Mean-field soft-spin Potts glass model: Statics and dynamics

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A soft-spin Potts glass model is introduced and solved in the mean-field limit. Near the glass-transition temperature T_c both the statics and the dynamics are considered with emphasis on establishing the connections between the two approaches. For some parameter values of the Hamiltonian, the Potts glass transition is continuous while for others it is discontinuous. For both cases there is a continuous freezing as T_c is approached from above. Connections are made between this work and previous work on p-spin ($p > 2$) spin-glass models as well as to lattice Potts glass models.

I. INTRODUCTION

In a recent paper we studied the statics and the dynamics of the mean-field p-spin ($p>2$) interaction spin-glass (SG) model.[1,2] The basic motivation was to investigate a class of models exhibiting spin-glass-like transitions where the Edwards-Anderson order parameter q_{EA} is discontinuous at the SG transition temperature T_c.[1-4]

In this paper we report similar work on a mean-field soft-spin Potts glass (PG) model. The basic motivation here is as follows. First, we want to establish the generic nature of the p-spin ($p>2$) work for SG models with discontinuous q_{EA}. For some parameter values we find that the PG model introduced here undergoes a glass transition where q_{EA} is continuous at a glass-transition temperature which we denote by T_g. For this case the dynamical theory and the usual static theory give the same T_g as in the mean-field Ising spin-glass model. The dynamical theory predicts a continuous slowing down as T_g is approached from above. For other parameter values, we show that the PG model used here undergoes a glass transition at a temperature T_A with q_{EA} discontinuous at T_A. However, as T_A is approached from above the spin autocorrelation function continuously slows down and freezes at T_A. In Ref. 5 T_A has been related to the temperature where nontrivial metastable solutions to the (TAP) Thouless-Anderson-Palmer equations[6] for the PG first exist. In non-mean-field models, the long-time dynamics is then governed by activated transitions between the different TAP states. In strictly mean-field models, however, activated transport cannot take place[7] because there is no distinction between surface and bulk free energies. Thus, transition between the TAP states involves cooperative motion of all the spins. As a consequence of this, and the infinite free-energy barriers between distinct TAP states, in the mean-field model T_A signals a dynamical transition from ergodic to nonergodic behavior. In the equilibrium replica-based calculations given here we give a prescription for obtaining T_A from the static theory. At a lower temperature $T_K < T_A$, the static theory predicts a true equilibrium transition with a discontinuous q_{EA} at T_K. At this transition temperature there is no latent heat but the specific heat is discontinuous. Elsewhere[5] T_K has been interpreted as a Kauzmann temperature[7] where the configurational entropy vanishes. Similar results were found previously for the p-spin model.[1] Second, we will show that the dynamics on the ergodic side as $T\to T_g^+$ or $T\to T_A^+$ is considerably richer than in the usual Sherrington and Kirkpatrick (SK) model or in the p-spin ($p>2$) SG model. We find that the exponent governing the continuous slowing down depends explicitly on a coupling constant characterizing the Hamiltonian and is therefore nonuniversal.

The plan of this paper is as follows. In Sec. II we introduce the model and derive the dynamical equations of motion for the quenched average (over the random bond interactions) correlation functions. In Sec. III an approximate solution to these equations is presented. For some parameter values continuous PG transition is obtained and for others the glass transition is discontinuous. The asymptotic results seem to be consistent with the results of a direct numerical integration of the dynamical equations. In Sec. IV replica methods are used to discuss the equilibrium PG transitions. Here we also discuss connections between the static and the dynamic approaches. In Sec. V a few additional comments are made, and in the Appendix a lengthy equation used in the text is quoted.

II. THE DYNAMICAL MODEL

A. The model

The usual lattice mean-field (infinite ranged) p-state PG Hamiltonian is[8-11]

$$H=-\sum_{a=1}^{p-1}\sum_{i<j}J_{ij}\phi_i^a\phi_j^a \; . \tag{2.1a}$$

Here $(a,b,c,d,\ldots)$ denote the $p-1$ spin components, (i,j) denote lattice sites, and $\{J_{ij}\}$ denotes the random interactions which are assumed to be Gaussian distributed,

$$P(J_{ij})=\left[\frac{N}{2\pi J^2}\right]^{1/2}\exp\left[-\frac{(J_{ij})^2N}{2J^2}\right] . \tag{2.1b}$$

The N dependence in Eq. (2.1b) is chosen such that a well-defined thermodynamic limit exists. The ϕ_i^a are the usual Potts variables[12] which are chosen from the set of p vectors $\{\mathbf{e}^l\}$ ($l=1,2,\ldots,p$), in a $(p-1)$-dimensional vector space,

$$e_a^l=\left[\frac{p-a}{p+1-a}\right]^{1/2}p^{1/2}\times\begin{cases}0, & l<a\\ 1, & l=a\\ -\dfrac{1}{p-a}, & l>a\end{cases} \tag{2.1c}$$

The static PG theory for the Hamiltonian given by Eq. (2.1a) is relatively simple to construct.[4,5] However, because we are mainly interested in the dynamical theory for the PG transition we consider here a soft-spin generalization of Eq. (2.1a). Standard field-theoretic methods can then be used to treat the relaxational dynamics of the PG transition. To proceed we first write down a ϕ^4 PG field theory. We require the ϕ^2 term to be random and frustrated according to Eq. (2.1a) and the ϕ^3 and ϕ^4 terms to have the correct Potts symmetry. The appropriate field theory is ($-\infty<\phi_i^a<\infty$)

$$\begin{aligned}\beta H=&-\beta\sum_{a=1}^{p-1}\sum_{i<j}J_{ij}\phi_i^a\phi_j^a+\frac{r_0}{2}\sum_{a=1}^{p-1}\sum_{i=1}^{N}\phi_i^a\phi_i^a\\&+\frac{g_3}{3}\sum_{a,b,c,=1}^{p-1}\sum_{i=1}^{N}Q_{abc}\phi_i^a\phi_i^b\phi_i^c\\&+\frac{1}{4}\sum_{a,b,c,d=1}^{p-1}\sum_{i=1}^{N}T_{abcd}\phi_i^a\phi_i^b\phi_i^c\phi_i^d ,\end{aligned} \tag{2.2a}$$

where $\beta=T^{-1}$ and the Boltzmann's constant is taken to be unity. The field theory given by Eq. (2.2a) may be viewed as the equivalent lattice representation of the continuum Lagrangian given by Zia and Wallace.[13] This is only true if the coordination number of the lattice is infinite, and this makes the field theory infinite ranged thus making it amenable to mean-field treatment. Note that the last three terms in Eq. (2.2a) are single-site terms. Q and T are the Potts coupling constants given by[13]

$$Q_{abc}=\sum_{l=1}^{p}e_a^le_b^le_c^l \tag{2.2b}$$

and

$$T_{abcd}=u_0S_{abcd}+f_0F_{abcd} , \tag{2.2c}$$

with

$$S_{abcd}=\tfrac{1}{3}(\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}) \tag{2.2d}$$

and

$$F_{abcd}=\sum_{l=1}^{p}e_a^le_b^le_c^le_d^l . \tag{2.2e}$$

Some identities which will be of use below are

$$\sum_{l=1}^{p}e_a^le_b^l=p\delta_{ab} , \tag{2.3a}$$

$$\sum_{a=1}^{p-1}e_a^le_a^{l'}=p\delta_{ll'}-1 , \tag{2.3b}$$

$$\sum_{l=1}^{p}e_a^l=0 . \tag{2.3c}$$

It is relevant to point out that the "hard" Hamiltonian given by Eq. (2.1a) cannot be simply recovered by a limiting process from Eq. (2.2a). Specifically, Eq. (2.2a) cannot be made to coincide with Eq. (2.1a), by an appropriate choice of r_0, g_3, and T_{abcd}. This is different from the usual soft-spin versions of the p-spin interaction SG models.[1] We therefore expect, and we will show, that the glass transitions for Eq. (2.2a) are only qualitatively similar to those for Eq. (2.1a). In particular, using the replica-symmetry-breaking scheme the "hard" model, the PG transition is discontinuous for $p\geq 4$ and continuous for $p<4$ (Refs. 8 and 9). Here we find that the dividing line for continuous and discontinuous transitions depends both on p and on the coupling constant g_3. This, however, seems to be the only apparent difference between the glass transitions predicted by the Hamiltonians given by Eqs. (2.1a) and (2.2a).

B. Dynamical equations

A purely relaxational dynamics is used for $\phi_i^a(t)$ and is assumed to be given by Langevin equation

$$\Gamma_0^{-1}\partial_t\phi_i^a(t)=-\frac{\delta(\beta H)}{\delta\phi_i^a(t)}+\xi_i^a(t) . \tag{2.4a}$$

Here Γ_0 is a bare kinetic coefficient, which sets the microscopic time scale, and $\xi_i^a(t)$ is a Gaussian random noise with zero mean and variance:

$$\langle\xi_i^a(t)\xi_j^b(t')\rangle_\xi=\frac{2}{\Gamma_0}\delta_{ij}\delta_{ab}\delta(t-t') . \tag{2.4b}$$

Equations (2.4) ensure a correct approach to equilibrium. In the dynamical calculation the physical quantities of interest are the spin-spin correlation function

$$C_{ab}(t-t')=\delta_{ab}C(t-t')=\frac{1}{N}\sum_{i=1}^{N}\langle\phi_i^a(t)\phi_i^b(t')\rangle \tag{2.5a}$$

and the linear response function

$$\begin{aligned}G_{ab}(t-t')&=\delta_{ab}G(t-t')\\&=\frac{1}{N}\sum_{i=1}^{N}\frac{\partial\langle\phi_i^a(t)\rangle}{\partial h_i^b(t')},\quad t>t' .\end{aligned} \tag{2.5b}$$

Here the angular brackets denote an average over ξ and the random interactions, and h_i is an external magnetic field. We have also used that C and G are diagonal in the vector indices a,b. Causality yields the relation,

$$G(t)=-\theta(t)\partial_tC(t) , \tag{2.5c}$$

with $\theta(t>0)=1$ and $\theta(t<0)=0$.

To carry out the averaging over the quenched random interactions we use the dynamical functional integral formulation of De Dominicis[14] and Janssen *et al.*[15] Since this procedure is now standard we only quote the results here. In the $N\to\infty$ limit, the mean-field equation of motion for $\phi_i^a(\omega)$, the Fourier transform of $\phi_i^a(t)$, averaged over the random interactions is

$$\phi_i^a(\omega)=G_0(\omega)[f_i^a(\omega)+h_i^a(\omega)]-G_0(\omega)g_3\sum_{b,c=1}^{p-1}Q_{abc}\int\frac{d\omega_1}{2\pi}\phi_i^b(\omega_1)\phi_i^c(\omega-\omega_1)$$
$$-G_0(\omega)\sum_{b,c,d=1}^{p-1}T_{abcd}\int\frac{d\omega_1\,d\omega_2}{(2\pi)^2}\phi_i^b(\omega_1)\phi_i^c(\omega_2)\phi_i^d(\omega-\omega_1-\omega_2)\ , \tag{2.6a}$$

with $G_0(\omega)$ a renormalized bare propagator

$$G_0^{-1}(\omega)=r_0-i\omega\Gamma_0^{-1}-\mu\int_0^\infty dt\, e^{i\omega t}G(t)\ , \tag{2.6b}$$

and

$$\mu=\beta^2J^2 \tag{2.6c}$$

and $f_i^a(\omega)$ a renormalized noise term

$$\langle f_i^a(\omega)f_j^b(\omega')\rangle=2\pi\delta(\omega+\omega')\delta_{ij}\delta_{ab}\times\left[\frac{2}{\Gamma^0}+\mu\int_{-\infty}^{+\infty}dt\, e^{i\omega t}C(t)\right]. \tag{2.6d}$$

III. APPROXIMATE SOLUTION OF DYNAMICAL EQUATIONS

In this section an approximate solution to the dynamical equations given by Eq. (2.6) is presented, using the causal relation given in Eq. (2.5c), and the glass transition predicted by them is discussed. In the dynamical approach our goal is to determine T_c and characterize the critical slowing down as $T\to T_c$ from above. Here T_c denotes either T_g or T_A. The behavior for $T<T_c$ is most easily examined using the equilibrium methods discussed in Sec. IV. It is important to note that unlike the case of the SK model,[16] we are not able to make general statements, valid to all orders in g_3, u_0, and f_0, about the critical behavior as T_A is approached from above. For the parameter values corresponding to the continuous PG transition it will be clear from the results of this section and Sec. IV that T_g can be easily determined in general. For parameter values where there is a discontinuous PG transition our results are more questionable. In order to control our approximations we consider here only parameter values where the discontinuities at T_A are themselves a small expansion parameter. The validity of our approximations are confirmed in Sec. IV where we make a connection with the equilibrium theory of the PG transition. These points and the reliability of the exponents characterizing the critical slowing down as $T\to T_c$ will be discussed further below. Finally, we point out that it seems likely that one could obtain an exactly solvable PG field theory by constructing a dynamical PG version of the exactly solvable ϕ^3 model suggested by Amit and Roginsky.[17] Although we have not pursued this approach here, it appears that such a theory would be equivalent to the present approximate theory.

We treat the g_3 and T terms in Eq. (2.6a) in the one-loop approximation. Corrections will be discussed below. An equation for

$$\hat{C}(\omega)=\int_0^\infty dt\, e^{i\omega t}C(t)\quad(\text{Im}\ \omega>0) \tag{3.1a}$$

can be derived from Eqs. (2.6), (2.5c), (2.2), and (2.3). In the ergodic phase we obtain

$$\hat{C}(\omega)=\frac{C(t=0)}{-i\omega+\overline{r}_0\Gamma(\omega)}\ , \tag{3.1b}$$

with the equal-time spin correlation function given by

$$C(t=0)=\frac{1}{\overline{r}_0}=\{r_0-\mu C(t=0)-2p^2(p-2)g_3^2C^2(t=0)+C(t=0)[u_0(p+1)+3f_0p^2]\}^{-1}\ . \tag{3.1c}$$

$\Gamma(\omega)$ is a renormalized kinetic coefficient:

$$\Gamma^{-1}(\omega)=\Gamma_0^{-1}+\mu\int_0^\infty dt\, e^{i\omega t}C(t)+2p^2(p-2)g_3^2\int_0^\infty dt\, e^{i\omega t}C^2(t)\ . \tag{3.1d}$$

Notice that when $p=2$ (the Ising limit), Eq. (3.1) reduces to those obtained by Sompolinski and Zippelius.[16] For use below we note that in the time domain $\phi(t)=C(t)/C(0)$ satisfies the equation

$$\nu_0^{-1}\dot\phi(t)+\phi(t)+\lambda_1\int_0^t dt_1\,\phi(t-t_1)\dot\phi(t_1)+\lambda_2\int_0^t dt_1\,\phi^2(t-t_1)\dot\phi(t_1)=0\ , \tag{3.2a}$$

with $\phi(t=0)=1$, $\nu_0\equiv\overline{r}_0\Gamma_0$, and the nonlinear coupling constants are given by

$$\lambda_1=\frac{\mu}{\overline{r}_0^{\,2}},\quad \lambda_2=\frac{2p^2(p-2)}{\overline{r}_0^{\,3}}g_3^2\ . \tag{3.2b}$$

Finally, we note that it is important that we consider $\hat{C}(\omega)$ rather than the Fourier transform of $C(t)$. As discussed in the context of the dynamics of the p-spin model,[1] the formulation based on $\hat{C}(\omega)$ enables us to take $C(t=0)$ to be continuous at T_A and avoid specifying the Langevin force correlation function in the nonergodic phase. This correlation function is determined by the $\hat{C}(\omega)$ and the fluctuation dissipation theorem. With this step we also avoid the usual unstable replica-symmetric (RS) solution. These points are directly related to the work of Houghton, Jain, and Young[18] on the Sk spin-glass model. These ideas were also used in our earlier work[1] on the p-spin model with $p>2$.

The interesting critical properties of the Eq. (3.2a) have been considered previously by Götze in a different context.[17] As a consequence we will be brief in describing the phase transitions predicted by Eqs. (3.1) and (3.2).

A. PG transition ($T = T_c = T_g$ or T_A)

We first locate T_c for the continuous transition and then the discontinuous transition. We then determine the critical properties as $T \to T_c^+$. First define the Edwards-Anderson order parameter:

$$\lim_{t\to\infty} C(t) \equiv q_{\mathrm{EA}} = q \equiv \frac{\bar{q}}{\bar{r}_0} . \tag{3.3a}$$

Assuming that $\hat{C}(\omega)$ has a time-persistent part with a nonzero q and a decaying part, Eqs. (3.1) and (3.2) yield the equation of state

$$q = \frac{1}{\bar{r}_0^2} \frac{\mu q + 2p^2(p-2)g_3^2 q^2}{1 + \dfrac{\mu q}{\bar{r}_0} + \dfrac{2p^2(p-2)g_3^2 q^2}{\bar{r}_0}} \tag{3.3b}$$

which can be written using Eq. (3.2b) as

$$\lambda_2 \bar{q}^{\,3} + \bar{q}^{\,2}(\lambda_1 - \lambda_2) = \bar{q}(\lambda_1 - 1) . \tag{3.3c}$$

(i) The continuous transition. Since $\bar{q}$ must be greater than zero, it is clear from Eq. (3.3c) that for small λ_2 there is a continuous transition at a critical temperature given by

$$\lambda_1 \to \lambda_{1c} = 1 = \frac{\mu_c}{\bar{r}_{0c}^{\,2}} . \tag{3.4}$$

An explicit T_g can then be determined from Eqs. (2.6c) and (3.1c). Here $\mu_c = J^2/(T_g)^2$ and $\bar{r}_{0c}$ is $\bar{r}_0$ at T_g. Near T_g Eq. (3.3c) predicts $\bar{q} \sim \bar{t}$ with $\bar{t} = 1 - T/T_g$, so that the exponent for $\bar{q}$ is unity as in the SK model. Here we will not be any more explicit with the coefficient for $\bar{q}$ since our equilibrium results (cf. Sec. IV) indicate that replica-symmetry-breaking (RSB) methods are needed to correctly determine $\bar{q}$ below T_g.

(ii) The discontinuous transition. The PG transition is continuous until the second term in Eq. (3.3c) becomes negative. Since $\lambda_1 = 1$ at the continuous transition the requirement is $\lambda_2 < 1$ implies a continuous transition while $\lambda_2 > 1$ implies a discontinuous transition. As mentioned above, in order to make the theory reliable the strength of the discontinuity is controlled by letting

$$\lambda_2 = 1 + \epsilon , \tag{3.5a}$$

with $\epsilon > 0$ but small. Since we find that the discontinuous transition occurs before the continuous one we let

$$\lambda_1 = 1 - \Delta , \tag{3.5b}$$

with $\Delta > 0$. Equations (3.5b) and (3.2b) will lead to an equation for T_A. With Eqs. (3.5), Eq. (3.3c) becomes

$$(1+\epsilon)\bar{q}^{\,3} - \bar{q}^{\,2}(\epsilon + \Delta) + \bar{q}\Delta = 0 . \tag{3.6a}$$

At T_A we will find $\Delta = O(\epsilon^2)$ and $\bar{q}_c = O(\epsilon)$ so that Eq. (3.6a) is consistently

$$\bar{q}^{\,3} - \bar{q}^{\,2}\epsilon + \bar{q}\Delta = 0 . \tag{3.6b}$$

The relevant nontrivial solution is

$$\bar{q} = \frac{\epsilon}{2} + \frac{1}{2}(\epsilon^2 - 4\Delta)^{1/2} . \tag{3.6c}$$

Therefore, there is a physical nontrivial solution at a critical temperature given by, here $\mu_c = J^2/T_A^2$,

$$\Delta \to \Delta_c = \frac{\epsilon^2}{4} \quad \text{or} \quad \frac{\mu_c}{\bar{r}_{0c}^{\,2}} = 1 - \frac{\epsilon^2}{4} . \tag{3.7a}$$

The Edwards-Anderson order parameter at T_A is given by

$$\bar{q} \to \bar{q}_c = \frac{\epsilon}{2} . \tag{3.7b}$$

Finally, we note that Eq. (3.2) gives a continuous freezing to a nontrivial value of q_{EA} as soon as a physical nontrivial long-time solution to Eq. (3.2) exists. Analytically, this follows from a local stability analysis of Eq. (3.2) similar to the one discussed in Ref. 1 for the p-spin interaction SG model. Below we confirm this fact numerically.

We conclude this subsection by discussing higher-loop corrections to Eqs. (3.3). The general theory, as well as Eq. (3.3), give an equation of state for q as a power series in q and in terms of coupling constants which in our approximate theory we have denoted by λ_1 and λ_2. Since we are at most considering a weakly discontinuous transition it follows in general we require an equation of state of $O(q^3)$ only [cf. Eq. (3.3c)].[4,5] The higher-loop corrections can then only renormalize the coupling constants in Eq. (3.3c). Since the hard-PG model has similar SG-like transitions[4,5] we conclude that these renormalizations do not remove the transitions predicted by Eq. (3.3c) and that our approximations are not serious. These observations also apply to the static theory given in Sec. IV.

B. Dynamics as $T \to T_c^+ = T_g^+$ or T_A^+

We first present a few analytic results for the critical behavior as $T \to T_c^+$ for both the continuous and the discontinuous transitions. Since the derivation of these asymptotic results rests on several assumptions, we have attempted to verify these by a numerical solution of the integral equation. The numerical results seem to lend support to the analytical analysis.

(i) Continuous transition. As shown earlier the line defining this transition in the λ_1, λ_2 plane is given by $\lambda_1 = 1$, $0 \le \lambda_2 \le 1$. Following Götze,[19] the neighborhood of this transition line is conveniently parametrized by a quantity x, which satisfies the equation $0 \le x \le 1$

$$\lambda(x) \equiv \Gamma^2 \left[\frac{1+x}{2} \right] \Big/ \Gamma(x) = \lambda_2 , \tag{3.8}$$

with Γ the gamma function. On the transition line, i.e., at $T = T_g$, the asymptotic behavior is

$$\phi(t \to \infty) \sim A / t^{(1-x)/2} , \tag{3.9a}$$

where A is a positive constant. Note that when $\lambda_2 = 0$, $x = 0$, and we recover the results for the SK model. For $\lambda_2 \to 1$, $x \to 1$ and $C(t \to \infty)$ decays very slowly. The zero-frequency kinetic coefficient, which controls the critical slowing down, vanishes as the T_g^+ is approached as

$$\Gamma(T,\omega=0) \sim |T-T_g|^{(1+x)/(1-x)} . \tag{3.9b}$$

The exponent $(1+x)/(1-x)$ increases dramatically as $\lambda_2 \to 1$. Furthermore, the critical exponent changes as one moves along the transition line thus yielding a line of critical points instead of an isolated transition point.

In order to verify the asymptotic results presented above the integral equation for $\phi(t)$ [cf. Eq. (3.2)] has been numerically solved for several values of the coupling constants λ_1. For the continuous transition λ_2 was taken to be 0.684 (corresponding to $x=\frac{1}{3}$) and the renormalized zero-frequency kinetic coefficient was determined numerically for several values of λ_1. In Fig. 1 a plot of Γ obtained numerically as well as analytically [see Eq. (3.9b)] as a function of $\Delta\lambda = (\lambda_{1c}-\lambda)$ is shown. It is clear that for $\lambda_{1c}-\lambda$ less than about 0.01, the results of the asymptotic analysis and that of the numerical solution coincide. It was also verified that the exponent, characterizing the vanishing of $\Gamma(T,\omega=0)$ depends on the value of x in a way consistent with Eq. (3.9b).

(ii) The discontinuous transition. The dynamics near the discontinuous transition, defined by the equation $\lambda_1 = 2\sqrt{\lambda_2}-\lambda_2$ with $1 \le \lambda_2 \le 4$, is much more complicated than the continuous transition. For the discontinuous transition the coupling constants λ_1 and λ_2 are parametrized according to

$$\lambda_1 = \{[2\lambda(x)-1]/\lambda^2(x)\} - \bar{t}\lambda(x)/\{1+[1-\lambda(x)]^2\} \tag{3.10}$$

and

$$\lambda_2 = \frac{1}{\lambda^2(x)} - \bar{t}\lambda(x)[1-\lambda(x)]/\{1+[1-\lambda(x)]^2\} , \tag{3.11}$$

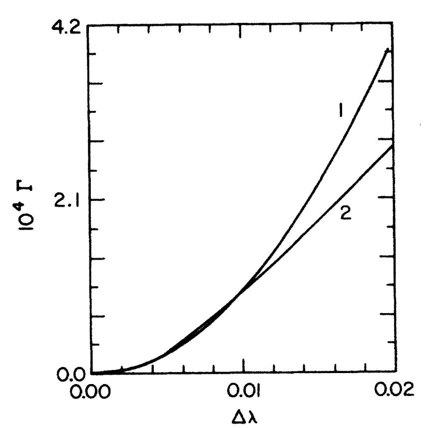

FIG. 1. Zero-frequency coefficient $\Gamma(T,\omega=0)$ for the continuous transition as a function of $\Delta\lambda(=\lambda_{1c}-\lambda)$. The parameter v_0 appearing in the integral equation [cf. Eq. (3.2a)] has been set to unity. The curve labeled 1 corresponds to the asymptotic result given by Eq. (3.9b) while curve 2 represents the numerically obtained values.

where $\bar{t} = T/T_A - 1$ is a measure of the distance from the transition line in the (λ_1,λ_2) plane.

At $T=T_A$, the asymptotic behavior of $\phi(t)$ can be obtained and is given by

$$\phi(t\to\infty) \sim \bar{q}_c + B/t^{(1-x)/2} , \tag{3.12}$$

with B being a positive constant. Note for this transition $\phi(t\to\infty)$ goes to nonzero value $\bar{q}_c$ at $T=T_A$ unlike in the case of the continuous transition. Using a series of rather complicated arguments, Götze has argued that the zero-frequency kinetic coefficient should vanish as $T\to T_A^+$ as[19]

$$\Gamma(T,\omega=0) \sim |T-T_A|^{(y-x)/(1-x)(y-1)} , \tag{3.13}$$

where y is the second solution to the equation

$$\lambda(x) = \lambda(y) , \tag{3.14}$$

with $1<y<3$.

For the discontinuous transition, the numerical solution shows a continuous slowing down as T_A is approached from above. Analysis of the data indicates that the zero-frequency kinetic coefficient $\Gamma(T,\omega=0)$ goes to zero algebraically as $T\to T_A^+$. The precise value of the nonuniversal exponent does not seem to be in agreement with that suggested by Götze.[19]

IV. EQUILIBRIUM DESCRIPTION OF THE PG TRANSITION

In this section we use equilibrium statistical mechanics and replica methods to discuss the PG transition for the Hamiltonian given by Eq. (2.2a). Our main aim is to relate the dynamical transitions discussed in Sec. III to equilibrium transition. In order to make the theory as exact as possibly we consider parameter values where the PG transition is continuous or only weakly discontinuous as in Sec. III. It is then possible to obtain the mean-field free energy in terms of an order-parameter expansion which can be consistently truncated at low order (at least near T_c).

The replica trick is used to perform the average over the quenched randomness.[20,8] The average free energy F is given by

$$-\beta F = \lim_{n\to 0} \frac{1}{n}(\bar{Z}^n - 1) . \tag{4.1a}$$

Here n denotes the number of replicas, bar denotes an average over the random interactions, and

$$Z^n = \int \left[\prod_{\alpha=1}^{n}\prod_{a,i} d^\alpha\phi_i^a\right] \exp\left[-\sum_{\alpha=1}^{n} \beta H(^\alpha\phi_i^a)\right] . \tag{4.1b}$$

Here $(\alpha,\beta,\ldots)$ denote replica indices and $H({}^{\alpha}\phi_i^a)$ is the replicated Hamiltonian given by Eq. (2.2a) with an additional replica superscript α. By performing the indicated averages in Eq. (4.1a) the free energy per site for $N\rightarrow\infty$ can be written as

$$\frac{\beta F}{N}=\max\lim_{n\rightarrow 0}\frac{1}{n}G[C,Q]\ . \tag{4.2a}$$

Here max denotes maximum and

$$G[C,Q]=\frac{\mu}{4}\sum_{\alpha=1}^{n}\sum_{a,b=1}^{p-1}[C_{ab}^{\alpha\alpha}]^2+\frac{\mu}{4}\sum_{\alpha\neq\beta}^{n}\sum_{a,b=1}^{p-1}[Q_{ab}^{\alpha\beta}]^2 -\ln\int\left[\prod_{\alpha,a}d\phi_a^\alpha\right]\exp(-H_{\text{eff}})\ , \tag{4.2b}$$

with

$$H_{\text{eff}}=\frac{r_0}{2}\sum_{\alpha=1}^{n}\sum_{a=1}^{p-1}\phi_a^\alpha\phi_a^\alpha+\frac{g_3}{3}\sum_{\alpha=1}^{n}\sum_{a,b,c=1}^{p-1}Q_{abc}\phi_a^\alpha\phi_b^\alpha\phi_c^\alpha +\frac{1}{4}\sum_{\alpha=1}^{n}\sum_{a,b,c=1}^{p-1}T_{abcd}\phi_a^\alpha\phi_b^\alpha\phi_c^\alpha\phi_d^\alpha-\frac{\mu}{2}\sum_{\alpha=1}^{n}\sum_{a,b=1}^{p-1}C_{ab}^{\alpha\alpha}\phi_a^\alpha\phi_b^\alpha-\frac{\mu}{2}\sum_{\alpha\neq\beta}^{n}\sum_{a,b=1}^{p-1}Q_{ab}^{\alpha\beta}\phi_a^\alpha\phi_b^\beta\ . \tag{4.3}$$

Here $C_{ab}^{\alpha\alpha}$ is the single-site equal-time spin correlation function and $Q_{ab}^{\alpha\beta}$ is the usual replica order parameter. In this paper we search for an isotropic PG phase, and we assume $C_{ab}^{\alpha\alpha}$ is independent of the replica index

$$C_{ab}^{\alpha\alpha}\equiv\delta_{ab}C\ , \tag{4.4a}$$

$$Q_{ab}^{\alpha\beta}\equiv\delta_{ab}Q^{\alpha\beta}\ . \tag{4.4b}$$

With this and Eq. (4.3) we obtain

$$G[C,Q]=\frac{\mu n}{4}(p-1)C^2+\frac{\mu}{4}(p-1)\sum_{\alpha\neq\beta}[Q^{\alpha\beta}]^2-\ln\int\left[\prod_{\alpha,a}d\phi_a^\alpha\right]\exp\left[-\sum_{\alpha=1}^{n}H_\alpha+\frac{\mu}{2}\sum_{\alpha\neq\beta}^{n}\sum_{a=1}^{p-1}Q^{\alpha\beta}\phi_a^\alpha\phi_a^\beta\right]\ , \tag{4.5a}$$

where H_α is the single-replica Hamiltonian:

$$H_\alpha=\tfrac{1}{2}(r_0-\mu C)\sum_{a=1}^{p-1}\phi_a^\alpha\phi_a^\alpha+\frac{g_3}{3}\sum_{a,b,c=1}^{p-1}Q_{abc}\phi_a^\alpha\phi_b^\alpha\phi_c^\alpha+\tfrac{1}{4}\sum_{a,b,c,d=1}^{p-1}T_{abcd}\phi_a^\alpha\phi_b^\alpha\phi_c^\alpha\phi_d^\alpha\ . \tag{4.5b}$$

We expand the last term in Eq. (4.5a) in powers of $Q^{\alpha\beta}$. The continuous transition can be characterized by truncating the expansions to third order in $Q^{\alpha\beta}$ while for the discontinuous transition terms up to $[Q^{\alpha\beta}]^4$ are needed. The expression for the free energy when fourth-order terms are included is long and is quoted in the Appendix. Here attention is focused on the continuous transition and thus expanding $G[C,Q]$ up to $O([Q^{\alpha\beta}]^3)$ yields

$$G[C,Q]=\frac{\mu n}{4}(p-1)C^2+\frac{\mu}{4}(p-1)\sum_{\alpha\neq\beta}[Q^{\alpha\beta}]^2-n\ln\int\left[\prod_{a}d\phi_a\right]\exp(-H)-\frac{\mu^2}{4}[G_2]^2(p-1)\sum_{\alpha\neq\beta}[Q^{\alpha\beta}]^2 -\frac{\mu^3}{6}[G_2]^3(p-1)\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}Q^{\alpha\beta}Q^{\beta\gamma}Q^{\gamma\alpha}-\frac{\mu^3}{12}[G_3]^2p^2(p-1)(p-2)\sum_{\alpha\neq\beta}[Q^{\alpha\beta}]^3+O(Q^4)\ . \tag{4.6}$$

Here H is the single-replica Hamiltonian given by Eq. (4.5b) and

$$G_{2ab}=\langle\phi_a\phi_b\rangle\equiv\delta_{ab}G_2\ , \tag{4.7a}$$

$$G_{3abc}=\langle\phi_a\phi_b\phi_c\rangle\equiv Q_{abc}G_3\ , \tag{4.7b}$$

where the angular brackets denote a statistical mechanical average with respect to H_α and the last equalities in Eqs. (4.7) follow from symmetry. It is worth noting that in the Ising spin glass the last term in Eq. (4.6), namely, $[Q^{\alpha\beta}]^3$, does not exist, and this is the critical difference between the behavior of the models. In what follows we evaluate G_3 at the tree level, and in general it vanishes when $p=2$, which is the Ising limit.

Using the free energy in Eq. (4.6), we can now discuss the continuous PG transition, find the condition when the transition is continuous, and compare these results with the dynamical theory presented in Sec. III. Previous work on the lattice PG (Ref. 4) and on the p-spin $(p>2)$ SG model[1,2] suggests that a replica-symmetric solution with nonzero q would be unstable everywhere and consequently cannot be used to locate the transition temperature. These studies also suggest that only one replica-symmetry-breaking ansatz[21] is needed to characterize the continuous PG transition. Consequently, the free energy depends only on two PG parameters: q and the break point $\bar{x}$. The different replicas overlap with strength q, or they do not, and the fraction of replicas that overlap is given by $y=1-\bar{x}$. The parameters q and $\bar{x}$ are determined variationally using Eq. (4.6), with $q(x)=q\theta(x-\bar{x})$, $0\leq\bar{x}\leq 1$. With this Eq. (4.6) becomes

$$\frac{G[C,Q]}{\mu n(p-1)}=\frac{C^2}{4}-\frac{1}{\mu(p-1)}\ln\int\left[\prod_a d\phi_a\right]e^{-H}$$
$$-\frac{1-\bar{x}}{4}q^2(1-\mu[G_2]^2)-\frac{\mu^2}{6}(G_2)^3(1-\bar{x})(2-\bar{x})q^3+\frac{\mu^2}{12}[G_3]^2p^2(p-2)(1-\bar{x})q^3+O(q^4)\ . \tag{4.8}$$

In order to establish a connection between the predictions of the equilibrium and the dynamical theories, we evaluate G_2 to one-loop order and G_3 in a tree approximation:

$$G_3=-2g_3[G_2]^3 \tag{4.9a}$$

and

$$G_2^{-1}=r_0-\mu C+G_2[u_0(p+1)+3f_0p^2]$$
$$-2g_3^2p^2(p-2)G_2^2$$
$$\equiv\bar{r}\ . \tag{4.9b}$$

Note that from Eq. (4.9b) and the variational equation for C, $\partial G/\partial C=0$, it follows that, at T_g,

$$C(T=T_g)=C_{0c}=G_2(T=T_g)=\frac{1}{\bar{r}_{0c}}=\frac{1}{\bar{r}_c}\ , \tag{4.9c}$$

with $\bar{r}_0$ given by Eq. (3.1c). Near T_g the variational equation for q yields

$$0=\left[1-\frac{\mu}{\bar{r}^2}\right]\bar{q}+\bar{q}^2(2-\bar{x}-\lambda_2)\quad(\bar{q}\equiv q/\bar{r}_0)\ , \tag{4.10a}$$

with λ_2 given by Eq. (3.2b). The variational equation for $\bar{x}$ yields (recall that by definition $0\le\bar{x}\le1$)

$$\bar{x}=\lambda_2\ . \tag{4.10b}$$

This in Eq. (4.10a) yields $\bar{q}=0$ or

$$\bar{q}=\frac{1}{2(1-\lambda_2)}\left[\frac{\mu}{\bar{r}^2}-1\right]\ . \tag{4.10c}$$

From these results we conclude the following. (1) There is a continuous PG transition at T_g given by $\mu_c=\bar{r}_{0c}^2$. This is in accord with the dynamical theory. (2) The nature of the PG transition changes at $\lambda_2=1$. This is also in agreement with the dynamical theory (3) $\bar{x}\neq0$ for $\lambda_2\neq0$. A nonzero $\bar{x}$ indicates RSB which is important here at $0(\bar{t}=1-T/T_g)$. For $p=2$, $\bar{x}=\lambda_2=0$, and we recover the usual SK result where there is no RSB ansatz at $O(\bar{t})$.[12,21]

We next consider the discontinuous PG transition which occurs at $\lambda_2=1+\epsilon$, $\epsilon\ll1$. For this case the discontinuity at T_c is of $O(\epsilon)$ and the q^4 terms in Eq. (4.8) are needed. From Eqs. (A1), (A2), and (A3) the free energy to $O(q^4)$ is given by the maximum of

$$\frac{G[C,Q]}{\mu n(p-1)}=\frac{C^2}{4}-\frac{1}{\mu(p-1)}\ln\int\left[\prod_a d\phi_a\right]\bar{e}^{H}-\frac{(1-\bar{x})}{4}q^2\left[1-\frac{\mu}{\bar{r}^2}\right]-\frac{\mu^2}{6\bar{r}^3}(1-\bar{x})q^3\left[2-\bar{x}-\frac{2g_3^2}{\bar{r}^3}p^2(p-2)\right]$$
$$+\frac{\mu^3q^4}{8\bar{r}^4}(1-\bar{x})\left[\bar{x}^2-3\bar{x}+3-(2-\bar{x})\frac{4g_3^2}{\bar{r}^3}p^2(p-2)\right]+O(q^5)\ . \tag{4.11}$$

At $T_c(=T_A$ or T_K, cf. below) the variational equation for q for $\epsilon\ll1$ is

$$0=(1-\lambda_1)\bar{q}+\bar{q}^2(2-\bar{x}-\lambda_2)$$
$$-\bar{q}^3[\bar{x}^2-3\bar{x}+3-(2-\bar{x})2\lambda_2] \tag{4.12a}$$

or

$$\bar{q}^3(1+\bar{x}-\bar{x}^2)+\bar{q}^2(1-\bar{x}-\epsilon)+\Delta\bar{q}=0\ . \tag{4.12b}$$

At T_c, $\bar{x}(T=T_c)=1$ since the free energies in the paramagnetic (PM) phase must equal the PG free energy. The critical $q(\equiv q_c)$ therefore satisfies

$$\bar{q}^3-\epsilon\bar{q}^2+\Delta\bar{q}=0\ . \tag{4.12c}$$

This result is identical to Eq. (3.6b) from the dynamical theory. The variational equation for $\bar{x}$ is

$$1-\bar{x}=\frac{3}{\bar{q}}\left[\frac{\epsilon\bar{q}}{6}-\frac{\Delta}{4}-\frac{\bar{q}^2}{8}\right]\quad(0\le\bar{x}\le1)\ . \tag{4.13}$$

With Eqs. (4.12c) and (4.13) we next discuss two distinct PG transitions. As already noted, Eq. (4.12c) is identical to Eq. (3.6b). Therefore Eq. (4.12c) first has a physical solution ($\bar{q}>0$ and real) at a transition temperature given by Eq. (3.7a) and the critical order parameter is given by Eq. (3.7b). $\bar{x}(T=T_A)=1$ so that $F_{\rm PG}=F_{\rm PM}$ as is required. For the discontinuous transition we have therefore established a *precise* connection with the dynamical theory. We next note the important fact that with $\bar{x}=1$ and Eqs. (3.7) Eq. (4.13) is not satisfied. Further, Eqs. (3.7) in Eq. (4.13) lead to an unphysical negative value of $1-\bar{x}$. This situation is identical to that obtained before for the p-spin ($p>2$) SG model: The dynamical transition corresponds to a transition where $\bar{q}$ is given by Eq. (4.12c) but where the variational equation for $\bar{x}$ is not satisfied and $\bar{x}$ is fixed to be at its physical end point $\bar{x}(T=T_A)=1$. Several comments are in order here. First note that Eqs. (4.12) have been obtained by dividing out a common factor of $1-\bar{x}$. For $\bar{x}=1$ this step is problematic. Nevertheless, Eq. (4.12c) is identical to the equation of state obtained from the dynamical theory. Physically, $\bar{x}=1$ indicates there is no replica overlap and that the order parameter describing the transition at T_A is $Q^{\alpha\alpha}$ which does not appear in Eqs. (4.2).[8] In Ref. 5 the equation of state given by Eq. (4.12c) is obtained from the TAP approach.[8] There it is also shown that the transition at T_A which is given by the mean-field dynamical theory represents freezing into a thermodynamically metastable state that has an infinite lifetime in mean-field theory. Here we have shown that T_A can also be obtained from usual equilibrium theory with some caveats. This point is discussed in more detail elsewhere.[5]

The true equilibrium PG transition occurs when both Eqs. (4.12c) and (4.13) are first satisfied. Denoting the critical parameters for this transition by primes and this equilibrium transition temperature by T_K one obtains

$$\Delta \to \Delta_c' = \tfrac{2}{9}\epsilon^2 \quad \text{or} \quad \frac{\mu_c'}{\bar{r}_{0c}^2} = 1 - \frac{2\epsilon^2}{9} \tag{4.14a}$$

and

$$\bar{q} \to \bar{q}_c' = \tfrac{2}{3}\epsilon \ . \tag{4.14b}$$

Note that $T_A > T_K$ and $\bar{q}_c < \bar{q}_c'$ as for the p-spin ($p>2$) SG model. The free energy for $T<T_K$ is

$$\frac{\beta F}{N\mu(p-1)} = \frac{C^2}{4} - \frac{1}{\mu(p-1)} \ln \int \left[\prod_a d\phi_a\right] e^{-\beta H} + \frac{(1-\bar{x})^2 \bar{q}^3}{6\bar{r}^2} , \tag{4.15}$$

with $1-\bar{x} \sim \bar{t}' = 1 - T/T_K$. As in the usual SG transitions $F_{\rm PG} > F_{\rm PM}$. From Eq. (4.15), it follows that there is no latent heat at T_K and that the specific heat is discontinuous at T_K. In Ref. 5 T_K is interpreted as a Kauzmann temperature where a configurational entropy vanishes.

In Ref. 1 a more complete discussion on the nature of these types of transitions for the p-spin interaction SG model has been given. The same comments also apply here. Finally, we remark that a complete replica-stability analysis[22] proves that the PG phases considered here are locally stable for T not too far from T_c.

V. DISCUSSION

We conclude this paper with the following remarks.

(1) Here we have gone to some length to establish the connections between the dynamical approach and the static approach. We have also been careful to treat only weakly discontinuous transitions where we have some control on our perturbative results. These two points are important. It is easy to define a field theory, do a one-loop approximation, and conclude from the dynamical approach that freezing takes place. However, it is also easy to construct field theories where such a conclusion is wrong. For example, in Sec. III if we set $\mu=0$ then we would still predict a (strongly) discontinuous freezing at a critical value of λ_2. However, the exact field theory is then a noncritical regular single-site Potts model which clearly does not undergo any phase transition. This point has been raised before by Siggia[23] in a different context.

A legitimate theory for both the spin-glass problem and the structural glass problem should be able to treat the dynamical and equilibrium properties on equal footing.[24,25] Even in the case of a kinetic-glass transition, where the system gets stuck in a very long-lived metastable state, we believe that this state should be describable by a static theory (at least in the sense of analytic continuation).

(2) In general it is known that in the PG model there are additional transitions at lower temperatures.[2,4] It would be interesting to physically interpret these additional transitions. In many glasses, i.e., structural glasses[7,26,27] and it has been suggested for KBr-KCN mixed crystals,[28] there are multiple transitions which are due to the freezing of different degrees of freedom.

(3) It is interesting to note that the Parisi-type solutions[21] of the PG as well as the p-spin ($p>2$) SG model is considerably simplier than the corresponding solution of the SK model.[11] Because of this simplification it should be easy to calculate[22] fluctuation effects around the upper critical dimension[11] ($d=6$, and for some quantities $d=8$) in the ordered phase of the PG. For these models it should be possible to explicitly determine whether or not a Parisi-type solution can exist for lower dimensions.[29]

For the case of a continuous PG transition a renormalization-group calculation for the dynamical theory would also be of interest.[22] It is possible that below some dimension fluctuations tend to make the dynamical exponents in Sec. III universal.[30] It has already been argued[31] that the nonuniversal dividing line between continuous and discontinuous PG transitions becomes universal below six dimensions.

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APPENDIX

In this appendix we quote some results which are used in the text. With one RSB ansatz and to $O(q^4)$ the free energy is given by Eq. (4.2a) with

$$G[C,Q] = \frac{\mu n}{4}(p-1)C^2 - \frac{\mu}{4}(p-1)(1-\bar{x})q^2 - n \ln \int \left[\prod_a d\phi_a\right] e^{-H} + \frac{n}{4}(1-\bar{x})(p-1)(\mu q)^2 G_2^2$$
$$+ \frac{n}{12}(1-\bar{x})(\mu q)^3 \sum_{abc} [G_{3abc}]^2 - \frac{n}{6}(1-\bar{x})(2-\bar{x})(p-1)(\mu q)^3 G_2^3 + \frac{n}{48}(1-\bar{x})(\mu q)^4 \sum_{abcd} [G_{4abcd}]^2$$

$$-\frac{n}{8}(1-\bar{x})(2-\bar{x})(\mu q)^4 G_2 \sum_{abc} [G_{3abc}]^2 - \frac{n}{8}(1-\bar{x})(2-\bar{x})(\mu q)^4 G_2^2 \sum_{ab} G_{4aabb}$$

$$+\frac{n}{32}(1-\bar{x})(2-\bar{x})(3-\bar{x})(\mu q)^4 G_2^4 (p-1)[1+4(p-1)] + \frac{n\bar{x}}{32}(1-\bar{x})^2(\mu q)^4 G_2^4 (p-1)^2 + O(q^5) . \quad \text{(A1)}$$

Here G_2 and G_{3abc} are given by Eqs. (4.7) and

$$G_{4abcd} = \langle \phi_a \phi_b \phi_c \phi_d \rangle . \quad \text{(A2)}$$

To obtain Eq. (4.11) we have used Eqs. (4.9a), (4.9b), and

$$G_{4abcd} \simeq G_2^3(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) . \quad \text{(A3)}$$

It is straightforward to establish that the tree corrections to Eq. (A3) do not affect our results if T is near T_A and to $O(\epsilon)$.

[1]T. R. Kirkpatrick and D. Thirumalai, Phys. Rev. Lett. **58**, 2091 (1987); Phys. Rev. B **36**, 5388 (1987).

[2]For an earlier work focusing on the static properties of p-spin model, see E. Gardner, Nucl. Phys. **B257** [FS14], 747 (1985).

[3]D. Gross and M. Mezard, Nucl. Phys. **B240**, 431 (1984).

[4]D. Gross, I. Kantor, and H. Sompolinsky, Phys. Rev. Lett. **55**, 304 (1985).

[5]T. R. Kirkpatrick and P. G. Wolynes, Phys. Rev. B **36**, 8552 (1987).

[6]D. Thouless, P. W. Anderson, and R. G. Palmer, Philos. Mag. **35**, 593 (1977).

[7]See, for example, J. Jackle, Rep. Prog. Phys. **49**, 171 (1986), and references therein.

[8]D. Elderfield and D. Sherrington, J. Phys. C **16**, 497 (1983); **16**, L971 (1983); **16**, L1169 (1983); D. Elderfield, J. Phys. A **17**, L517 (1984).

[9]E. J. S. Lage and A. Erzan, J. Phys. C **16**, 873 (1983); E. J. S. Lage and J. M. Nunes da Silva, *ibid.* **17**, L593 (1984).

[10]For a recent review of spin glasses, see K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986).

[11]For a review of the regular Potts model, see F. Y. Wu, Rev. Mod. Phys. **54**, 235 (1982).

[12]M. Chang, Phys. Rev. B **28**, 6592 (1983).

[13]R. K. P. Zia and D. J. Wallace, J. Phys. A **8**, 1495 (1975).

[14]C. De Dominicis, J. Phys. C **1**, 2475 (1976); C. De Dominicis and L. Pelit, Phys. Rev. B **18**, 353 (1978); C. De Dominicis, *ibid.* **18**, 353 (1978).

[15]H. K. Janssen, Z. Phys. B **23**, 377 (1976); R. Bausch, H. K. Janssen, and H. Wagner, *ibid.* **24**, 113 (1976).

[16]H. Sompolinsky and A. Zippelius, Phys. Rev. B **25**, 6860 (1982); J. A. Hertz and R. A. Klemm, *ibid.* **20**, 316 (1979).

[17]D. J. Amit and D. V. I. Roginsky, J. Phys. A **12**, 689 (1979).

[18]A. Houghton, S. Jain, and A. P. Young, Phys. Rev. B **28**, 2630 (1983). For a related discussion in another context see the appendices in W. Götze and L. Sjogren, J. Phys. C **17**, 5759 (1984).

[19]W. Götze, Z. Phys. B **56**, 139 (1984).

[20]S. F. Edwards and P. W. Anderson, J. Phys. F **5**, 965 (1975).

[21]The replica-symmetry-breaking ansatz due to Parisi is used to discuss the equilibrium transitions. See G. Pairis, Phys. Rev. Lett. **43**, 1754 (1979); J. Phys. A **13**, L115 (1980); **13**, 1101 (1980); **13**, 1887 (1980).

[22]T. R. Kirkpatrick and D. Thirumalai (unpublished).

[23]E. D. Siggia, Phys. Rev. A **32**, 3135 (1985).

[24]T. R. Kirkpatrick and D. Thirumalai (unpublished).

[25]T. R. Kirkpatrick and P. G. Wolynes, Phys. Rev. A **35**, 3072 (1987).

[26]For a theoretical discussion of these transitions, see W. Götze and L. Sjogren, Z. Phys. B **65**, 415 (1987).

[27]J. Wong and C. A. Angell, *Glass Structure by Spectroscopy* (Dekker, New York, 1976).

[28]K. H. Michel and J. M. Rowe, Phys. Rev. B **22**, 1417 (1980); A. Loidl, K. Knorr, R. Feile, and J. K. Kyems, Phys. Rev. Lett. **51**, 1954 (1984).

[29]A. J. Bray and M. A. Moore, J. Phys. C **17**, L613 (1984); D. S. Fisher and D. A. Huse, Phys. Rev. Lett. **56**, 1601 (1986); A. Bovier and J. Frohlich, J. Stat. Phys. **44**, 347 (1986).

[30]D. S. Fisher and H. Sompolinsky, Phys. Rev. Lett. **54**, 1063 (1985).

[31]Y. Y. Goldschmidt, Phys. Rev. B **31**, 4369 (1985).