG) method will improve the results of the MFRG method. In order to illustrate the EFRG approximate method, we study herein the bond-diluted Ising model on a square and a simple-cubic lattice.

## II. BOND-DILUTED ISING FERROMAGNET ON A SOURCE LATTICE

The effective Hamiltonian is

$$\beta\mathcal{H}=-\sum_{(i,j)}K_{ij}S_iS_j\ , \tag{1}$$

where $i,j$ are nearest neighbors and coupling $K_{ij}\equiv\beta J_{ij}$ is an independent random variable with probability distribution

$$p(K_{ij})=p\delta(K_{ij}-K)+(1-p)\delta(K_{ij})\ . \tag{2}$$

We consider two finite clusters with $N'=1$ ($s_1$) and $N=2$ ($s_1$ and $s_2$) spins, respectively.[3] For a ferromagnetic Ising system, each boundary spin is fixed to $b'$ and $b$ for the $N'$ and $N$ spin clusters, respectively. In the effective field for $N'=1$ cluster (EF 1), the spin $s_1$ interacts with $z=4$ nearest-neighbor sites via a coupling $K'$. Each external site $j$ contributes with a symmetry breaking field $K'_{1j}b'_j=K'_{1j}b'$.[3] According to the exact Callen identity,[19] the magnetization

$$\langle s_1\rangle_1=\left\langle\tanh\left[\sum_{j\neq1}^{z}K'_{1j}S'_j\right]\right\rangle$$
$$=\left\langle\prod_{j\neq1}^{z}[\cosh(DK')+s'_j\sinh(DK')]\right\rangle\tanh\beta x\bigg|_{x=0}\ , \tag{3}$$

where $D\equiv\partial/\partial x$ is a differential operator.[15] In a quenched random-bond system, the disorder lies in the exchange bonds, and, hence, it is necessary to take the random average $\langle\cdots\rangle_r$ over all possible bond configurations. One gets

$$\langle\langle s_1\rangle_1\rangle_r=G_1(K',p')b'+O(b'^3)\ , \tag{4}$$

where

$$\begin{aligned}G_1(K',p')=&\tfrac{1}{2}p'^4(\tanh4K'+2\tanh2K')\\&+3p'^3(1-p')(\tanh3K'+\tanh K')\\&+6p'^2(1-p')^2\tanh2K'\\&+4p'(1-p')^3\tanh K'\ .\end{aligned} \tag{5}$$

In the effective field for the $N=2$ cluster (EF 2), the spins $s_1$ and $s_2$ interact directly via a coupling $K$ and both $s_1$ and $s_2$ interact with three nearest-neighbor sites whose spins are fixed to the value $b$. The Hamiltonian for the two-spins cluster is

$$\beta\mathcal{H}_2=-K_{12}s_1s_2-s_1\sum_{j\neq2}K_{1j}s_j-s_2\sum_{j\neq1}K_{2j}s_j\ . \tag{6}$$

The average magnetization

$$\langle s_1\rangle_2=\operatorname*{Tr}_{(s_1,s_2)}s_1e^{-\beta\mathcal{H}_2}\Big/\operatorname*{Tr}_{(s_1,s_2)}e^{-\beta\mathcal{H}_2}$$
$$=\left\langle\exp\left[D\sum_{j\neq2}K_{1j}s_j+D'\sum_{j\neq1}K_{2j}s_j\right]\right\rangle f(x,y)\bigg|_{x=y=0}\ . \tag{7}$$

 

Where $D' \equiv \partial/\partial y$ is also a differential operator, the function $f(x,y)$ is given by

$$f(x,y)=\frac{\tanh(x+y)}{1+e^{-2K}\cosh(x-y)/\cosh(x+y)}+\frac{\tanh(x-y)}{1+e^{2K}\cosh(x+y)/\cosh(x-y)} . \quad (8)$$

Averaging over the random-bond configurations and noting that in the vicinity of the critical temperature $b\to 0$, we can only take the linear terms of $b$:

$$\langle\langle s_1\rangle_2\rangle_r=G_2(K,p)b , \quad (9)$$

where

$$G_2(K,p)=3[p^7A_1+p^6(1-p)(3A_2+2A_3)+p^5(1-p)^2(3A_4+6A_5+A_6)+p^4(1-p)^3(A_7+6A_8+3A_9)+p^3(1-p)^4(2A_{10}+3A_{11})+p^2(1-p)^5A_{12}]+\tfrac{3}{4}p^3(\tanh 3K+\tanh K)+3p^2(1-p)\tanh 2K+3p(1-p)^2\tanh K . \quad (10)$$

These coefficients $A_1-A_{12}$ are given in Appendix A. Combining EF 1 and EF 2 with the rescaling assumption, one gets the recursion relation,

$$\frac{\partial}{\partial b'}\langle\langle s_1\rangle_1\rangle_r\,|_{b'=0}=\frac{\partial}{\partial b}\langle\langle s_1\rangle_2\rangle_r\,|_{b=0} , \quad (11)$$

namely

$$G_1(K',p')=G_2(K,p) . \quad (12)$$

The fixed-point equation associated with (12) for $p'=p$ is

$$G_1(K_c,p)=G_2(K_c,p) , \quad (13)$$

where $K_c$ is the critical coupling for a given value of $p$. For the pure Ising system $p'=p=1$, the critical coupling is calculated by the EFRG method $K_c^{\rm EFRG}=0.3579$. It can be compared with other results: $K_c^{\rm MK}=0.69$ (MK decimation), $K_c^{\rm MFRG}=0.346$, $K_c^{\rm TSRG}=0.536$, and the series value $K_c^{\rm exact}=0.441$.[14] Our method also allows the calculation of a finite percolation threshold for $K_c\to\infty$. The result is $p'=p=p_c^{\rm EFRG}=0.4345$. It should be noted that the present EFRG method gives substantial improvement for the result of the MFRG method $p_c^{\rm MFRG}=0.333$, and it can be compared with $p_c^{\rm MK}=0.618$, $p_c^{\rm TSRG}=0.475$ and series value $p_c^{\rm exact}=0.5$.[14]

## III. BOND-DILUTED ISING FERROMAGNET ON A SIMPLE CUBIC LATTICE

Similar procedures can be carried out to study the critical behavior of the bond-diluted Ising ferromagnet on a simple-cubic lattice. The average magnetization for two clusters can be obtained

$$\langle\langle s_1\rangle_1\rangle_r=G_1^{\rm sc}(K',p')b' ,\qquad \langle\langle s_1\rangle_2\rangle_r=G_2^{\rm sc}(K,p)b , \quad (14)$$

where

$$G_1^{\rm sc}(K',p')=\tfrac{3}{16}p'^6(\tanh 6K'+5\tanh 2K'+4\tanh 4K')+\tfrac{15}{8}p'^5(1-p')(\tanh 5K'+3\tanh 3K'+2\tanh K')+\tfrac{15}{2}p'^4(1-p')^2(\tanh 4K'+2\tanh 2K')+15p'^3(1-p')^3(\tanh 3K'+\tanh K')+15p'^2(1-p')^4\tanh 2K'+6p'(1-p')^5\tanh K' , \quad (15)$$

$$G_2^{\rm sc}(K,p)=5p^{11}B_1+p^{10}(1-p)(20B_2+25B_3)+p^9(1-p)^2(30B_4+100B_5+50B_6)+p^8(1-p)^3(20B_7+150B_8+200B_9+50B_{10})+p^7(1-p)^4(5B_{11}+100B_{12}+300B_{13}+200B_{14}+25B_{15})+p^6(1-p)^5(25B_{16}+200B_{17}+300B_{18}+100B_{19}+5B_{20})+p^5(1-p)^6(50B_{21}+200B_{22}+150B_{23}+20B_{24})+p^4(1-p)^7(50B_{25}+100B_{26}+30B_{27})+p^3(1-p)^8(25B_{28}+20B_{29})+p^2(1-p)^9(5B_{30})+p^5(1-p)[\tfrac{5}{16}(\tanh 5K+3\tanh 3K+2\tanh K)]+p^4(1-p)^2[\tfrac{5}{2}(\tanh 4K+2\tanh 2K)]+p^3(1-p)^3[\tfrac{15}{2}(\tanh 3K+\tanh K)]+p^2(1-p)^4(10\tanh 2K)+p(1-p)^5(5\tanh K) . \quad (16)$$

The coefficients $B_1-B_{30}$ are given in Appendix B.

By the use of the recursion relation for the coupling constants and the corresponding fixed-point equation, we can obtain the critical coupling $K_c^{\rm EFRG}(p=p'=1)=0.2061$ and a finite percolation threshold $p_c^{\rm EFRG}=0.2589$. These results considerably improve the results for critical properties, compared with other real-space renormalization group, MFRG, and TSRG methods that are used: $K_c^{\rm MK}=0.261$, $K_c^{\rm MFRG}=0.203$, $K_c^{\rm TSRG}=0.246$, and series value $K_c^{\rm exact}=0.221$; $p_c^{\rm MK}=0.28$, $p_c^{\rm MFRG}=0.20$, $p_c^{\rm TSRG}=0.243$, and $p_c^{\rm exact}=0.25$. It is worth mentioning that the EFRG method can be used to obtain very good estimates for the critical coupling and critical concentration.

In conclusion, we have presented an effective-field renormalization-group method and applied it to the bond-diluted Ising model. It leads to considerable improvements over other frequently used real-space renormalization group and MFRG methods, and can be compared with the TSRG method without involving many more computational works. It should be noted that the EFRG method is very powerful and can be applied to other disordered systems and surface problems, etc.

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## APPENDIX A

We present the coefficients $A_1-A_{12}$:

$$A_1=\{\cosh DK\cosh^2 D'K\sinh D'K+\cosh^3 D'K\cosh^2 DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$A_2=\{\cosh^2 DK\cosh^2 D'K\sinh D'K+\cosh^2 D'K\cosh^2 DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$A_3=\{\cosh^3 DK\cosh D'K\sinh D'K+\cosh^3 D'K\cosh DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$A_4=\{\cosh DK\cosh^2 D'K\sinh D'K+\cosh D'K\cosh^2 DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$A_5=\{\cosh^2 DK\cosh D'K\sinh D'K+\cosh^2 D'K\cosh DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$A_6=\{\cosh^3 DK\sinh D'K+\cosh^3 D'K\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$A_7=\{\cosh^2 D'K\sinh D'K+\cosh^2 DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$A_8=\{\cosh DK\cosh D'K\sinh D'K+\cosh D'K\cosh DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$A_9=\{\cosh^2 DK\sinh D'K+\cosh^2 D'K\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$A_{10}=\tfrac{1}{2}\{\sinh 2D'K+\sinh 2DK\}f(x,y)|_{x=y=0}\,,$$

$$A_{11}=\{\cosh DK\sinh D'K+\cosh D'K\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$A_{12}=\{\sinh D'K+\sinh DK\}f(x,y)|_{x=y=0}\,.$$

## APPENDIX B

We present the coefficients $B_1-B_{30}$:

$$B_1=\{\cosh^5 DK\cosh^4 D'K\sinh D'K+\cosh^5 D'K\cosh^4 DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$B_2=\{\cosh^5 DK\cosh^3 D'K\sinh D'K+\cosh^5 D'K\cosh^3 DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$B_3=\{\cosh^4 DK\cosh^4 D'K\sinh D'K+\cosh^4 D'K\cosh^4 DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$B_4=\{\cosh^5 DK\cosh^2 D'K\sinh D'K+\cosh^5 D'K\cosh^2 DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$B_5=\{\cosh^4 DK\cosh^3 D'K\sinh D'K+\cosh^4 D'K\cosh^3 DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$B_6=\{\cosh^3 DK\cosh^4 D'K\sinh D'K+\cosh^3 D'K\cosh^4 DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$B_7=\{\cosh^5 DK\cosh D'K\sinh D'K+\cosh^5 D'K\cosh DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$B_8=\{\cosh^4 DK\cosh^2 D'K\sinh D'K+\cosh^4 D'K\cosh^2 DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$B_9=\{\cosh^3 DK\cosh^3 D'K\sinh D'K+\cosh^3 D'K\cosh^3 DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$B_{10}=\{\cosh^2 DK\cosh^4 D'K\sinh D'K+\cosh^2 D'K\cosh^4 DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$B_{11}=\{\cosh^5 DK\sinh D'K+\cosh^5 D'K\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$B_{12}=\{\cosh^4 DK\cosh D'K\sinh D'K+\cosh^4 D'K\cosh DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$B_{13}=\{\cosh^3DK\cosh^2D'K\sinh D'K+\cosh^3D'K\cosh^2DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$B_{14}=\{\cosh^2DK\cosh^3D'K\sinh D'K+\cosh^2D'K\cosh^3DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$B_{15}=\{\cosh DK\cosh^4D'K\sinh D'K+\cosh D'K\cosh^4DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$B_{16}=\{\cosh^4DK\sinh D'K+\cosh^4D'K\sinh DK\}f(x,y)|_{x=y=0},\quad B_{17}=A_3,\quad B_{18}=A_2\,,$$

$$B_{19}=\{\cosh DK\cosh^3D'K\sinh D'K+\cosh D'K\cosh^3DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$B_{20}=\{\cosh^4D'K\sinh D'K+\cosh^4DK\sinh DK\}f(x,y)|_{x=y=0},\quad B_{21}=A_6\,,$$

$$B_{22}=\{\cosh^2DK\cosh D'K\sinh D'K+\cosh^2D'K\cosh DK\sinh DK\}f(x,y)|_{x=y=0},\quad B_{23}=A_4\,,$$

$$B_{24}=\{\cosh^3D'K\sinh D'K+\cosh^3DK\sinh DK\}f(x,y)|_{x=y=0}\,,$$

$$B_{25}=A_9,\quad B_{26}=A_8,\quad B_{27}=A_7,\quad B_{28}=A_{11},\quad B_{29}=A_{10},\quad B_{30}=A_{12}\,.$$

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