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Derivation and generalization of the Suhl spin-wave instability relations

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The Suhl spin-wave instability relations are shown to be derivable using linear-stability theory and the method of averaging. This makes rigorous Suhl's early work on formulas for the critical radio frequency field for spin-wave instabilities as well as reformulating the problem in more mathematical terms. It also makes possible several generalizations and extensions including formulas for spin waves with frequencies near the usual detuned frequency, and a direct application of second-order averaging theory to show that the first-order results here and in Suhl's original work are very accurate within the infinite time-averaging approximations used. Appendixes on the full equations of motion, including the Landau-Lifshitz damping, and the complete expressions for the Jacobian from the variational equations are also given.

INTRODUCTION

The subject of spin-wave instabilities in ferromagnetic and ferrimagnetic materials in a rf driving field has seen a revival recently.[1–8] This is primarily because magnetic materials undergoing such instabilities, like yttrium iron garnet, have been shown to exhibit many of the interesting nonlinear types of behavior such as period doubling, quasiperiodic motion, and chaos.[1–8] To this date the original work of Suhl[9] and the later rederivation by Schlömann[10] and Akhiezer *et al.*[11] remain the main references for the expressions for the onset of spin-wave instabilities in these materials. Some time later Patton,[12] using the same approach as Suhl, derived instability relations which include anisotropy.

I show in this paper that it is possible to derive these relationships in greater generality and with greater rigor and to extend the results to a higher order of approximation. This is done using linear-stability theory[13–15] and the method of averaging,[16–18] first and second order. This shows that Suhl's work was actually a form of infinite-time averaging applied to a variational equation of linear stability derived from a simplified form of the classical equations of motion.[9] The results are given for the complete equations of motion which include the Landau-Lifshitz[19] damping. The results also include instability relations for spin-wave frequencies near the usual detuned frequency.[9] For spin waves at frequencies near the usual detuned frequency, the second-order correction to the first-order averaging result is zero. In addition, the complete equations of motion, which include terms not present in Refs. 9 and 10, are given up to third order, as well as the Jacobian expressions for the stability and critical rf field analysis.

Recently Sneddon[20] has shown, from a somewhat different point of view than the one here, that there is a relationship between linear stability theory and ferromagnetic instabilities as studied by Suhl[9] and Schlömann.[10] Here the point is to show what mathematical concepts apply to the instability problem, how to rephrase the problem to fit the rigorous mathematical scheme and, in some cases, to extend the results of instability theory.

EQUATIONS OF MOTION

The equations of motion are well described in Suhl's paper[9] and they will only be covered in a cursory manner here, except for the final result of the full equations of motion which appear in Appendix A.

The primary equation describing the time evolution of the magnetization is taken to be the Landau-Lifshitz equation[19] with the Landau-Lifshitz form for the damping

$$\frac{d\mathbf{m}}{dt} = -\gamma\mathbf{m}\times\mathbf{H}_{\text{eff}} - \alpha\mathbf{m}\times(\mathbf{m}\times\mathbf{H}_{\text{eff}}) , \quad (1)$$

where $\mathbf{m}=\mathbf{M}(\mathbf{r},t)/|\mathbf{M}|$ is the "reduced magnetization" and $\mathbf{H}_{\text{eff}}$ is the effective field described below. Since $|\mathbf{M}|$ is a constant of the motion, $|\mathbf{m}|=1$. The first term on the right-hand side of Eq. (1) is the gyroscopic force causing precession in the linear limit. The second term is a phenomenological damping term. This adds a "force" which tends to cause $\mathbf{m}$ to align with $\mathbf{H}_{\text{eff}}$ in the absence of a driving field. The damping term is an attempt to describe, in simple terms, the actual interaction of $\mathbf{m}$ with phonons and, in metals, eddy currents, both of which extract energy from the motion of the magnetization. This damping can be described in more physically realistic terms using magnetoelastic equations and Maxwell's equations,[21] although their inclusion greatly complicates the equations of motion.

The effective field $\mathbf{H}_{\text{eff}}$ has four parts: the exchange term, the dipole term, the static applied field, and the applied rf field. The anisotropy is neglected for now.

The exchange term is $\beta M_S\nabla^2\mathbf{m}$, where $M_S=|\mathbf{M}|$ is the saturation value of the magnetization. The exchange constant here is related to another common version by $\beta=H_{ex}l^2/M_S$ (l is the lattice constant).

The dipole term is $-4\pi M_S\sum_{\mathbf{k}}\mathbf{k}(\mathbf{k}\cdot\mathbf{m}_{\mathbf{k}})e^{i\mathbf{k}\cdot\mathbf{r}}/k^2$, where $\mathbf{m}_{\mathbf{k}}$ is the Fourier coefficient of $\mathbf{m}(\mathbf{r})$. This comes about from the magnetostatic approximation[9,11] to the boundary conditions. The sum in the dipole term is over nonzero $\mathbf{k}$ spin-wave modes. For the $\mathbf{k}=0$ mode the dipole term is $4\pi M_S(N_T m_{x0}, N_T m_{y0}, N_z m_{z0})$, where N_T and N_z are the demagnetizing factors.

The driving or pumping rf field is $\mathbf{h}_{\rm rf}=(h\cos\omega t, h\sin\omega t, 0)$, with ω being the driving frequency and the static field is $\mathbf{H}_0=(0,0,H_0)$.

Because $|\mathbf{m}(\mathbf{r},t)|=1$, the motion takes place on a sphere for each $\mathbf{r}$. The number of equations of motion can be reduced from three to two. There are many ways to do this.[9,11,22] Here, the standard approach is used which is simple projection onto the x-y plane. Assume the z component of $\mathbf{m}\approx 1$. This implies that m_x and m_y are small and all terms of order 4 or more are dropped in the following. Then, write the x and y components of $\mathbf{m}$ in the usual complex form and expand the z component of $\mathbf{m}$ in terms of the new complex variables:

$$m_\pm = m_x + i m_y, \quad m_z = \sqrt{(1-m_+m_-)} \approx 1 - \tfrac{1}{2} m_+ m_-$$

Express $\mathbf{m}$ in terms of its Fourier series (spin-wave mode expansion), $\mathbf{m}=\sum_{\mathbf{k}} \mathbf{m}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}$, and define $a_{\mathbf{k}}=m_{\mathbf{k}x}+im_{\mathbf{k}y}$. Then $m_+=\sum_{\mathbf{k}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}$, $m_-=\sum_{\mathbf{k}} a^*_{-\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}$, and $m_z \approx 1-\frac{1}{2}\sum_{\mathbf{k},\mathbf{k}'} a_{\mathbf{k}'} a^*_{\mathbf{k}'-\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}$. Other relations among $a_{\mathbf{k}}$, $a^*_{\mathbf{k}}$, and $\mathbf{m}$ follow from these.

These latter definitions and relations combined with Eq. (1) lead to the following infinite system of complex ordinary differential equations:

$$\frac{da_{\mathbf{k}}}{dt} = iA_{\mathbf{k}} a_{\mathbf{k}} + iB_{\mathbf{k}} a^*_{-\mathbf{k}} + P_{\mathbf{k}}(a_{\mathbf{k}}, a^*_{\mathbf{k}}) \tag{2}$$

and their complex conjugates. In Eq. (2) $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$ are defined as in Suhl[9] and are given in Appendix B here. The term $P_{\mathbf{k}}(a_{\mathbf{k}}, a^*_{\mathbf{k}})$ is a polynomial of degrees two and three in $a_{\mathbf{k}}$ and $a^*_{\mathbf{k}}$. The expressions for these quantities in Eq. (2) are given in Appendix A. The full Landau-Lifshitz damping terms are included.

The linear parts of Eq. (2) give rise to the well-known spin-wave frequency dispersion relations. In the linear limit the $\mathbf{k}=0$ (uniform mode) spin wave is stable and the $\mathbf{k}\neq 0$ spin-wave modes decay to their thermal values. The uniform mode amplitudes a_0 and a_0^* have magnitudes on the order of h, the rf field. Thus at small rf power levels the uniform mode precesses with frequency ω, the driving frequency, and is a stable solution to Eq. (1). It is the nonlinear part $P_{\mathbf{k}}$ which causes the instabilities and, presumably, other interesting effects seen at higher power rf levels. The next section examines the nonlinear contribution as the rf power increases by using linear-stability theory[13-15] and the method of averaging for ordinary differential equations (ODE's).[16-18]

STABILITY RELATIONS FOR SPIN WAVES

A. The instability equation and its solution

Linear-stability theory examines the time evolution of a small perturbation to a known solution to an ODE. Specifically, whether the perturbation grows or diminishes determines whether the known solution is unstable or stable, respectively. In the case of spin-wave mode equations, the question is whether a perturbation of the $\mathbf{k}\neq 0$ modes increases or decreases in magnitude during one oscillation of the rf field. Since small perturbations of the $a_{\mathbf{k}}$'s are considered, it is appropriate to examine the "linearization" of Eq. (2) using the Jacobian of the right-hand side of Eq. (2).

By calculating the Jacobian, the equation of motion for the linear stability of $a_{\mathbf{k}}$ is

$$\frac{da_{\mathbf{k}}}{dt} = \sum_{\mathbf{k}} J^1_{\mathbf{k}\mathbf{k}'} a_{\mathbf{k}'} + \sum_{\mathbf{k}} J^2_{\mathbf{k}\mathbf{k}'} a^*_{-\mathbf{k}'} , \tag{3}$$

where

$$J^1_{\mathbf{k}\mathbf{k}'} = iA_{\mathbf{k}}\delta_{\mathbf{k}\mathbf{k}'} + \frac{\partial P_{\mathbf{k}}}{\partial a_{\mathbf{k}'}}, \quad J^2_{\mathbf{k}\mathbf{k}'} = iB_{\mathbf{k}}\delta_{\mathbf{k}\mathbf{k}'} + \frac{\partial P_{\mathbf{k}}}{\partial a^*_{-\mathbf{k}'}} . \tag{4}$$

The Jacobian is evaluated on the orbit for the uniform mode motion, that is, $a_{\mathbf{k}}=a^*_{\mathbf{k}}=0$ if $\mathbf{k}\neq 0$ and $a_0=a_0(t)$ is a solution of the equation of motion for the $\mathbf{k}=0$ mode. When this is done the Jacobian reduces to 2×2 block structure in the pairs of variables $a_{\mathbf{k}}$ and $a^*_{-\mathbf{k}}$. The block for the a_0 mode is diagonal. Since the transformation $k\to -k$ does not affect $J^1_{\mathbf{k}\mathbf{k}'}$ or $J^2_{\mathbf{k}\mathbf{k}'}$, the 2×2 block becomes

$$J = \begin{bmatrix} J^1_{\mathbf{k}\mathbf{k}} & J^2_{\mathbf{k}\mathbf{k}} \\ J^{2*}_{\mathbf{k}\mathbf{k}} & J^{1*}_{\mathbf{k}\mathbf{k}} \end{bmatrix} , \tag{5}$$

where the J's are functions of a_0 and t only. The full expressions for the J's are given in Appendix B.

The 2×2 block structure makes it possible to obtain approximate solutions to the instability equations (3). This was implicit in Suhl's original paper[9] and in others who followed Suhl's approach.[10,11] The uniform-mode equation can be solved in some approximation giving a_0 as a function of t and, therefore J as solely a function of t. This means the instability equations can be written compactly as

$$\dot{a} = J(t)a , \tag{6}$$

where a is the column vector $(a_{\mathbf{k}}, a^*_{-\mathbf{k}})^T$. The matrix $J(t)$ is periodic in t with period $2\pi/\omega$.

First, separate the linear and nonlinear terms in J: $J=L+U$ where

$$L = \begin{bmatrix} A_{\mathbf{k}} & B_{\mathbf{k}} \\ B^*_{\mathbf{k}} & A^*_{\mathbf{k}} \end{bmatrix} \quad U = \begin{bmatrix} \dfrac{\partial P_{\mathbf{k}}}{\partial a_{\mathbf{k}}} & \dfrac{\partial P_{\mathbf{k}}}{\partial a^*_{-\mathbf{k}}} \\ \dfrac{\partial P^*_{\mathbf{k}}}{\partial a_{\mathbf{k}}} & \dfrac{\partial P^*_{\mathbf{k}}}{\partial a_{-\mathbf{k}}{}^*} \end{bmatrix} .$$

Now use a variation of parameters approach to write $a(t)=e^{Lt}c(t)$. Then, the equation of motion for c is

$$\frac{dc}{dt} = e^{-Lt} U e^{Lt} c \equiv Ec . \tag{7}$$

The matrix U is of the order of h (or a_0) which is the small parameter in the problem. This means Eq. (7) is in a form that is suitable for solution by the method of averaging.[16,17] This approximates E by its infinite-time average $\bar{E}$, where

$$\bar{E}=\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau E(t)dt\ .$$

The approximate solutions are $c(t)=e^{\bar{E}t}c(0)$ and, therefore, $a(t)=e^{Lt}e^{\bar{E}t}a(0)$. The expression for $a(t)$ can now be put ino a more tractable form. This is done below.

B. Eigenvalues and critical rf fields

Instabilities exhibit themselves by the existence of an eigenvalue of $e^{LT}e^{\bar{E}T}$ which is greater than 1, where T is the period of one oscillation of the uniform mode a_0. In other words, small perturbations in $a(t)$ will grow during each period. Let S diagonalize L, $SLS^{-1}=L_D$. S is the Holstein-Primakoff[23] transformation. Then, since eigenvalues are preserved under similarity transformations, the eigenvalues of $e^{LT}e^{\bar{E}T}$ are the same as the eigenvalues of $e^{L_D T}e^{\bar{E}T}$, where $\bar{E}'=S\bar{E}S^{-1}$. In the diagonalized matrix

$$L_D=\begin{bmatrix}\Omega_{\mathbf{k}} & 0\\ 0 & \Omega_{\mathbf{k}}^*\end{bmatrix},$$

the $\Omega_{\mathbf{k}}$'s are the complex frequencies of the $\mathbf{k}\neq 0$ spin wave, where the real part of $\Omega_{\mathbf{k}}$, $\eta_{\mathbf{k}}$, determines the damping for the linear equations and the imaginary part, $\omega_{\mathbf{k}}$, is the usual spin-wave frequency (see Appendix B). The same form of $\Omega_{\mathbf{k}}$ results for all standard damping mechanisms in the Landau-Lifshitz equation (LLE).

For simplicity, let the matrix $F=SUS^{-1}$. Then using the definitions of L_D and $\bar{E}'$,

$$\bar{E}'=\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau\begin{bmatrix}F_{11} & F_{12}e^{-2i\omega_{\mathbf{k}}}\\ F_{12}^*e^{2i\omega_{\mathbf{k}}} & F_{11}^*\end{bmatrix}dt\ . \quad (8)$$

The F matrix depends on t through various factors of a_0 and a_0^* and the rf field. The rf field dependence is simply $he^{i\omega t}$. Since a_0 is periodic it is also expandable in a Fourier series[10] in $e^{in\omega t}$. In the linear limit a_0 is simply proportional to $he^{i\omega t}$. This all means that F is always expandable in a Fourier series

$$F=\sum_n F^{(n)}e^{in\omega t}\ . \quad (9)$$

The immediate result of averaging in Eq. (10) is that on the diagonal only $F_{11}^{(0)}$ and its complex conjugate remain and off the diagonal only $F_{12}^{(n)}$ and its complex conjugate remain, with the Suhl restriction that $\omega_{\mathbf{k}}=n\omega/2$.

Note that only $a(T)$ is needed to determine instability, where $T=2\pi/\omega$. Because of the Suhl restriction, $\Omega_{\mathbf{k}}T=in\pi+\eta_{\mathbf{k}}T$, this means the first exponential factor for the time averaged solution for $a(T)$ becomes simply $(-1)^n e^{\eta_{\mathbf{k}}T\mathbf{I}}$, where $\mathbf{I}$ is the 2×2 unit matrix. Now, combine the exponentials to get

$$a(t)\sim(-1)^n e^{\eta_{\mathbf{k}}T\mathbf{I}+\bar{E}'T} \quad (10)$$

up to a similarity transformation. The sign and magnitude of the eigenvalues of $\bar{E}'$ relative to the $\eta_{\mathbf{k}}$ values will determine the stability of the uniform mode: if $\nu_{\mathbf{k}}$ is an eigenvalue of $\bar{E}'$, then instability of the uniform mode results when $\nu_{\mathbf{k}}+\eta_{\mathbf{k}}>0$. The critical field is determined by the equality $\nu_{\mathbf{k}}+\eta_{\mathbf{k}}=0$.

The above relations can be generalized somewhat by adding and subtracting a constant diagonal matrix $X=\mathrm{diag}(ih^2\chi,-ih^2\chi)$ in the equation of motion for $a(t)$, where h is the magnitude of the rf pumping field. The reason for the inclusion of the h^2 term is apparent below in the $F^{(0)}$ relationship. This generalization changes the above formulas by replacing F_{11} by $F_{11}+ih^2\chi$ and $\omega_{\mathbf{k}}$ by $\omega_{\mathbf{k}}-h^2\chi$. The advantage of this is that it allows variation of the Suhl frequency restriction and the stability of spin-waves nearby in frequency can be examined. Viewed in another way, the variable χ controls the detuning.[9] The remainder of the formulas in this section contain this extended feature.

The eigenvalues of $\bar{E}'$ are

$$\nu_{\mathbf{k}}=\mathrm{Re}(F_{11}^{(0)})\pm\{-[\mathrm{Im}(F_{11}^{(0)})+h^2\chi]^2+|F_{12}^{(n)}|^2\}^{1/2}\ , \quad (11)$$

where Re( ) and Im ( ) mean real and imaginary parts. Two things are immediately apparently. The real part of $F_{11}^{(0)}$ can affect the stability and the relative magnitudes of $\mathrm{Im}(F_{11}^{(0)})+h^2\chi$ and $F_{12}^{(n)}$ can affect the stability. Each is an ostensibly independent contribution. Early work by Suhl[9] and Schlömann[10] effectively took $h^2\chi=-\mathrm{Im}(F_{11}^{(0)})$, i.e., a particular detuning of the spin-wave frequency. Here the detuning is variable and explicit. Equation (13) shows that the most unstable spin wave, that which will grow exponentially at smallest rf field, will have the detuned frequency of Suhl, i.e., $h^2\chi=\mathrm{Im}(F_{11}^{(0)})$. Equation (13) allows for an estimate of the instability criteria for spin waves whose frequencies are near this detuned frequency.

The F matrix Fourier components are all proportional to powers of the rf field. This allows a general derivation of relations for the critical rf field, h_{crit}. Write $F^{(n)}=h^n f^{(n)}$ and $F^{(0)}=h^2 f^{(0)}$, where $f^{(n)}$ is independent of h and is just $F^{(n)}$ evaluated at $h=1$. Then for the first-order Suhl instability ($n=1$),

$$h_{\mathrm{crit}}=\left[\frac{f_{12}^{(1)2}}{2}\pm\frac{1}{2}[f_{12}^{(1)4}-4(f_{11}^{(0)}+\chi)^2\eta_{\mathbf{k}}^2]^{1/2}\right]^{1/2}, \quad (12)$$

and for the second-order Suhl instability ($n=2$),

$$h_{\mathrm{crit}}=\frac{\sqrt{|\eta_{\mathbf{k}}|}}{[f_{12}^{(2)2}-(f_{11}^{(0)}+\chi)^2]^{1/4}}\ . \quad (13)$$

When $\chi=f_{11}^{(0)}$, Eqs. (12) and (13) reduce to the usual "detuned" Suhl relations,[9,10]

$$h_{\mathrm{crit}}=\frac{\eta_{\mathbf{k}}}{f_{12}^{(1)}}\quad (n=1,\ \text{first order})\ ,$$

$$h_{\mathrm{crit}}=\left[\frac{|\eta_{\mathbf{k}}|}{f_{12}^{(2)}}\right]^{1/2}\quad (n=2,\ \text{second order})\ .$$

C. Second-order averaging results

The averaging technique was implicit in Suhl's work.[9] Here it is explicit and leads directly to the Suhl restrictions on the spin-wave frequency with some generalization. One advantage of the method of averaging is that it can, in principle, be done to any order and it can also provide error estimates.[17,18] The results in the previous section were derived using first-order averaging. This section uses a simple approach based on general nth-order averaging[17,18] to obtaining a higher-order correction for the instability relations and, furthermore, shows that this will be small.

In the first-order averaging, the matrix E, which helps define the vector field for the equation of motion for $c(t)$, is replaced by an averaged matrix $\bar{E}$ which allows an approximate solution to $c(t)$ to be found, say $c^{(1)}(t)=e^{\bar{E}t}c(0)$. The next logical step is to find a higher-order correction to $c^{(1)}$, say $c^{(2)}(t)$. Let $c^{(2)}=c^{(1)}+\varepsilon P$, where ε is of the order of the small parameter in the problem (either h or a_0 here) and P is to be found. Then

$$\dot{c}^{(2)}=\dot{c}^{(1)}+\epsilon\dot{P}=Ec^{(2)}\approx Ec^{(1)}(t)=Ee^{\bar{E}t}c^{(1)}(0)\ , \qquad (14)$$

where $E\sim\gamma h\sim\varepsilon$ at most and higher-order terms have been dropped. Solving for P gives

$$P=\frac{1}{\varepsilon}\int_0^T(E-\bar{E})e^{\bar{E}t}dt\,c^{(1)}(0)\ . \qquad (15)$$

Now $\bar{E}T\sim 2\pi\gamma h/\omega\ll 1$. Therefore, the exponential can be expanded and only the lowest-order term which might contribute to the integral be retained. Equation (15) becomes

$$P=\frac{1}{\varepsilon}\int_0^T(E-\bar{E})dt\,c^{(1)}(0)\ . \qquad (16)$$

If E were periodic in general, with period T, then P would always be zero. Recall that $E=e^{-Lt}Ue^{Lt}$, where the eigenvalues of L are $\Omega_{\mathbf{k}}=i(\omega_{\mathbf{k}}-h^2\chi)+\eta$. The inclusion of the $h^2\chi$ term allows second-order averaging results to be calculated in a simple way for all spin waves with frequencies near $n\omega/2$ for the particular nth-order instability being investigated.

By writing $E=S^{-1}e^{-L_Dt}Fe^{L_Dt}S$, where S and F are the same matrices as in Sec. B, the equation for P becomes

$$P=\frac{1}{\varepsilon}S^{-1}\int_0^T dt\begin{bmatrix} F_{11}-F_{11}^{(0)} & F_{12}e^{-2i(\omega_{\mathbf{k}}-h^2\chi)}-F_{12}^{(n)} \\ F_{12}^{*}e^{2i(\omega_{\mathbf{k}}-h^2\chi)}-F_{12}^{(n)*} & F_{11}^{*}-F_{11}^{(0)*}\end{bmatrix}Sc^{(1)}(0)\ , \qquad (17)$$

where the order of the instability is assumed to be n. It is now easy to see that $P=0$. The diagonal terms become zero upon integration because of the periodic nature of F_{11}; only $F_{11}^{(0)}$ survives and this cancels with the other $F_{11}^{(0)}$ term. On the off-diagonal, recall that χ is chosen so that $\omega_{\mathbf{k}}-h^2\chi=n\omega/2$. This causes all terms in the Fourier expansion of F_{12} to drop out, except the nth term, which then cancels with the $F_{12}^{(n)}$ term. Hence, the relations among the eigenvalues at the onset of instability and especially the critical field formulas are unchanged to second order in the method of averaging. This may partly explain why, despite seemingly severe approximations, the Suhl critical field relations are so accurate.[24–26]

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APPENDIX A: THE EQUATIONS OF MOTION

In the following $\lambda=\gamma+i\alpha$ and for any symbol w, $w_\pm=w_x\pm iw_y$. All other symbols are consistent with Ref. 9. Note that all terms which collectively have the coefficient α and the second-order term in h_+ are not present in other derivations of the equations of motion.[9–11]

$$\begin{aligned}\frac{da_{\mathbf{k}}}{dt}=&-i\lambda h_+\delta_{\mathbf{k}0}+i\lambda M_S\left[H_0+\beta|\mathbf{k}|^2+2\pi\frac{k_+k_-}{|\mathbf{k}|^2}-4\pi N_z+4\pi N_T\delta_{\mathbf{k}0}\right]a_{\mathbf{k}}+2\pi i\lambda M_S\frac{k_+^2}{|\mathbf{k}|^2}a^*_{-\mathbf{k}}-\frac{\alpha h_-}{2}\sum_{\mathbf{k}'}a_{\mathbf{k}'}a_{\mathbf{k}-\mathbf{k}'}\\ &+i\lambda M_S\sum_{\mathbf{k}'}\left\{\left[\frac{h_+}{2M_S}-2\pi\left[\frac{k_z'k_+'}{|\mathbf{k}'|^2}+\frac{k_zk_+}{|\mathbf{k}|^2}\right]\right]a_{\mathbf{k}-\mathbf{k}'}a^*_{-\mathbf{k}'}-2\pi\frac{k_z'k_-'}{|\mathbf{k}'|^2}a_{\mathbf{k}'}a_{\mathbf{k}-\mathbf{k}'}\right\}\\ &+i\lambda M_S\sum_{\mathbf{k}'\mathbf{k}''}\left[\left[\frac{\beta}{2}(|\mathbf{k}|^2-2\mathbf{k}\cdot\mathbf{k}')+2\pi\frac{(k-k')_z^2}{|\mathbf{k}-\mathbf{k}'|^2}-\pi\frac{k_+'k_-'}{|\mathbf{k}'|^2}\right]a_{\mathbf{k}'}a_{\mathbf{k}''}a^*_{\mathbf{k}'+\mathbf{k}''-\mathbf{k}}\right.\\ &\qquad\left.-\pi\frac{(k_+')^2}{|\mathbf{k}'|^2}a_{\mathbf{k}''}a^*_{-\mathbf{k}'}a^*_{\mathbf{k}'+\mathbf{k}''-\mathbf{k}}+2\pi N_z\delta_{\mathbf{k}''\mathbf{k}}a_{\mathbf{k}''}a_{\mathbf{k}'}a^*_{\mathbf{k}'}-2\pi N_T\delta_{\mathbf{k}''0}a_{\mathbf{k}''}a_{\mathbf{k}'}a^*_{\mathbf{k}'-\mathbf{k}}\right]\\ &+\alpha M_S\sum_{\mathbf{k}'\mathbf{k}''}\left[\left[\frac{\beta}{2}|\mathbf{k}'|^2+\pi\frac{k_+'k_-'}{|\mathbf{k}'|^2}+\frac{H_0}{2M_S}-2\pi N_z\right]a_{\mathbf{k}''}a_{\mathbf{k}-\mathbf{k}'-\mathbf{k}''}a^*_{-\mathbf{k}'}\right.\\ &\qquad\left.+\pi\frac{(k_-')^2}{|\mathbf{k}'|^2}a_{\mathbf{k}'}a_{\mathbf{k}''}a_{\mathbf{k}-\mathbf{k}'-\mathbf{k}''}+2\pi N_T\delta_{\mathbf{k}''0}a^*_{\mathbf{k}''}a_{\mathbf{k}'}a_{\mathbf{k}'-\mathbf{k}}\right]\ .\end{aligned}$$

APPENDIX B: EXPRESSIONS FOR THE JACOBIAN

Below are expressions for the Jacobian of the equation of motion in Appendix A, evaluated on the uniform mode ($a_0 \neq 0$, $a_{\mathbf{k}}=0$, if $\mathbf{k} \neq 0$). In addition, the Holstein-Primakoff transformation, S, and the dispersion relation for the spin waves are given. Together all these expressions enable the computation of h_{crit}.

$$J^1_{\mathbf{kk}} = i\lambda \left[H_0 + \beta |\mathbf{k}|^2 M_S + 2\pi M_S \frac{k_+ k_-}{|\mathbf{k}|^2} - 4\pi N_z M_S \right] - \left[2\pi i \lambda M_S \frac{k_z k_-}{|\mathbf{k}|^2} + \alpha h_- \right] a_0 + i\lambda \left[\frac{h_+}{2} - 2\pi \frac{k_+ k_z}{|\mathbf{k}|^2} \right] a_0^*$$
$$+ \alpha\pi M_S \frac{k_-^2}{|\mathbf{k}|^2} a_0^2 + \alpha[4\pi M_S (N_T - N_z) + H_0] a_0 a_0^* + \pi i \lambda \left[2M_S (N_z - N_T) - M_S \frac{k_+ k_-}{|\mathbf{k}|^2} + 2M_S \frac{k_z^2}{|\mathbf{k}|^2} \right] a_0 a_0^* ,$$

$$J^2_{\mathbf{kk}} = 2\pi i \lambda \frac{k_+^2}{|\mathbf{k}|^2} M_S + i\lambda \left[\frac{h_+}{2} - 4\pi M_S \frac{k_z k_+}{|\mathbf{k}|^2} \right] a_0 + \alpha \left[\frac{\beta}{2} M_S |\mathbf{k}|^2 + \tfrac{1}{2}(H_0 - 2\pi N_z M_S) + \pi M_S \frac{k_+ k_-}{|\mathbf{k}|^2} \right] a_0^2$$
$$+ i\lambda \left[\frac{\beta}{2} M_S |\mathbf{k}|^2 - 2\pi N_T M_S + 2\pi M_S \frac{k_z^2}{|\mathbf{k}|^2} \right] a_0^2 + \pi i \lambda M_S \frac{k_+^2}{|\mathbf{k}|^2} a_0 a_0^* .$$

In order to calculate h_{crit} the matrix $F = SUS^{-1}$ must be calculated. The matrix U is defined by U_{11} equals the terms containing a_0 in $J^1_{\mathbf{kk}}$, U_{12} equals the terms containing a_0 in $J^2_{\mathbf{kk}}$, $U_{22} = U^*_{11}$, and $U_{21} = U^*_{12}$. The matrix S is given by

$$S_{11} = \left[\frac{A_{\mathbf{k}} + \omega_{\mathbf{k}}}{2\omega_{\mathbf{k}}} \right]^{1/2}, \quad S_{12} = \left[\frac{A_{\mathbf{k}} - \omega_{\mathbf{k}}}{2\omega_{\mathbf{k}}} \right]^{1/2} e^{2i\phi_{\mathbf{k}}}$$

with

$$\omega_{\mathbf{k}} = (A_{\mathbf{k}}^2 - |B_{\mathbf{k}}|^2)^{1/2} \quad \text{and} \quad e^{2i\phi_{\mathbf{k}}} = B_{\mathbf{k}} / |B_{\mathbf{k}}| ,$$

and $S_{21} = S^*_{12}$ and $S_{22} = S^*_{11}$. $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$ are part of the linear terms in the equations of motion (see text and Ref. 9) and $\omega_{\mathbf{k}}$ is the spin-wave frequency. The matrix F can now be constructed using these relations and an approximate solution for the trajectory of the uniform mode $a_0(t)$. By substituting in for $a_0(t)$ the Fourier coefficients $F^{(n)}$ can be calculated. A particularly simple case is the asymptotic linear solution for $a_0(t)$, then

$$a_0 = \frac{\lambda h e^{i\omega t}}{\lambda H_0 - \omega} \quad (\lambda = \gamma + i\alpha) ,$$

and $F^{(0)}$ is the matrix made up from all the Jacobian terms in $a_0(t) a_0^*(t)$ plus the one linear term in $h_+ a_0(t)$, $F^{(1)}$ is the matrix made from Jacobian terms in $a_0(t)$, and $F^{(2)}$ is the matrix made from Jacobian terms in $a_0^2(t)$.

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