Exact solutions for Ising-model even-number correlations on planar lattices

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A systematic and unifying method is developed and demonstrated for obtaining exact solutions of n-site (n even integer) Ising correlations on various planar lattices. The scheme, which is exceedingly more simple than using solely traditional Pfaffian techniques, embodies five mapping theorems in alliance with algebraic correlation identities. In the theoretical framework, the triangular Ising model plays an overarching role. In particular, considering a select 7-site cluster of the triangular Ising model, the knowledge of all its 11 even-number correlations defined upon this cluster (where only four of the correlations need to be actually calculated by Pfaffian procedures) is shown to be sufficient for determining exactly all honeycomb, decorated-honeycomb, and kagomé Ising model even-number correlations upon their correspondingly select 10-site, 19-site, and 9-site clusters, respectively. The relative ease and direct applicability of the present approach are highlighted not only by the resulting large numbers of n-site (n even-integer) correlation solutions (e.g., approximately 85 and 50 for the honeycomb and kagomé Ising models, respectively) and their large $n_{\max}$ values ($n_{\max}=8,10,18$ for the kagomé, honeycomb, and decorated-honeycomb Ising models, respectively) but also by the realization that the exact solutions for Ising multisite correlations upon the kagomé lattice (one of the four regular lattices in two dimensions) are apparently the first to explicitly appear in the literature beyond its nearest-neighbor pair correlation (energy).

I. INTRODUCTION

The study of correlation behavior among the various degrees of freedom comprising an interacting many-body system in thermodynamic equilibrium has leading importance for the basic understanding of cooperative effects exhibited by such systems. Since correlation functions are structured using thermal expectation values of *products* of localized variables, they clearly offer a more detailed description than thermodynamic for the order and symmetry present in the system. The impetus for the modern synoptic view of critical phenomena was, in fact, the recognition of the fundamental role the anomalously long-ranged spatial correlations played near a critical point resulting in scaling theories and the renormalization-group approach towards problems in phase transitions and particle physics. Besides examining the asymptotically large-distance behavior of correlation functions, it is also desirable to obtain solutions for more spatially compact, short-distance-type correlations. These smaller-scale correlations have varied applications (at criticality and otherwise) with Ising-type examples being found in the analyses of local equilibrium properties in the vicinity of isolated defects,[1] in the theory of both transport coefficients[2] and thermodynamic response functions,[3] in the investigations of inelastic neutron scattering,[4] percolation phenomena,[5] and many other problems.

The theoretical task of explicitly evaluating correlations and more completely understanding their physical consequences has been especially aided through the years by careful examinations upon the Ising model which yet retains its hallmark importance while continuing to provide definite guidance and insights on the many fascinating questions and facets of phase transitions and critical phenomena. Actually, the two-dimensional Ising model in a zero magnetic field is the only realistic microscopic model of cooperative phenomena for which correlation solutions have been found exactly. Planar Ising-model correlations have traditionally been calculated largely through direct use of Pfaffian techniques[6] with exact and explicit multisite correlation solutions, therefore, being restricted in practice to small even numbers of closely neighboring lattice sites. In more recent times, fruitful connections with quantum field theory have enabled multisite correlations to be computed at criticality for planar Ising models in the continuum limit approximation to the transfer matrix resulting in a one-dimensional fermion field theory.[7] Attention should also be directed towards the work of McCoy, Tracy, and Wu[8] who have obtained exact solutions for arbitrary n-site Ising correlations on the square lattice where the integral forms of the solutions are convenient for ascertaining their large-distance behavior, and to the work of Bariev[9] who has derived exact solutions for planar Ising-model correlations making use of Kadanoff's hierarchy of local Ising operators.[10] Even more currently, focusing primarily upon the pair correlations of the square Ising model in a zero magnetic field ("Onsager lattice"), various authors have extended our knowledge and methodology of exact solutions for Ising-model correlations. In particular, Au-Yang and Perk[11] have demonstrated that the values of the pair correlations at criticality can be conveniently obtained from the quadratic difference equations of Hirota's Toda lattice form, Ghosh and Shrock[12] have developed and analyzed exact and explicit solutions for various short- and intermediate-range spin-spin correlations (diagonal, off axis, and row) in terms of elliptic integrals, while Yamada[13] expressed the pair correlation in a simple determinantal form called a "generalized Wronskian." Also, applied to Ising correlations near a critical point, the theory of conformal invariance has provided new perspectives and results where the reader is referred to a recent review article by Cardy.[14]

The present paper pursues yet another approach. One first proves five extended transformation theorems (extended in the sense that the theorems apply beyond partition functions to multisite correlations) which successively map unknown planar Ising-multisite correlations upon linear combinations of those Ising correlations already known on other planar lattices. The theorems are direct and systematic in their application, and in alliance with algebraic correlation identities, the method demonstrates that only a few localized Ising correlations on the triangular lattice actually need to be calculated by Pfaffian procedures in order to obtain large numbers of exact solutions for localized Ising correlations upon the honeycomb and kagomé lattices as well as upon other irregular (bond-decorated) planar lattices. One emphasizes that the present approach is exceedingly more simple than using solely Pfaffian procedures which are known to become progressively lengthy and arduous as either the number of sites under consideration or the distances between these sites increase. Among the results of the present investigations is the presumably first finding of exact and explicit solutions for Ising-multisite correlations upon the kagomé lattice which is one of the four regular lattices in two dimensions, and the finding of exact solutions for n-site (n even integer) localized correlations where the maximal values of n are considerably larger than existing literature values, e.g., in the present work, $n_{\max}=8$, 10, 18 for the kagomé, honeycomb, and decorated-honeycomb Ising models, respectively.

The literature on the Ising model[15] and its applications is very extensive. The model has been used not only to represent certain kinds of highly anisotropic magnetic crystals but also as an extremely useful paradigm for various other physical systems, e.g., as a lattice model for fluids, alloys, adsorbed monolayers, for biological and chemical systems, and even in field theories of elementary particles (lattice gauge theories describing the quark structure of hadrons). As a general rule, in all such applications, a more complete understanding of thermostatistical behavior requires information on relevant Ising-correlation solutions which forms a major motivation for the present studies. One also remarks that, since an Ising model is defined upon a discrete lattice and Ising variables are dichotomic (± 1), such types of models considered upon *large* but *finite* lattices are particularly suitable for Monte Carlo calculations and numerical investigations using special-purpose computers designed for the simulation of Ising models. Finally, along with unifying seemingly diverse areas of research, the distinctive pedagogical merits of the Ising model enable it to be aptly acknowledged a veritable "daystar" in the realm of statistical physics as students and teachers alike of the subject can assuredly attest.

II. BASIC GENERATING EQUATIONS FOR ISING-CORRELATION IDENTITIES ON THE HONEYCOMB AND TRIANGULAR LATTICES

Consider initially the honeycomb lattice structure (periodic array of regular hexagons) shown in Fig. 1. One

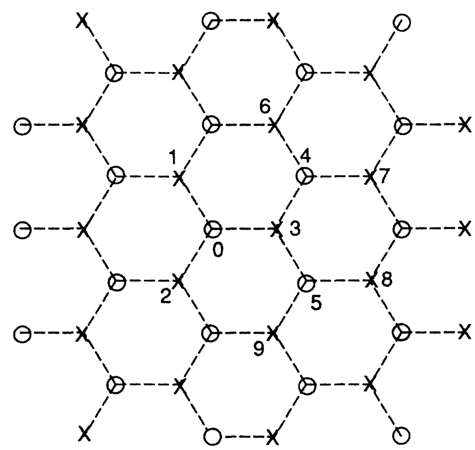

FIG. 1. The honeycomb lattice and its decomposition into two interlacing sublattices ("circled" ○ sites and "crossed" × sites, respectively) where each sublattice is triangular. Ten honeycomb sites are specifically enumerated for use throughout the paper.

defines the honeycomb Ising-model ferromagnet on such a lattice of N_h sites as the (dimensionless) Hamiltonian

$$\mathcal{H}_h=-K\sum_{\langle i,j\rangle}\sigma_i\sigma_j\ , \tag{2.1}$$

where each site-localized Ising variable $\sigma_r=\pm 1$, $\sum_{\langle i,j\rangle}$ designates the summation over all distinct nearest-neighbor pairs of lattice sites, and $K>0$ is the (dimensionless) strength parameter of the ferromagnetic interaction. Letting the set of all Ising variables $\{\sigma_0,\sigma_1,\ldots,\sigma_{N_h-1}\}\equiv\sigma$, the magnetic canonical partition function Z_h is given by the usual trace formula over all degrees of freedom of the system:

$$Z_h=\mathrm{Tr}_\sigma e^{-\mathcal{H}_h}\ . \tag{2.2}$$

A class of correlation identities considered in the present paper is a set of linear algebraic equations with coefficients dependent only upon the interaction parameters.[16] To develop such identities systematically, one now proceeds to derive their basic generating equation.[17] Let $[f]$ be any function of the honeycomb Ising variables $\sigma_1,\sigma_2,\ldots,\sigma_{N_h-1}$ (*excluding* σ_0, the origin-site variable in Fig. 1). Similarly letting $\mathcal{H}'_h,\mathrm{Tr}'_\sigma$ denote a *restricted* (dimensionless) Hamiltonian and trace operation, respectively, which *exclude* σ_0, one can construct the canonical thermal average $\langle\sigma_0[f]\rangle$ as

$$\begin{aligned}Z_h\langle\sigma_0[f]\rangle&=\mathrm{Tr}_\sigma e^{-\mathcal{H}_h}\sigma_0[f]\\&=\mathrm{Tr}_\sigma e^{-\mathcal{H}'_h+K\sigma_0(\sigma_1+\sigma_2+\sigma_3)}\sigma_0[f]\\&=\mathrm{Tr}'_\sigma e^{-\mathcal{H}'_h}[f]\mathrm{Tr}_{\sigma_0}e^{K\sigma_0(\sigma_1+\sigma_2+\sigma_3)}\sigma_0\\&=\mathrm{Tr}_\sigma e^{-\mathcal{H}_h}[f]\frac{\mathrm{Tr}_{\sigma_0}e^{K\sigma_0(\sigma_1+\sigma_2+\sigma_3)}\sigma_0}{\mathrm{Tr}_{\sigma_0}e^{K\sigma_0(\sigma_1+\sigma_2+\sigma_3)}}\end{aligned} \tag{2.3}$$

thereby yielding

$$\langle \sigma_0[f]\rangle = \left\langle [f]\frac{\mathrm{Tr}_{\sigma_0} e^{K\sigma_0(\sigma_1+\sigma_2+\sigma_3)}}{\mathrm{Tr}_{\sigma_0} e^{K\sigma_0(\sigma_1+\sigma_2+\sigma_3)}}\right\rangle , \quad \sigma_0 \notin [f] \tag{2.4}$$

having written the standard definition of canonical thermal average (2.3). To further develop expression (2.4), one has that

$$\frac{\mathrm{Tr}_{\sigma_0} e^{K\sigma_0(\sigma_1+\sigma_2+\sigma_3)}\sigma_0}{\mathrm{Tr}_{\sigma_0} e^{K\sigma_0(\sigma_1+\sigma_2+\sigma_3)}} = \tanh K(\sigma_1+\sigma_2+\sigma_3) = A(\sigma_1+\sigma_2+\sigma_3) + B\sigma_1\sigma_2\sigma_3 \tag{2.5}$$

where

$$A = \tfrac{1}{4}(\tanh 3K + \tanh K), \quad B = \tfrac{1}{4}(\tanh 3K - 3\tanh K) \tag{2.6}$$

use having been made of the fact that any Ising variable σ_l satisfies $\sigma_l^{2n+1}=\sigma_l$, $\sigma_l^{2n}=1$, $n=0,1,2,\ldots$. Substituting (2.5) into (2.4), one obtains

$$\langle \sigma_0[f]\rangle = A\langle [f](\sigma_1+\sigma_2+\sigma_3)\rangle + B\langle [f]\sigma_1\sigma_2\sigma_3\rangle, \quad \sigma_0\notin[f] . \tag{2.7}$$

One can consider the case of a triangular lattice (× sites, say, in Fig. 1) in a similar fashion. The triangular Ising model ferromagnet is defined upon such a lattice of N_t sites as the (dimensionless) Hamiltonian

$$\mathcal{H}_t = -R\sum_{\langle k,l\rangle}\sigma_k\sigma_l , \tag{2.8}$$

where again each site-localized Ising variable $\sigma_q=\pm 1$, $\sum_{\langle k,l\rangle}\cdots$ denotes summation over all distinct nearest-neighbor pairs of lattice sites, and $R>0$ is the (dimensionless) strength parameter of the ferromagnetic interaction. The magnetic canonical partition function Z_t of the triangular Ising model is now defined by

$$Z_t = \mathrm{Tr}_\times e^{-\mathcal{H}_t} , \tag{2.9}$$

where the notation $\mathrm{Tr}_\times$ signifies that the trace operation is taken over the degrees of freedom of all N_t ×-site Ising variables. In Fig. 1, calling site 3 the origin site, the sites 1, 2, 6, 7, 8, 9 then become its six nearest-neighboring sites. Using the same form of derivation as previously, i.e., descriptively, due to the commuting nature of classical Ising variables, one is permitted to "split, rearrange, then reconstitute" quantities inside the trace operation, and once again utilizing the dichotomic (±1) nature of Ising variables, one obtains

$$\begin{aligned}\langle 3[g]\rangle = {} & C\langle (1+2+6+7+8+9)[g]\rangle \\ & + D\langle (126+127+128+129+167+168+169+178+179+189+267+268+269+278+279+289 \\ & \quad +678+679+689+789)[g]\rangle \\ & + E\langle (12678+12679+12689+12789+16789+26789)[g]\rangle, \quad 3\notin[g] ,\end{aligned} \tag{2.10}$$

where, for notational simplicity, only the numeric site labels of the Ising variables are entered, and where the coefficients are given by

$$\begin{aligned} C &= \tfrac{1}{32}(\tanh 6R + 4\tanh 4R + 5\tanh 2R) , \\ D &= \tfrac{1}{32}(\tanh 6R - 3\tanh 2R) , \\ E &= \tfrac{1}{32}(\tanh 6R - 4\tanh 4R + 5\tanh 2R) . \end{aligned} \tag{2.11}$$

Equations (2.7) and (2.10) are the *basic generating equations* for developing linear algebraic identities among Ising-multisite correlations upon the honeycomb and triangular lattice structures, respectively.

III. STAR-TRIANGLE-TYPE RELATIONSHIPS AND HONEYCOMB ISING CORRELATIONS AS LINEAR COMBINATIONS OF TRIANGULAR ISING CORRELATIONS

The star-triangle (Y-Δ) transformation[18] (see Fig. 2) is due to Onsager and relates the honeycomb and triangular Ising models by showing that their magnetic canonical partition functions $Z_{h,K}$ and $Z_{t,R}$, respectively, differ only by a known multiplicative constant, i.e.,

$$Z_{h,K} = \Delta^{N_h/2} Z_{t,R} , \tag{3.1}$$

where

$$\Delta^4 = e^{4R}(e^{4R}+3)^2 , \tag{3.2}$$

$N_h(=2N_t)$ is the total number of honeycomb lattice sites, and

$$2\cosh 2K = e^{4R}+1 \tag{3.3}$$

relates the (dimensionless) interaction parameters K,R of the honeycomb and triangular lattices, respectively. For emphases, writing corresponding (dimensionless) interac-

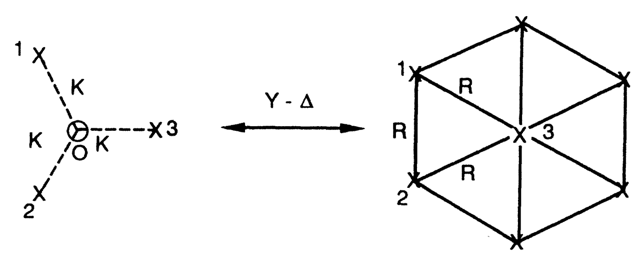

FIG. 2. The star-triangle (Y-Δ) transformation equates, aside from a known multiplicative constant, the honeycomb and triangular Ising-model partition functions $Z_{h,K}$, $Z_{t,R}$, respectively, where their (dimensionless) interaction parameters K, R are simply related.

tion parameters explicitly as subscripts on Hamiltonians, partition functions, and thermal averages will be a useful and frequent notation in this and following sections. Using the site enumerations in Fig. 1, the Y-Δ transformation (3.1) is based upon the realization that the partial trace evaluation

$$\mathrm{Tr}_{\sigma_0} e^{K\sigma_0(\sigma_1+\sigma_2+\sigma_3)} = \Delta e^{R(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3)} \tag{3.4}$$

is similarly valid for each and every "circled" σ variable appearing within the total trace operation (2.2) defining $Z_{h,K}$.

Using these correspondence concepts and making convenient reference to Fig. 1, one can straightforwardly prove the following theorem connecting honeycomb and triangular Ising-model select correlations.

Theorem 1. Consider the honeycomb lattice decomposition depicted in Fig. 1, and let $[r]$ be any function of Ising variables containing *only* "crossed" $\times$ sites (or *only* "circled" $\circ$ sites). Then

$$\langle [r] \rangle_{h,K} = \langle [r] \rangle_{t,R} ,$$

where $\langle [r] \rangle_{h,K}$ and $\langle [r] \rangle_{t,R}$ denote canonical thermal averages pertaining to the honeycomb and triangular Ising-model (dimensionless) Hamiltonians $\mathcal{H}_{h,K}$ and $\mathcal{H}_{t,R}$, respectively, with their (dimensionless) interaction parameters K and R related by $2\cosh 2K = e^{4R}+1$.

Proof. For definiteness, let $[r]$ be any function of Ising variables containing solely "crossed" $\times$ sites, and adopt the notation $\mathrm{Tr}_{\times}$ to signify the partial-trace operation over the degrees of freedom associated with all the "crossed" $\times$ sites of the honeycomb lattice, and $\mathrm{Tr}_{\circ,\times}$ to signify the trace operation over the degrees of freedom associated with the totality of honeycomb sites. Then,

$$\langle [r] \rangle_{h,K} = \frac{\mathrm{Tr}_{\circ,\times}[r] e^{-\mathcal{H}_{h,K}}}{Z_{h,K}} = \frac{\Delta^{N_h/2}\mathrm{Tr}_{\times}[r] e^{-\mathcal{H}_{t,R}}}{\Delta^{N_h/2} Z_{t,R}} = \langle [r] \rangle_{t,R} , \tag{3.5}$$

where use has been made of (3.1) and the relationship (3.4) for every "circled" $\circ$ site of the honeycomb lattice. Noting that a parallel line of argument can be similarly constructed for the case where $[r]$ is any function of Ising variables comprising solely "circled" $\circ$ sites, this completes the proof of the theorem.

The above registry theorem will have significant consequences throughout the remainder of the investigation and analysis of Ising-multisite correlations on planar lattices. In general, any honeycomb lattice thermal average associated with a configuration of sites which are in registry with the sites of the triangular lattice can now be equated to the corresponding triangular lattice thermal average, and vice versa, in the sense of the Y-Δ relationships. It should thus be clear that the thermal-average symbol for any correlation that satisfies the registry Theorem 1 does not actually require subscripts $\langle \cdots \rangle_{h,K}$ or $\langle \cdots \rangle_{t,R}$ since either context is correct.

As an extension, the next theorem, which utilizes the previous Theorem 1, will enable *any* honeycomb Ising correlation to be systematically developed into a linear combination of triangular Ising correlations.

Theorem 2. Any honeycomb Ising-model correlation can be represented as a linear combination of triangular Ising-model correlations.

Proof. Since Ising variables are classical and, therefore, commute, one can rearrange as desired the ordering of any product of Ising variables within a thermal-average symbol thus conveniently enabling an arbitrary choice of the origin site in the basic generating equation (2.7) for honeycomb Ising-correlation identities. Prescriptively, therefore, all "circled" $\circ$ sites can be systematically eliminated from within any honeycomb Ising-model thermal-average symbol through repeated application of the basic generating equation (2.7) whereupon every ensuing honeycomb Ising correlation then directly corresponds by Theorem 1 to some individual triangular Ising correlation. This completes the proof of the theorem.

As an example which illustrates the above simple superposition procedure, considering the following four-site correlation, one finds that

$$\begin{aligned}\langle 0349 \rangle_{h,K} &= \langle 0439 \rangle_{h,K} \\ &= \langle [A(1+2+3)+B123][A(3+6+7)+B367]39 \rangle_{h,K} \\ &= A^2\langle 19+29+39+69+79+1369+1379+2369+2379 \rangle_{h,K} \\ &\quad + AB\langle 1239+1269+1279+1679+2679+3679 \rangle_{h,K} + B^2\langle 123679 \rangle_{h,K} \\ &= A^2\langle 19+29+39+69+79+1369+1379+2369+2379 \rangle_{t,R} \\ &\quad + AB\langle 1239+1269+1279+1679+2679+3679 \rangle_{t,R} + B^2\langle 123679 \rangle_{t,R} ,\end{aligned} \tag{3.6}$$

where the coefficients A,B are given by (2.6) and, for convenience of notation in (3.6), one has written in an obvious fashion only the numeric site labels within the thermal-average symbols (see again Fig. 1). Of course, recognizing the symmetry of the triangular lattice, expression (3.6) can be further simplified by equating geometrically equivalent correlations, e.g.,

$$\langle 29 \rangle_{t,R} = \langle 39 \rangle_{t,R}, \quad \langle 19 \rangle_{t,R} = \langle 79 \rangle_{t,R} ,$$

$$\langle 1369 \rangle_{t,R} = \langle 2369 \rangle_{t,R} = \langle 2379 \rangle_{t,R} = \langle 3679 \rangle_{t,R} ,$$

and so forth.

IV. KAGOMÉ ISING CORRELATIONS AS LINEAR COMBINATIONS OF HONEYCOMB ISING CORRELATIONS

The kagomé lattice (Japanese woven bamboo pattern) is a periodic array of equilateral triangles and regular hexagons (see Fig. 3) thus also called the 3–6 lattice. The lattice is regular (all sites equivalent, all bonds equivalent) and may be termed "close packed" since it contains elementary polygons having an odd number of sides (viz., triangles). One recognizes that the kagomé lattice has the same coordination number 4 as the square lattice, the latter being "loose packed." One defines the kagomé Ising-model ferromagnet on such a lattice of N_k sites as the (dimensionless) Hamiltonian

$$\mathcal{H}_k = -Q \sum_{\langle m,n \rangle} \mu_m \mu_n \ , \tag{4.1}$$

where each site-localized Ising variable $\mu_s = \pm 1$, $\sum_{\langle m,n \rangle} \cdots$ designates summations over all distinct nearest-neighbor pairs of lattice sites, and $Q > 0$ is the (dimensionless) strength parameter of the ferromagnetic interaction. Letting the set of all Ising variables

$$\{\mu_0, \mu_1, \ldots, \mu_{N_k - 1}\} \equiv \mu \ ,$$

the magnetic canonical partition function Z_k is given as usual by

$$Z_k = \mathrm{Tr}_{\mu} e^{-\mathcal{H}_k} \ . \tag{4.2}$$

Towards proving that kagomé Ising correlations can be mapped upon linear combinations of honeycomb Ising correlations, one first introduces the *decorated-honeycomb* lattice which is the lattice formed by the previous honeycomb lattice supplemented with lattice points at the centers of all bonds (see Fig. 4). The resulting bond-decorated lattice is irregular since all sites are no longer equivalent. The decorated-honeycomb Ising-model ferromagnet is then defined by the (dimensionless) Hamiltonian

$$\mathcal{H}_d = -L \sum_{\langle p,q \rangle} \sigma_p \mu_q \ , \tag{4.3}$$

where σ_p, μ_q are Ising variables localized on an original honeycomb site p and "solid-circled" decoration site q, respectively, $\sum_{\langle p,q \rangle} \cdots$ designates summation over all distinct nearest-neighbor pairs of lattice sites, and $L > 0$ is the (dimensionless) strength parameter of the ferromagnetic interaction. The magnetic canonical partition function Z_d is given again by the standard trace formula

$$Z_d = \mathrm{Tr}_{\mu,\sigma} e^{-\mathcal{H}_d} \ . \tag{4.4}$$

Small portions of the honeycomb, decorated-honeycomb, and kagomé Ising models are depicted in Fig. 5, and these models are connected by the decoration-iteration (I) and star-triangle (Y-Δ) transformations,[18] respectively. More specifically, $Z_{d,L}$ can be connected to $Z_{k,Q}$ by a star-triangle (Y-Δ) transformation upon all σ variables, whereupon the remaining "solid-circled" μ variables and their joining dotted lines are the localized Ising variables and bonds, respectively, of the kagomé Ising model, with the results that

$$Z_{d,L} = \Delta_1^{N_h} Z_{k,Q} \ , \tag{4.5a}$$

where, in (4.5a),

$$\Delta_1^4 = e^{4Q}(e^{4Q}+3)^2 \tag{4.5b}$$

and

$$e^{4Q} = 2\cosh 2L - 1 \ . \tag{4.5c}$$

Although the star-triangle (Y-Δ) transformation was

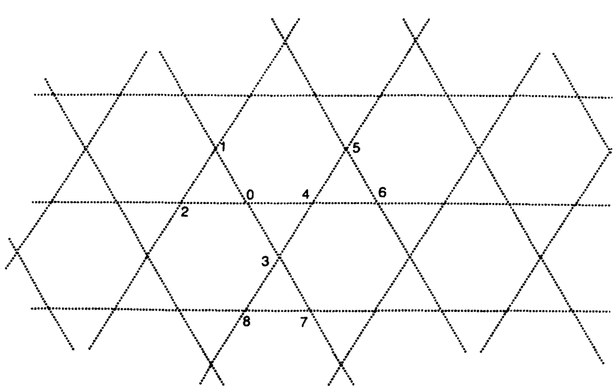

FIG. 3. The kagomé lattice. Nine sites are specifically enumerated for later use.

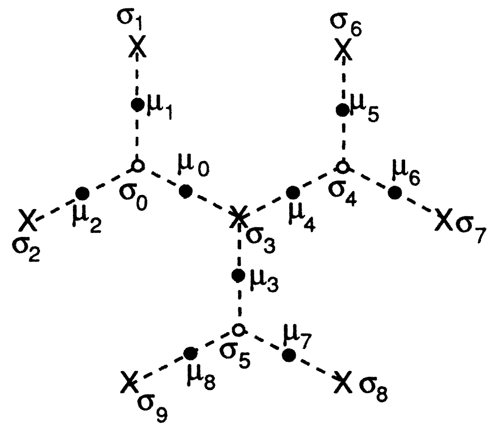

FIG. 4. A portion of the decorated-honeycomb lattice where ten σ variables $\sigma_0, \sigma_1, \ldots, \sigma_9$ and nine μ variables $\mu_0, \mu_1, \ldots, \mu_8$ are specified for later use.

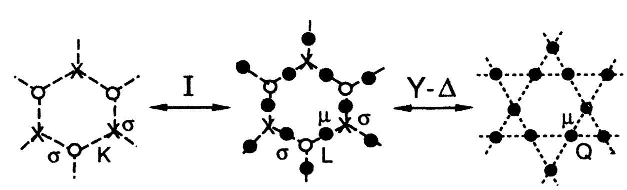

FIG. 5. The honeycomb, decorated-honeycomb, and kagomé Ising models are connected by transformation theory. The decoration-iteration (I) and star-triangle (Y-Δ) transformations equate, aside from known multiplicative constants, the partition functions $Z_{d,L}$ to $Z_{h,K}$ and $Z_{d,L}$ to $Z_{k,Q}$, respectively, where their (dimensionless) interaction parameters K, L, Q are simply related.

discussed in an earlier context of Sec. III, the so-called decoration-iteration (I) transformation is now used to connect the decorated-honeycomb and honeycomb Ising models. The decoration-iteration (I) transformation (diagrammatically illustrated in Fig. 6) enables a central decoration μ spin coupled to two neighboring σ spins to be replaced by a single-interaction bond connecting the latter two σ spins. More precisely, having called L and K the (dimensionless) interaction parameters of the decorated-honeycomb and honeycomb Ising models, respectively, one can show that

$$Z_{d,L} = I^{S_h} Z_{h,K} \;, \tag{4.6a}$$

where $S_h(=3N_h/2)$ is the total number of bonds on the honeycomb lattice, and where

$$I = 2e^{K} \tag{4.6b}$$

and

$$e^{2K} = \cosh 2L \;. \tag{4.6c}$$

The decoration-iteration (I) transformation is based upon the realization that the partial-trace evaluation

$$\mathrm{Tr}_{\mu_0} e^{L\mu_0(\sigma_1+\sigma_2)} = I e^{K\sigma_1\sigma_2} \tag{4.7}$$

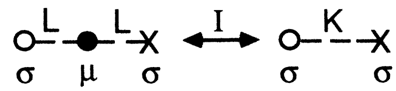

FIG. 6. The decoration-iteration (I) transformation enables a bond-decoration μ spin coupled via L to two neighboring σ spins to be replaced by a single-interaction bond K between the latter two σ spins.

is similarly valid for each and every "solid-circled" μ variable appearing within the total trace operation (4.4) defining $Z_{d,L}$.

Using the decorated-honeycomb Ising model in a mediating role, one proceeds to prove three theorems which taken together will enable any correlation of the kagomé Ising model to be mapped upon a linear combination of honeycomb Ising-model correlations.

Theorem 3.

$$\langle \mu_l \mu_m \cdots \mu_r \rangle_{k,Q} = \langle \mu_l \mu_m \cdots \mu_r \rangle_{d,L} \;,$$

where the corresponding (dimensionless) interaction parameters are related by $e^{4Q} = 2\cosh 2L - 1$.

Proof:

$$\begin{aligned}\langle \mu_l \mu_m \cdots \mu_r \rangle_{d,L} &= Z_{d,L}^{-1} \mathrm{Tr}_{\mu,\sigma} \mu_l \mu_m \cdots \mu_r e^{-\mathcal{H}_{d,L}} = Z_{d,L}^{-1} \mathrm{Tr}_{\mu} \mu_l \mu_m \cdots \mu_r \mathrm{Tr}_{\sigma} e^{-\mathcal{H}_{d,L}} \\ &= Z_{d,L}^{-1} \mathrm{Tr}_{\mu} \mu_l \mu_m \cdots \mu_r \Delta_1^{N_h} e^{-\mathcal{H}_{k,Q}} \\ &= Z_{k,Q}^{-1} \mathrm{Tr}_{\mu} \mu_l \mu_m \cdots \mu_r e^{-\mathcal{H}_{k,Q}} = \langle \mu_l \mu_m \cdots \mu_r \rangle_{k,Q} \;,\end{aligned}$$

where $e^{4Q} = 2\cosh 2L - 1$, having applied star-triangle-type transformations upon all σ variables and having used expressions (4.5a) and (4.5c) thus completing the proof of the theorem.

Theorem 4. $\langle \mu_l \mu_m \cdots \mu_r \rangle_{d,L} = M^n \langle (\sigma_j+\sigma_k)(\sigma_q+\sigma_s) \cdots (\sigma_t+\sigma_u) \rangle_{d,L}$, where the left-hand side (lhs) μ product contains n factors, $M = \frac{1}{2}\tanh 2L$, and on the rhs, σ_j, σ_k are the nearest-neighbor Ising variables of μ_l; σ_q, σ_s are the nearest-neighboring Ising variables of μ_m and so forth.

Proof. Let $[f]$ be any $(n-1)$ product of decorated-honeycomb μ variables *excluding* μ_l. Then

$$\begin{aligned}\langle \mu_l [f]_{d,L} \rangle &= Z_{d,L}^{-1} \mathrm{Tr}_{\mu,\sigma} e^{-\mathcal{H}_{d,L}} \mu_l [f] \\ &= Z_{d,L}^{-1} \mathrm{Tr}_{\mu,\sigma} e^{-\mathcal{H}'_{d,L} + L\mu_l(\sigma_j+\sigma_k)} \mu_l [f] \\ &= Z_{d,L}^{-1} \mathrm{Tr}_{\mu} \mathrm{Tr}_{\sigma} e^{-\mathcal{H}_{d,L}} [f] \frac{\mathrm{Tr}_{\mu_l} e^{L\mu_l(\sigma_j+\sigma_k)} \mu_l}{\mathrm{Tr}_{\mu_l} e^{L\mu_l(\sigma_j+\sigma_k)}} \\ &= Z_{d,L}^{-1} \mathrm{Tr}_{\mu} \mathrm{Tr}_{\sigma} e^{-\mathcal{H}_{d,L}} [f] \tanh L(\sigma_j+\sigma_k) = \langle [f] \tanh L(\sigma_j+\sigma_k) \rangle_{d,L} \;,\end{aligned}$$

where the above notation $\mathcal{H}'_{d,L}$ denotes the *restricted* Hamiltonian which excludes μ_l. Using the identity relation $\tanh L(\sigma_j+\sigma_k) = M(\sigma_j+\sigma_k)$, where $M = \frac{1}{2}\tanh 2L$, one obtains

$$\langle \mu_l [f] \rangle_{d,L} = M \langle [f](\sigma_j+\sigma_k) \rangle_{d,L} \;.$$

This procedure can be continued recitatively until all the $(n-1)$ μ variables belonging to $[f]$ are eliminated which completes the proof of the theorem

Theorem 5. $\langle \sigma_i \sigma_n \sigma_p \cdots \sigma_v \rangle_{d,L} = \langle \sigma_i \sigma_n \sigma_p \cdots \sigma_v \rangle_{h,K}$, where the corresponding (dimensionless) interaction parameters are related by $\cosh 2L = e^{2K}$.

Proof:

$$\begin{aligned}\langle \sigma_i\sigma_n\sigma_p \cdots \sigma_v \rangle_{d,L} &= Z_{d,L}^{-1}\mathrm{Tr}_{\mu,\sigma}\sigma_i\sigma_n\sigma_p \cdots \sigma_v e^{-\mathcal{H}_{d,L}} \\ &= Z_{d,L}^{-1}\mathrm{Tr}_{\sigma}\sigma_i\sigma_n\sigma_p \cdots \sigma_v \mathrm{Tr}_{\mu} e^{-\mathcal{H}_{d,L}} \\ &= Z_{d,L}^{-1}\mathrm{Tr}_{\sigma}\sigma_i\sigma_n\sigma_p \cdots \sigma_v I^{S_h} e^{-\mathcal{H}_{h,K}} \\ &= Z_{h,K}^{-1}\mathrm{Tr}_{\sigma}\sigma_i\sigma_n\sigma_p \cdots \sigma_v e^{-\mathcal{H}_{h,K}} = \langle \sigma_i\sigma_n\sigma_p \cdots \sigma_v \rangle_{h,K} \; ,\end{aligned}$$

where $\cosh 2L = e^{2K}$, having applied decoration-iteration-type transformations upon all μ variables and having used expressions (4.6a) and (4.6c), thus completing the proof of the theorem.

The unifying scheme of the present section IV and previous section III should now become clear. Systematically, Theorem 3 maps a kagomé Ising correlation upon a μ-type decorated-honeycomb Ising correlation whereupon Theorem 4 then maps this latter correlation upon a linear combination of σ-type decorated-honeycomb Ising correlations. Each of the latter correlations is then equated to a honeycomb Ising correlation by Theorem 5. Since the previous Sec. III established by using Theorems 1 and 2 that any honeycomb Ising correlation can itself be mapped upon a linear combination of triangular Ising correlations, one sees that the triangular Ising model plays the role of a "canopy or umbrella" in the sense that knowing *all* its correlations on a select cluster of sites is sufficient to determine *all* honeycomb, decorated-honeycomb, and kagomé Ising correlations upon their respective sites which are appropriately located within or upon the original "canopy" cluster of triangular lattice sites (see Fig. 7). Specifically in Fig. 7, knowledge of all triangular lattice correlations upon its 7-site cluster is sufficient to determine all honeycomb lattice correlations upon its 10-site cluster, all decorated-honeycomb lattice correlations upon its 19-site cluster, and all kagomé lattice correlations upon its 9-site cluster. This fact that all such Ising correlations can now be simply and systematically mapped upon *triangular* Ising-model correlations underscores the desirability of obtaining exact solutions for the latter and thus forms the motivation for the next section.

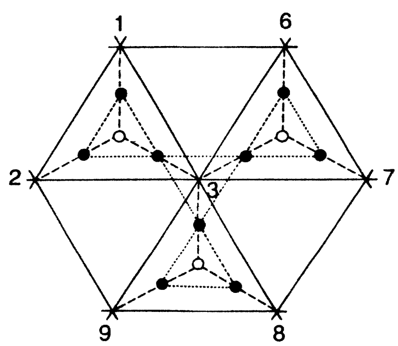

FIG. 7. For the calculation of Ising correlations, the triangular lattice (solid bonds) may be viewed as enveloping the honeycomb (dashed bonds), kagomé (dotted bonds), and decorated-honeycomb (dashed bonds) lattices. Seven sites 1, 2, 3, 6, 7, 8, 9 of the triangular lattice are specifically enumerated for select applications of the present theory.

V. EXACT SOLUTIONS FOR TRIANGULAR ISING-MODEL CORRELATIONS $u_1, u_2, \ldots, u_{11}$

The 11 (nonequivalent) even-number correlations $u_1, u_2, \ldots, u_{11}$ (diagrammatically represented in Fig. 8) are exhaustive in number upon the 7-site cluster of the triangular lattice previously shown in Fig. 7 and are sufficient as a spanning set to obtain all the even-number correlations of the other considered Ising models upon their appropriate multisite clusters enveloped in Fig. 7. Of course, each of the 11 correlations $u_1, u_2, \ldots, u_{11}$ could individually be calculated exactly by traditional Pfaffian techniques, but such methods knowingly become lengthy and laborious. Rather, the present section will show[19] that only four correlations u_1, u_6, u_7, u_{11} (three pair and a sextet) actually need to be calculated by Pfaffian procedures whereupon the remaining seven correlations u_2, u_3, u_4, u_5, u_8, u_9, u_{10} can then be found by more simple methods using both linear and nonlinear

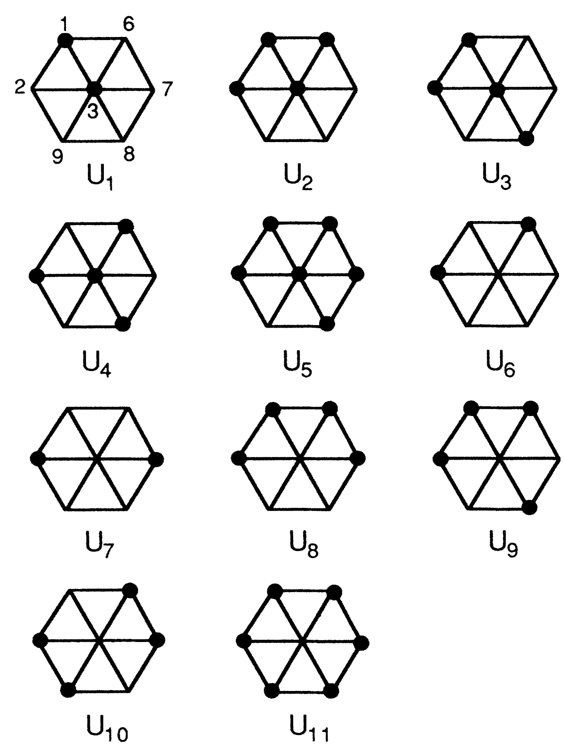

FIG. 8. Diagrammatic representation of the triangular lattice even-number correlations u_i, $i = 1, \ldots, 11$. These correlations exhaust all such (nonequivalent) possibilities defined upon sites 1, 2, 3, 6, 7, 8, 9 enumerated in Fig. 1 or Fig. 7, and constitute a spanning set in the context of the present paper.

algebraic correlation identities on the honeycomb and triangular lattices together with Theorems 1 and 2 proven in Sec. III.

To begin, assume that u_1, u_6, u_7, u_{11} have been calculated by Pfaffian methods, and for later use, recall that each of these triangular Ising-model correlations can be equated to its corresponding correlation of the honeycomb Ising model by the registry Theorem 1.

Consider next the honeycomb lattice sites enumerated in Fig. 1 as $0,1,2,\ldots,7$. For handy reference here and throughout the remainder of the paper, Table I specifies all possible honeycomb Ising-model even-number correlations defined upon this eight-site cluster $0,1,2,\ldots,7$. Using the basic generating equation (2.7), one develops the

TABLE I. Definitions of honeycomb Ising-model even-number correlations defined upon the cluster of eight sites $0,1,\ldots,7$ in Fig. 1. (Note that the subscripts 29, 44, 50, 52, and 53 have not been used in the listing.)

$x_1=\langle 01\rangle,\langle 02\rangle,\langle 03\rangle,\langle 34\rangle,\langle 35\rangle,\langle 46\rangle,\langle 47\rangle$
$x_2=\langle 04\rangle,\langle 05\rangle,\langle 12\rangle,\langle 13\rangle,\langle 16\rangle,\langle 23\rangle,\langle 36\rangle,\langle 37\rangle,\langle 45\rangle,\langle 67\rangle$
$x_3=\langle 06\rangle,\langle 14\rangle,\langle 25\rangle,\langle 57\rangle$
$x_4=\langle 07\rangle,\langle 15\rangle,\langle 24\rangle,\langle 56\rangle$
$x_5=\langle 0145\rangle,\langle 0245\rangle,\langle 0367\rangle,\langle 0456\rangle,\langle 0457\rangle,\langle 1234\rangle,\langle 1235\rangle,\langle 3567\rangle$
$x_6=\langle 0124\rangle,\langle 0125\rangle,\langle 0356\rangle,\langle 0357\rangle,\langle 0467\rangle,\langle 1345\rangle,\langle 2345\rangle,\langle 4567\rangle$
$x_7=\langle 0567\rangle,\langle 1245\rangle$
$x_8=\langle 0123\rangle,\langle 0345\rangle,\langle 3467\rangle$
$x_9=\langle 0134\rangle,\langle 0235\rangle,\langle 0346\rangle,\langle 3457\rangle$
$x_{10}=\langle 0135\rangle,\langle 0234\rangle,\langle 0347\rangle,\langle 3456\rangle$
$x_{11}=\langle 012345\rangle,\langle 034567\rangle$
$x_{12}=\langle 17\rangle,\langle 26\rangle$
$x_{13}=\langle 1236\rangle,\langle 1367\rangle$
$x_{14}=\langle 27\rangle$
$x_{15}=\langle 1237\rangle,\langle 2367\rangle$
$x_{16}=\langle 1267\rangle$
$x_{17}=\langle 0136\rangle,\langle 1346\rangle$
$x_{18}=\langle 0146\rangle$
$x_{19}=\langle 012367\rangle,\langle 123467\rangle$
$x_{20}=\langle 0126\rangle,\langle 0236\rangle,\langle 1347\rangle,\langle 1467\rangle$
$x_{21}=\langle 0137\rangle,\langle 2346\rangle$
$x_{22}=\langle 0167\rangle,\langle 1246\rangle$
$x_{23}=\langle 012467\rangle$
$x_{24}=\langle 0237\rangle,\langle 2347\rangle$
$x_{25}=\langle 0267\rangle,\langle 1247\rangle$
$x_{26}=\langle 0127\rangle,\langle 2467\rangle$
$x_{27}=\langle 0147\rangle,\langle 0246\rangle$
$x_{28}=\langle 012346\rangle,\langle 013467\rangle$
$x_{30}=\langle 0247\rangle$
$x_{31}=\langle 012347\rangle,\langle 023467\rangle$
$x_{32}=\langle 1356\rangle$
$x_{33}=\langle 0156\rangle,\langle 1456\rangle$
$x_{34}=\langle 1357\rangle,\langle 2356\rangle$
$x_{35}=\langle 1256\rangle,\langle 1567\rangle$
$x_{36}=\langle 013456\rangle$
$x_{37}=\langle 0256\rangle,\langle 1457\rangle$
$x_{38}=\langle 0157\rangle,\langle 2456\rangle$
$x_{39}=\langle 012356\rangle,\langle 134567\rangle$
$x_{40}=\langle 012456\rangle,\langle 014567\rangle$
$x_{41}=\langle 013567\rangle,\langle 123456\rangle$
$x_{42}=\langle 013457\rangle,\langle 023456\rangle$
$x_{43}=\langle 2357\rangle$
$x_{45}=\langle 1257\rangle,\langle 2567\rangle$
$x_{46}=\langle 0257\rangle,\langle 2457\rangle$
$x_{47}=\langle 023567\rangle,\langle 123457\rangle$
$x_{48}=\langle 012357\rangle,\langle 234567\rangle$
$x_{49}=\langle 012457\rangle,\langle 024567\rangle$
$x_{51}=\langle 023457\rangle$
$x_{54}=\langle 123567\rangle$
$x_{55}=\langle 012567\rangle,\langle 124567\rangle$
$x_{56}=\langle 01234567\rangle$

identity

$$x_1 = A + (2A + B)x_2 , \quad (5.1)$$

where one has defined the pair correlations

$$x_1 = \langle 01 \rangle, \quad x_2 = \langle 04 \rangle \quad (5.2)$$

having, for simplicity of notation, entered only the numeric labels of the sites in the thermal-average symbols themselves taken to mean $\langle \cdots \rangle_{h,K}$. Inspecting (5.1), one immediately concludes that, using the known literature value for x_2 (energy u_1 of the triangular Ising model by the registry Theorem 1), one obtains the value for x_1 (energy of the honeycomb model). Consequently, x_1 and x_2 are now considered to be known.

To continue, one makes use of a class of *nonlinear* algebraic correlation identities which does *not* explicitly contain the interaction parameters. This class of identities was derived using largely abstract graph theory by Groeneveld, Boel, and Kasteleyn[20] (GBK) and forms a generalization of the Griffiths-Kelly-Sherman (GKS) inequalities.[21] The GBK nonlinear algebraic identities express those even-numbered correlations defined upon a so-called ***boundary sequence*** **of sites (i.e., descriptively, the sites in order of appearance around an elementary polygon)** of an arbitrary planar simple Ising model (in zero magnetic field) in terms of products of pair correlations alone. In particular, the following four-site honeycomb correlations can be decomposed as

$$\langle 1034 \rangle = \langle 10 \rangle \langle 34 \rangle + \langle 14 \rangle \langle 03 \rangle - \langle 13 \rangle \langle 04 \rangle ,$$

$$\langle 1036 \rangle = \langle 10 \rangle \langle 36 \rangle + \langle 03 \rangle \langle 16 \rangle - \langle 06 \rangle \langle 13 \rangle ,$$

$$\langle 1046 \rangle = \langle 10 \rangle \langle 46 \rangle + \langle 16 \rangle \langle 04 \rangle - \langle 14 \rangle \langle 06 \rangle ,$$

or, respectively,

$$x_9 = x_1^2 - x_2^2 + x_1 x_3 , \quad (5.3a)$$

$$x_{17} = 2x_1 x_2 - x_2 x_3 , \quad (5.3b)$$

$$x_{18} = x_1^2 + x_2^2 - x_3^2 , \quad (5.3c)$$

where one has defined the honeycomb correlations

$$x_3 = \langle 06 \rangle, \quad x_9 = \langle 1034 \rangle ,$$
$$x_{17} = \langle 0136 \rangle, \quad x_{18} = \langle 1046 \rangle . \quad (5.4)$$

Again using the basic generating equation (2.7), one develops the linear identities

$$x_3 = A(2x_2 + x_{12}) + Bx_{13} , \quad (5.5a)$$

$$x_{17} = A(2x_2 + x_{13}) + Bx_{12} , \quad (5.5b)$$

where one has defined the honeycomb correlations

$$x_{12} = \langle 26 \rangle, \quad x_{13} = \langle 1236 \rangle . \quad (5.6)$$

Substituting (5.5a) and (5.5b) into (5.3b), the nonlinear identity (5.3b) becomes

$$A(2x_2 + x_{13}) + Bx_{12} = x_2[2x_1 - A(2x_2 + x_{12}) - Bx_{13}] . \quad (5.7)$$

Since the sites appearing within the honeycomb thermal-average symbols for x_{12} and x_{13} can be registered upon the triangular lattice, x_{12} and x_{13} can be equated to the corresponding triangular lattice thermal averages for u_6 and u_2, respectively, and vice versa, in the sense of the registry Theorem 1. However, as originally stated, x_{12} ($\equiv u_6$) is assumed to be known through use of Pfaffian techniques thus enabling x_{13} ($\equiv u_2$) to be determined from (5.7). Consequently, substituting these known solutions for x_{12}, x_{13} into (5.5a) determines x_3. The fact that x_1, x_2, x_3 are now known easily determines x_9, x_{18} from (5.3a) and (5.3c). The latter known solutions for x_9, x_{18} are next used as follows. Theorem 2 enables one to write

$$\langle 1034 \rangle = x_9$$
$$= A^2(1 + 5x_2 + x_{12} + x_{13} + x_{15})$$
$$+ AB(2x_2 + x_{12} + x_{13} + x_{14} + x_{16}) + B^2 x_{15} \quad (5.8a)$$

and

$$\langle 1046 \rangle = x_{18}$$
$$= A^2(1 + 5x_2 + 2x_{13} + x_{16})$$
$$+ 2AB(x_2 + x_{12} + x_{15}) + B^2 x_{14} , \quad (5.8b)$$

where one has defined the correlations

$$x_{14} = \langle 27 \rangle, \quad x_{15} = \langle 1237 \rangle, \quad x_{16} = \langle 1267 \rangle . \quad (5.9)$$

In Eqs. (5.8a) and (5.8b) the two-site thermal average x_{14} ($\equiv u_7$ by the registry Theorem 1) is assumed to be known by Pfaffian techniques as earlier stated. Therefore, solutions can be found for the remaining two unknowns x_{15} and x_{16} from the pair of equations (5.8a) and (5.8b) since all other correlations appearing in (5.8a) and (5.8b) are already known.

To review, assuming that u_1 ($\equiv x_2$), u_6 ($\equiv x_{12}$), u_7 ($\equiv x_{14}$) have each been obtained by Pfaffian procedures, one has obtained solutions for u_2 ($\equiv x_{13}$), u_3 ($\equiv x_{15}$), u_8 ($\equiv x_{16}$) by more simple procedures. One now proceeds towards the goal of securing exact solutions for *all* members of the spanning set $u_1, u_2, \ldots, u_{11}$ by methods again more simple than employing Pfaffian methods alone. Using the basic generating equation (2.10), the following correlation identities are developed:

$$u_1 = C(1 + 2u_1 + 2u_6 + u_7)$$
$$+ 2D(2u_1 + 2u_6 + u_7 + 2u_8 + 2u_9 + u_{10})$$
$$+ E(2u_8 + 2u_9 + u_{10} + u_{11}) , \quad (5.9a)$$

$$u_2 = C(2u_1 + u_6 + 2u_8 + u_9)$$
$$+ D(1 + 2u_1 + 4u_6 + 3u_7 + 2u_8 + 4u_9 + 3u_{10} + u_{11})$$
$$+ E(2u_1 + u_6 + 2u_8 + u_9) , \quad (5.9b)$$

$$u_3 = (C + E)(u_1 + u_6 + u_7 + u_8 + u_9 + u_{10})$$
$$+ D(1 + 4u_1 + 4u_6 + u_7 + 4u_8 + 4u_9 + u_{10} + u_{11}) , \quad (5.9c)$$

$$u_4=3(C+E)(u_6+u_9)+D(1+6u_1+3u_7+6u_8+3u_{10}+u_{11}) , \quad (5.9d)$$

$$u_5=C(2u_8+2u_9+u_{10}+u_{11})+2D(2u_1+2u_6+u_7+2u_8+2u_9+u_{10})+E(1+2u_1+2u_6+u_7) , \quad (5.9e)$$

where the triangular lattice even-number correlations are defined as

$$u_1=\langle 13\rangle,\ u_2=\langle 1236\rangle,\ u_3=\langle 1369\rangle,\ u_4=\langle 1379\rangle ,\ u_5=\langle 123678\rangle,\ u_6=\langle 17\rangle,\ u_7=\langle 18\rangle,\ u_8=\langle 1267\rangle ,\ u_9=\langle 1268\rangle,\ u_{10}=\langle 1278\rangle,\ u_{11}=\langle 126789\rangle , \quad (5.10)$$

C, D, E are the known interaction-dependent coefficients (2.11), and diagrammatic representations of the correlations (5.10) have formerly been shown in Fig. 8. Subtracting (5.9c) from (5.9b) gives

$$u_2-u_3=2(C-2D+E)(u_1-u_7+u_8-u_{10}) . \quad (5.11)$$

Equation (5.11) directly gives the solution for u_{10} since all other correlations appearing in the equation have been previously determined. Since, as stated earlier, the sextet correlation u_{11} is assumed to be known by Pfaffian procedures, Eq. (5.9a) itself now determines u_9 since all other correlations in this linear identity are already known; then Eqs. (5.9d) and (5.9e) directly determine the remaining correlations u_4 and u_5, respectively. Our goal has finally been achieved, namely, assuming that four correlations u_1, u_6, u_7, u_{11} are known *a priori* by Pfaffian methods, the remaining seven correlations u_2, u_3, u_4, u_5, u_8, u_9, u_{10} are now also known having been deduced by very much simpler procedures than Pfaffian techniques. Again, as depicted in Fig. 7, one emphasizes that this knowledge of the triangular lattice correlations $u_1, u_2, \ldots, u_{11}$ upon its 7-site cluster is sufficient to determine *all* honeycomb, decorated-honeycomb, and kagomé even-number correlations upon their 19-site, and 9-site clusters, respectively—truly large numbers of exact solutions for localized multisite correlations of planar Ising models.

VI. SOME SELECT RESULTS

This section illustrates the procedures of the present theory by finding exact solutions for a few select even-number localized correlations of the kagomé Ising model in terms of the spanning correlations of the triangular Ising model. Considering, first, the nearest-neighbor pair correlation (energy), one has

$$\begin{aligned}
\langle \mu_0\mu_1\rangle_{k,Q} &= \langle \mu_0\mu_1\rangle_{d,L} \quad \text{(by Theorem 3)}\\
&= M^2\langle(\sigma_0+\sigma_3)(\sigma_0+\sigma_1)\rangle_{d,L} \quad \text{(by Theorem 4)}\\
&= M^2\langle(\sigma_0+\sigma_3)(\sigma_0+\sigma_1)\rangle_{h,K} \quad \text{(by Theorem 5)}\\
&= M^2+M^2(2\langle\sigma_0\sigma_1\rangle_{h,K}+\langle\sigma_1\sigma_3\rangle_{h,K})\\
&= M^2+M^2\{2\langle[A(\sigma_1+\sigma_2+\sigma_3)+B\sigma_1\sigma_2\sigma_3]\sigma_1\rangle_{h,K}+\langle\sigma_1\sigma_3\rangle_{h,K}\} \quad \text{[using (2.7)]}\\
&= M^2+M^2\{2A+[2(2A+B)+1]\langle\sigma_1\sigma_2\rangle_{h,K}\}\\
&= M^2+M^2\{2A+[2(2A+B)+1]u_1\} \quad \text{(by Theorem 1)} .
\end{aligned} \quad (6.1)$$

Next choosing to consider a select quartet correlation, one obtains

$$\begin{aligned}
\langle \mu_0\mu_1\mu_2\mu_3\rangle_{k,Q} &= \langle \mu_0\mu_1\mu_2\mu_3\rangle_{d,L} \quad \text{(by Theorem 3)}\\
&= M^4\langle(\sigma_0+\sigma_3)(\sigma_0+\sigma_1)(\sigma_0+\sigma_2)(\sigma_3+\sigma_5)\rangle_{d,L} \quad \text{(by Theorem 4)}\\
&= M^4\langle(\sigma_0+\sigma_3)(\sigma_0+\sigma_1)(\sigma_0+\sigma_2)(\sigma_3+\sigma_5)\rangle_{h,K} \quad \text{(by Theorem 5)}\\
&= M^4(1+4x_1+4x_2+x_3+x_4+x_5+x_6+x_8+x_9+x_{10}) .
\end{aligned} \quad (6.2)$$

Consulting Table I and using the superposition Theorem 2 together with Figs. 1 and 8, one writes

$$x_1=\langle 01\rangle_{h,K}=A+(2A+B)u_1 , \quad (6.3a)$$

$$x_2=\langle 04\rangle_{h,K}=\langle 12\rangle_{h,K}=u_1 , \quad (6.3b)$$

$$x_3=\langle 06\rangle_{h,K}=A(2u_1+u_6)+Bu_2 , \quad (6.3c)$$

$$x_4=\langle 07\rangle_{h,K}=A(u_1+u_6+u_7)+Bu_3 , \quad (6.3d)$$

$$x_5=\langle 0145\rangle_{h,K}=\langle 0367\rangle_{h,K}=A(u_1+u_2+u_3)+Bu_8 , \quad (6.3e)$$

$$x_6=\langle 0124\rangle_{h,K}=A^2(4u_1+u_2+u_3+2u_6+u_7)+AB(2u_1+u_2+u_3+u_8)+B^2u_1+AB , \quad (6.3f)$$

$$x_8=\langle 0123\rangle_{h,K}=3Au_1+B , \quad (6.3g)$$

$$x_9=\langle 0134\rangle_{h,K}$$
$$=A^2(5u_1+u_2+u_3+u_6)$$
$$+AB(2u_1+u_2+u_6+u_7+u_8)+B^2u_3+A^2\ , \tag{6.3h}$$

$$x_{10}=\langle 0135\rangle_{h,K}$$
$$=\langle 0234\rangle_{h,K}$$
$$=A^2(4u_1+u_2+u_3+u_6+u_7)$$
$$+AB(3u_1+u_3+u_6+u_8)+B^2u_2+A^2\ . \tag{6.3i}$$

For calculational convenience, one notes above that prior to elimination of all "circled" ○ sites within a thermal-average symbol by the superposition Theorem 2, it was desirable initially to make minimal the number of such sites by using symmetry arguments, e.g., $\langle 04\rangle_{h,K}=\langle 12\rangle_{h,K}$ and $\langle 0145\rangle_{h,K}=\langle 0367\rangle_{h,K}$. In developing expressions (6.3), the reader is also aided by reviewing the illustrative example (3.6) of the superposition Theorem 2. Substituting (6.3) into (6.2), one has succeeded in writing the four-site kagomé Ising correlation $\langle\mu_0\mu_1\mu_2\mu_3\rangle_{k,Q}$ in terms of the known spanning correlations $u_1,\ldots,u_{11}$ of the triangular Ising model.

The exact solution curves for the above kagomé pair $\langle\mu_0\mu_1\rangle_{k,Q}$ and quartet $\langle\mu_0\mu_1\mu_2\mu_3\rangle_{k,Q}$ correlations as well as the sextet correlation $\langle\mu_0\mu_1\mu_2\mu_3\mu_4\mu_5\rangle_{k,Q}$ are displayed in Fig. 9. As expected, each exact solution curve is a continuous, monotonically-decreasing function of temperature and possesses a weak singularity of energy type $\epsilon\ln\epsilon$ (where $\epsilon\equiv|T-T_c|/T_c$) at the critical temperature T_c.

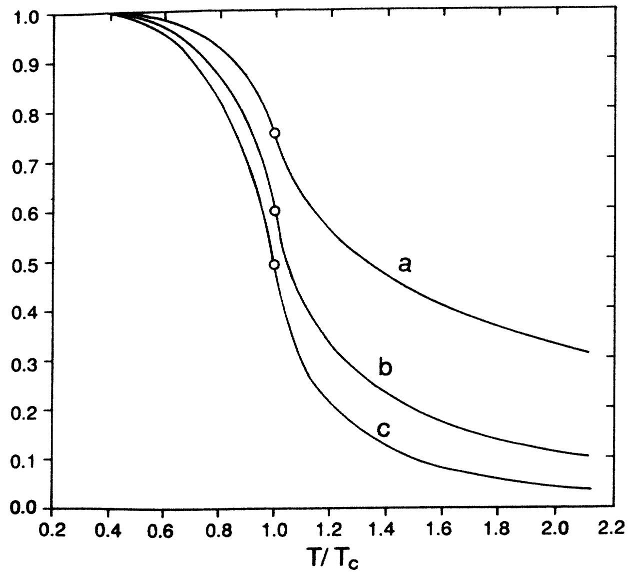

FIG. 9. Exact solution curves for kagomé Ising-model correlations (a) $\langle\mu_0\mu_1\rangle_{k,Q}$, (b) $\langle\mu_0\mu_1\mu_2\mu_3\rangle_{k,Q}$, and (c) $\langle\mu_0\mu_1\mu_2\mu_3\mu_4\mu_5\rangle_{k,Q}$, vs (reduced) temperature $Q_c/Q(=T/T_c)$, where $Q_c=\frac{1}{4}\ln(3+2\sqrt{3})=0.4665\ldots$. Vertical inflection points shown encircled exist at the critical temperature T_c. (Note the differing restricted ranges of the scales.)

VII. SUMMARY AND CONCLUSIONS

The present investigations have established a systematic and unifying method for finding exact solutions of localized Ising correlations on various planar lattices. In the theoretical formulation, the triangular Ising model satisfied an enveloping strategy. In particular, knowledge of all 11 even-number correlations upon a select 7-site cluster of the triangular Ising model was shown sufficient to determine *all* honeycomb, decorated-honeycomb, and kagomé Ising model even-number correlations upon their correspondingly select 10-site, 19-site, and 9-site clusters, respectively. The numbers of such multisite correlations are very considerable, e.g., approximately 85 and 50 for the honeycomb and kagomé Ising models, respectively, and for such n-site (n even integer) correlations, $n_{\max}=8$, 10, 18 for the kagomé, honeycomb, and decorated-honeycomb Ising models, respectively, where the latter $n_{\max}$ values are significantly greater than existing literature values. As demonstrated in the present paper, the method of mapping theorems in conjunction with algebraic correlation identities is exceedingly simpler than using traditional Pfaffian methods exclusively; in fact *only* *four* even-number correlations (three pair and a sextet) of the triangular Ising model were actually calculated by the comparatively lengthy Pfaffian techniques. The relative ease and direct applicability of the present method are highlighted not only by the resulting large numbers of n-site (n even integer) correlations and large $n_{\max}$ values, but also by the realization that the exact solutions for localized Ising-multisite correlations on the kagomé lattice (one of the four regular lattices in two dimensions) are presumably the first to explicitly appear in the literature beyond the knowledge of its nearest-neighbor pair correlation (or energy).

Finally, one notes that the numerical precision of all correlation solutions is the same as the numerical precision of the spanning correlations $u_1,u_2,\ldots,u_{11}$ of the triangular Ising model since each of the former solutions can now be viewed as a linear combination of the latter spanning solutions with known and well-behaved expansion coefficients thus allaying any concerns regarding possible numerical instabilities or similarly unusual computational difficulties in computer programs.

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