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NMR in superfluid ^{3}He in a confined geometry

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NMR in superfluid ^{3}He in a thin slab is studied in the Ginzburg-Landau regime. The predicted qualitative difference in response from the bulk case of both the A and B phases suggests that NMR can be a very sensitive experimental probe of the order parameter in severely confined geometries. We present numerical solutions for the NMR frequencies.

I. INTRODUCTION

The presence of walls deforms the order parameter of superfluid ^{3}He from its bulk form.[1–5] Near T_c one can calculate the effect of the walls with Ginzburg-Landau theory. Recent experiments on the flow of ^{3}He in confined geometries[6–10] (dimension of the order of the coherence length) have measured two distinct critical currents, the upper-critical-current density agrees with the theoretical value.[2,4,5] However, such experiments are not very sensitive to the detailed form of the order parameter. Moreover, the latest exhaustive theoretical results on the order parameter in a semi-infinite geometry have again studied only the superfluid mass and spin currents.[11,12]

The present paper demonstrates that nuclear magnetic resonance (NMR) provides a much better measure of the order parameter. As shown below, the resonance frequency of the lowest spin mode is substantially different for the A and B phases and depends strongly on the slab width.

II. THEORY

The calculations are based on Leggett's theory[13,14] generalized to include the effect of gradient terms in the free energy.[15,16] In the linearized approximation (relevant to continuous-wave NMR), the equations reduce to a Schrödinger-like eigenvalue equation with coefficients that depend on the static value of the order parameter.[17] These equations are then solved numerically.

We first determine the precise form of the static order parameter (see Sec. II A). Next, the Leggett equations for the change in the spin vector due to the applied oscillating magnetic field are derived in terms of the general static order parameter (Sec. II B). Then, in Sec. III, we specialize the Leggett equations to the A and B phases and solve them.

A. Form of the static order parameter

It is worthwhile to consider the relative magnitudes of the various contributions to the total free energy. The dominant bulk term is[18]

$$\begin{aligned}F_0 = &-\alpha A^*_{\mu i} A_{\mu i} + \beta_1 A^*_{\mu i} A^*_{\mu i} A_{\nu j} A_{\nu j} \\ &+\beta_2 A^*_{\mu i} A_{\mu i} A^*_{\nu j} A_{\nu j} + \beta_3 A^*_{\mu i} A^*_{\nu i} A_{\nu j} A_{\mu j} \\ &+\beta_4 A^*_{\mu i} A_{\nu i} A^*_{\nu j} A_{\mu j} + \beta_5 A^*_{\mu i} A_{\nu i} A_{\nu j} A^*_{\mu j} \ . \end{aligned} \tag{1}$$

The next most important contribution comes from the slow spatial variations of the order parameter. This is written as[14]

$$F_K = K_1 \partial_i A_{\mu i} \partial_j A_{\mu j} + K_2 \partial_j A^*_{\mu i} \partial_j A_{\mu i} + K_3 \partial_j A^*_{\mu i} \partial_i A_{\mu j} \ . \tag{2}$$

In the weak-coupling limit, K_1, K_2, and K_3 are equal. Moreover, the temperature-dependent coherence length is defined by $(K_2/\alpha)^{1/2} \equiv \xi(T)$, where $\xi(T) = (\frac{3}{5})^{1/2} \xi_0 (1-T/T_c)^{1/2}$ and T_c is the critical temperature. From Greywall's data[19] we find that for zero pressure, $\xi_0 = 6.4 \times 10^{-8}$ m and $T_c = 0.93$ K. Finally, we must include the magnetic energy term and the nuclear dipole term if we are to describe the NMR response of the superfluid. These are, respectively,

$$F_Z = g_Z H_\mu H_\nu A^*_{\mu i} A_{\nu i} \tag{3}$$

and

$$F_D = g_D (A^*_{\mu\mu} A_{\lambda\lambda} + A^*_{\mu\lambda} A_{\lambda\mu} - \tfrac{2}{3} A^*_{\mu\lambda} A_{\mu\lambda}) \ , \tag{4}$$

where the characteristic field strength[20] is $H_D \equiv (g_D/g_Z)^{1/2} \approx 25$ G and the characteristic length is $L_D \equiv (K_2/g_D)^{1/2} \approx 6$ μm. Thus $\alpha/g_D = (K/\xi^2 g_D) \approx 1\times 10^5 (1-T/T_c)$; for $(T_c - T)/T_c \gg 10^{-5}$, we may treat the dipole term as a perturbation in any calculation of the static order parameter. Similarly, since $(L_D H_D/\xi H)^2 \approx 6\times 10^7 (1-T/T_c)/H^2$ G^2 for strong fields (typically, $H \approx 600$ G) and $(T_c - T)/T_c \gg 10^{-2}$, the magnetic field may also be taken to be a small perturbation. (We shall see below that these two perturbations lift most of the degeneracy of the order parameter.) Throughout our analysis we shall consider a thin slab geometry with the two infinite planes lying perpendicular to the x axis, with the static magnetic field in the x direction. Moreover, we shall express all dimensional lengths and wave numbers in terms of the coherence length $\xi(T)$.

1. A-like phase

We first derive the Ginzburg-Landau equations for the A-phase order parameter. The bulk form of the A phase is well known, but in the presence of walls it is no longer the equilibrium state. As our ansatz we take

$$A_{\mu i} = \Delta_A \exp(iqz) \hat{d}_\mu [a_2(x)\hat{y} + i a_3(x) \hat{z}]_i \ . \tag{5}$$

where the functions $a_2(x)$ and $a_3(x)$ are determined by minimizing the bulk and gradient terms of the free energy, and the exponential describes uniform flow in the z direction with dimensionless momentum q. In bulk we take $a_2(x)=a_3(x)=1$, which gives $\Delta_A^2=\alpha/4\beta_{245}$. This ansatz satisfies the boundary condition that the $\hat{l}$ vector (equal to $\hat{\mathbf{y}}\times\hat{\mathbf{z}}$) be normal to the wall.[21] The vector $\hat{\mathbf{d}}$ is still arbitrary. The Ginzburg-Landau equations are obtained by substituting this form into F_0+F_K and varying with respect to $a_2(x)$ and $a_3(x)$. We find

$$a_2''-a_2(q^2-1)-\tfrac{1}{2}a_2[(a_2^2+a_3^2)+\beta(a_2^2-a_3^2)]=0\ , \quad (6)$$

$$a_3''-a_3(3q^2-1)-\tfrac{1}{2}a_3[(a_2^2+a_3^2)+\beta(a_3^2-a_2^2)]=0\ , \quad (7)$$

where $\beta=\beta_{13}/\beta_{245}$.

A relaxation technique is used to solve the equations.[22,23] Clearly, for zero flow ($q=0$) the equations are degenerate, whence $a_2=a_3$.[24] The $\hat{\mathbf{d}}$ vector is fixed by including the effects of the magnetic and dipole perturbations as well as the flow. The applied static magnetic field $H_0\hat{\mathbf{x}}$ forces the $\hat{\mathbf{d}}$ vector to lie in the y-z plane, while the dipole energy tends to align $\hat{\mathbf{d}}$ along the direction of flow. If there is no flow, then the $\hat{\mathbf{d}}$ vector has no preferred direction in the y-z plane. Hence, the lowest energy configuration is always obtained with $\hat{\mathbf{d}}$ in the z direction, which remains the equilibrium state if the applied oscillating rf field is in the x or y direction.

Thus the A-phase static order parameter is

$$A_{\mu i}=\Delta_A\exp(iqz)\delta_{\mu z}[a_2(x)\hat{\mathbf{y}}+ia_3(x)\hat{\mathbf{z}}]_i\ , \quad (8)$$

with $a_2(x)$ and $a_3(x)$ given by the solution of (6) and (7).

2. *B-like phase*

The B-phase order parameter is intrinsically more complicated than the A-phase order parameter. As in Ref. 5 we take as our ansatz

$$A_{\mu i}=\Delta_B\exp(iqz)R_{\mu\nu}A^0_{\nu i}\ , \quad (9)$$

where

$$A^0=\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \quad (10)$$

and

$$R_{\mu\nu}=\delta_{\mu\nu}\cos\theta+\hat{n}_\mu\hat{n}_\nu(1-\cos\theta)-\epsilon_{\mu\nu i}\hat{n}_i\sin\theta\ . \quad (11)$$

In bulk, the normalization is such that $a_1=a_2=a_3=1$ for $q=0$ and $a_2=a_3$ remains generally true for zero flow. The constant Δ_B is given by $\Delta_B^2=\alpha/2(3\beta_{12}+\beta_{345})$. Thuneberg *et al.*[12] have shown recently that this assumption is indeed the correct form for A^0: all off-diagonal elements in A^0 are zero except in the presence of flow when very small imaginary off-diagonal terms appear. We substitute this into the total free energy and vary it to obtain the Ginzburg-Landau equations.

The leading perturbation to F_0+F_K is the magnetic energy F_Z. For reasonably strong magnetic fields ($1\ll H/H_D\ll L_D/\xi$) without flow, the $\hat{\mathbf{n}}$ vector is forced to lie parallel to the static magnetic field. This step still leaves the rotation angle θ to be determined. We can show that the derivatives of θ are negligible relative to θ itself. A straightforward calculation for zero flow ($a_2=a_3$) shows that θ satisfies the differential equation

$$\frac{\partial}{\partial x}\left[a_3^2\frac{\partial\theta}{\partial x}\right]=-\frac{\xi^2}{L_D^2}a_3\sin\theta(a_1+4a_3\cos\theta)\ , \quad (12)$$

where the a_i ($i=1,2,3$) satisfy differential equations that include gradients of θ as factors. We see that the right-hand side is of the order of $\xi^2/L_D^2\ll 1$ so we expect only small variations in the value of θ across the slab. In order to verify this expectation explicitly, we may solve the equations for a_1 and a_3 in the absence of the gradient θ terms,[24] substitute the solution into Eq. (12), and see whether the solution for θ is approximately constant. As expected, numerical studies show that to a very good approximation θ is indeed constant except very close to the walls. A variational calculation with θ constant gives

$$\cos\theta=-\frac{1}{2}\frac{\int dx\, a_1(a_2+a_3)}{\int dx\,(a_2+a_3)^2}\ . \quad (13)$$

In the case of zero flow, $a_2=a_3$ and we recover the result of Fujita *et al.*[3] Thus, the static form of the order parameter is completely determined.

We would like to consider briefly the form of the order parameter in the case of large flow in a wide channel. The essential change is in the structure of the rotation matrix: the original $R(\hat{\mathbf{x}},\theta)$ is augmented by an additional rotation $R(\hat{\mathbf{y}},\frac{1}{2}\pi)$, which minimizes the integrated magnetic free energy whenever $\int dx\,(a_1^2-a_3^2)$ is positive. As a result the $\hat{\mathbf{n}}$ vector is no longer in the x direction, with the consequence that the angle θ differs substantially from the value in (13). These changes give a three-dimensional (3D) order parameter which is qualitatively different from the form (9), and we therefore expect a discontinuous change with increasing flow to a very different NMR response. we will not pursue this topic here since to our knowledge, there presently do not appear to be any experiments planned which could measure this effect.

B. The Leggett equations for a confined geometry

In this section we shall derive the Leggett equations for the spin vector of superfluid ^{3}He in a confined geometry. The analysis follows closely the approach of Buchholtz[16] and is not novel. The one new feature is that the static order parameter will appear explicitly as coefficients in the differential equations for the spin vector. This, as we shall show, renders the NMR frequency a sensitive function of the detailed form of the order parameter. We begin with the Heisenberg equations of motion for the spin density $\mathbf{S}$ and the order parameter $A_{\mu i}$,

$$i\hbar\frac{\partial S_i}{\partial t}=[S_i,\hat{H}]\ , \quad (14)$$

$$i\hbar\frac{\partial A_{\mu i}}{\partial t}=[A_{\mu i},\hat{H}]\ , \tag{15}$$

where,

$$\hat{H}=\int dx\left[\frac{\gamma^2}{2\chi}S^2-\gamma S_i H_i+F_D+F_K\right]\ . \tag{16}$$

Here $\mathbf{H}$ is the magnetic field, χ is the normal-state magnetic susceptibility, and $\gamma=2.04\times10^4$ $(\mathrm{G\,sec})^{-1}$. The commutation relations are given by $[S_i,S_j]=i\hbar\epsilon_{ijk}S_k$ and $[S_i,A_{\mu j}]=i\hbar\epsilon_{i\mu\nu}A_{\nu j}$.

In this paper, we shall consider the linearized form of the Leggett equations only. Thus, writing $\mathbf{H}=\mathbf{H}_0+\mathbf{H}'(t)$, $\mathbf{S}=\mathbf{S}_0+\mathbf{S}'$, and $A_{\mu i}=A^0_{\mu i}+A'_{\mu i}(t)$, where $A^0_{\mu i}$ is the static order parameter and $\mathbf{H}_0$ the constant applied field, and using $\gamma\mathbf{S}_0=\chi\mathbf{H}_0$, we find, from Eq. (15),

$$\frac{\partial A'_{\mu i}}{\partial t}=\gamma\epsilon_{\mu\nu\sigma}A^0_{\nu i}\left[H'_\sigma-\frac{\gamma}{\chi}S'_\sigma\right]\ . \tag{17}$$

Next, we differentiate Eq. (14) and substitute for $A'_{\mu i}$ from Eq. (17). This gives the required equation for $\mathbf{S}'$,

$$\frac{\partial^2 S'_k}{\partial t^2}-\gamma\epsilon_{kmn}\left[\frac{\partial S'_m}{\partial t}(H_0)_n+(S_0)_m\frac{\partial H'_n}{\partial t}\right]+(\mathcal{L}^D_{km}+\mathcal{L}^G_{km})\left[\frac{\gamma}{\chi}S'_m-H'_m\right]=0\ , \tag{18}$$

where $\mathcal{L}^D$ and $\mathcal{L}^G$ are functions of the static order parameter $A^0_{\mu i}$ (from now on we shall drop the superscript for clarity), namely

$$\begin{aligned}\mathcal{L}^D_{km}&=\mathrm{Re}2g_D\gamma[A^*_{ii}A_{km}+A^*_{ki}A_{im}\\&\quad+\epsilon_{kil}\epsilon_{mpq}(A^*_{il}A_{pq}+A^*_{iq}A_{pl})\\&\quad-\delta_{km}(A^*_{ii}A^*_{jj}+A^*_{ij}A_{ji})]\ ,\\ \mathcal{L}^G_{km}&=\mathrm{Re}2K\gamma\epsilon_{kpq}[\epsilon_{prm}A^*_{rn}(2A_{qs,sn}+A_{qn,ss})\\&\quad+2\epsilon_{qrm}A^*_{pn}(A_{rs,sn}+A_{rs,s}\partial_n\\&\quad+A_{rs,n}\partial_s+A_{rs}\partial^2_s)\\&\quad+\epsilon_{qrm}A^*_{pn}(A_{rn,ss}+2A_{rn,s}\partial_s+A_{rn}\partial^2_s)]\end{aligned} \tag{19}$$

(where $\partial_n\equiv\partial/\partial x_n$ and the commas denote differentiation, e.g., $A_{qs,sn}\equiv\partial^2A_{qs}/\partial x_s\partial x_n$). Finally, we Fourier transform these equations with respect to time to obtain a set of eigenvalue equations for the resonance frequency ω,

$$\omega^2S'_k-i\gamma\omega\epsilon_{kmn}[S'_m(H_0)_n+(S_0)_mH'_n]-(\mathcal{L}^D_{km}+\mathcal{L}^G_{km})\left[\frac{\gamma}{\chi}S'_m-H'_m\right]=0\ . \tag{20}$$

Thus, Eq. (20) together with the boundary condition[21]

$$\hat{\mathbf{n}}\cdot\frac{\partial\mathbf{S}'}{\partial x}\bigg|_{\text{boundary}}=0 \tag{21}$$

(where $\hat{\mathbf{n}}$ is the unit vector normal to the wall) determine the problem completely. In order to solve these equations we need consider the homogeneous part only. Moreover, we can neglect the off-diagonal terms of the operators $\mathcal{L}^D$ and $\mathcal{L}^G$ in the strong-field limit, i.e., $(\omega_0/\omega_L)^2\ll1$ where ω_0 is the longitudinal resonance frequency and $\omega_L=\gamma H_0$ is the Larmor frequency. Then the longitudinal NMR response is given by the x component of Eq. (20) while the transverse NMR response is found from the y and z components of (20) (see Ref. 17).

III. SOLUTION OF THE LEGGETT EQUATIONS

We shall now proceed to solve the Leggett equations for the A and B phases in a confined geometry. The results are summarized by Figs. 1–3. Although we present results for $T=0.9$ K only, numerical studies show that they are essentially unchanged for $T=0.7$ and 0.8 K.

A. A-like phase

From Eqs. (20) and (8) we find for longitudinal and transverse NMR, respectively,

$$-\frac{1}{2}\frac{L_D^2}{\xi^2}\frac{\partial}{\partial x}\left[(a_2^2+a_3^2)\frac{\partial S'_x}{\partial x}\right]+(a_2^2-a_3^2)S'_x-\frac{\omega^2}{\Omega_A^2}S'_x=0\ , \tag{22}$$

$$-\frac{1}{2}\frac{L_D^2}{\xi^2}\frac{\partial}{\partial x}\left[(a_2^2+a_3^2)\frac{\partial S'_y}{\partial x}\right]-a_3^2S'_y+\frac{\omega_L^2-\omega^2}{\Omega_A^2}S'_y=0\ , \tag{23}$$

where $\Omega_A^2=4g_D\gamma^2\Delta_A^2/\chi$.

Consider first the equation (22) for longitudinal NMR. For zero flow, $a_2=a_3$ and we expect no longitudinal reso-

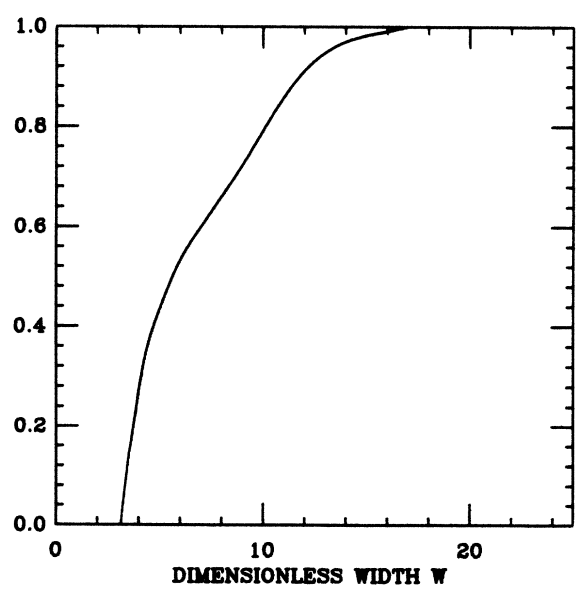

FIG. 1. A-phase transverse resonance frequency of the lowest spin mode as a function of dimensionless width, $\omega^2=\omega_L^2+\lambda_t\Omega_A^2$.

nance (the gradient terms are too small to give a measurable effect). However, finite flow results in a significant positive-squared longitudinal resonance shift. Indeed, at critical flow (for weak coupling) and neglecting the gradient terms) one finds $\omega_l^2 = \frac{4}{9}\Omega_A^2$. The gradient terms would tend to reduce this value.

Next, we see from Eq. (23) that there is no qualitative difference for transverse NMR between the case for zero flow and finite flow. Furthermore, since experiments are more likely to investigate zero-flow NMR, we shall present results for this situation only.[25] Introduce the parameter λ_t such that the transverse resonance frequency ω_t is given by $\omega_t^2 = \omega_L^2 + \lambda_t \Omega_A^2$. To leading order in $(\Omega_A^2/\omega_L)^2$, Eq. (22) reduces to

$$\frac{\partial}{\partial x}\left[a_3^2 \frac{\partial S_y'}{\partial x}\right] + \left[\frac{g_D \xi^2}{K}\right](a_3^2 + \lambda_t) S_y' = 0 \ . \tag{24}$$

In order to solve this, and all subsequent equations, we use a multiple, variable step-size shooting method.[26] The results of the calculation are given in Fig. 1. The expected downward shift is obtained for large width W. Indeed, for this orientation, $\hat{\mathbf{d}}\cdot\hat{\mathbf{l}} = 0$, the bulk value is $\lambda_t = -1$ assuming that the $\hat{\mathbf{l}}$ vector remains spatially constant, i.e., the slab width is much less than 10 μm.[24] Next, at the critical width $W = \pi$ [the actual width is $W_0 = W\xi(T)$] the frequency shift vanishes (in keeping with the fact that there is no longer any superfluid phase). Finally, the small dip in the curve between $W \approx 7$ and 12 is a direct consequence of the form of the static order parameter: a plot of the square of the average amplitude of the components of the static order parameter against width also shows this behavior. (This is also seen in the results for the B-phase longitudinal NMR.)

B. B-like phase

Again, we shall solve the zero-flow case ($q = 0$) only. The equation for the longitudinal NMR response is obtained by substituting for $A_{\mu i}^0$ in Eq. (20) from Eq. (9). We find

$$\begin{gathered}\frac{\partial}{\partial x}\left[2a_3^2 \frac{\partial S_x'}{\partial x}\right] - \left[\frac{g_D \xi^2}{K}\right](\tilde{\mathcal{L}}_{11}^D - \tfrac{15}{2}\lambda_l) S_x' = 0 \ , \\ \tilde{\mathcal{L}}_{11}^D = 8a_3^2(1 - 2\cos^2\theta) - 2a_1 a_3 \cos\theta \ ,\end{gathered} \tag{25}$$

where the longitudinal resonance frequency ω_l is given by $\omega_l^2 = \lambda_l \Omega_B^2$ with $\Omega_B^2 = 15 g_D \gamma^2 \Delta_B^2 / \chi$. The results are shown in Fig. 2. Clearly, the most interesting feature is the sharp drop in λ_l at $W \approx 7$, which is indicative of the 3D-to-planar-phase transition.[5] The theory gives $\lambda_l = \frac{4}{3}$ for a bulk planar state. We note that the bulk 3D value of $\lambda_l = 1$ is attained at $W \approx 50$.

The calculation of the transverse NMR response is slightly more complicated. We can, however, simplify the equations considerably if we introduce the new variables $S_+ = S_y' + iS_z'$ and $S_- = S_y' - iS_z'$. Then, because $a_2 = a_3$ for $q = 0$, we obtain

$$\frac{\partial}{\partial x}\left[(3a_1^2 + a_3^2)\frac{\partial S_+}{\partial x}\right] - \left[\frac{g_D \xi^2}{K}\right](\tilde{\mathcal{L}}_{33}^D - \tfrac{15}{4}\lambda_t) S_+ = 0 \ , \tag{26}$$

where

$$\tilde{\mathcal{L}}_{33}^D = a_3^2 - 2a_1^2 - 5a_1 a_3 \cos\theta - 4a_3^2 \cos^2\theta \ .$$

Here, the transverse resonance frequency ω_t is $\omega_t^2 = \omega_L^2 + \lambda_t \Omega_B^2$ to leading order in $(\Omega_B/\omega_L)^2$. In the present case, the 3D-to-planar-phase transition is even more apparent in the nonlinear dependence of λ_t on W (see Fig. 3). Indeed, the cusp at $W \approx 7$ is also seen in the

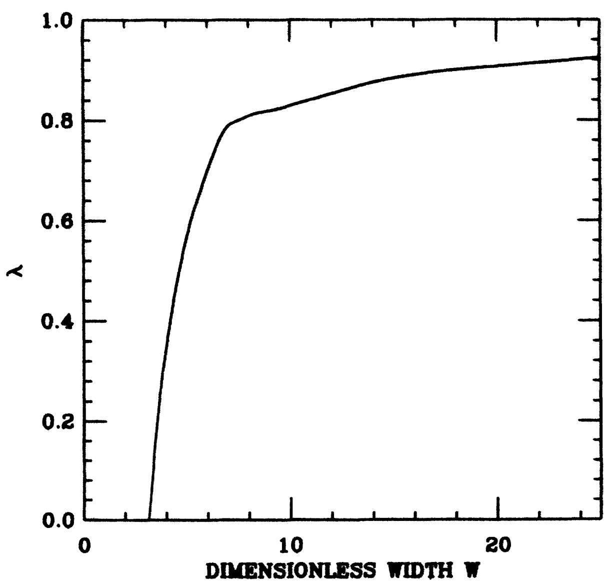

FIG. 2. B-phase longitudinal resonance frequency of the lowest spin mode as a function of dimensionless width, $\omega^2 = \lambda_l \Omega_B^2$.

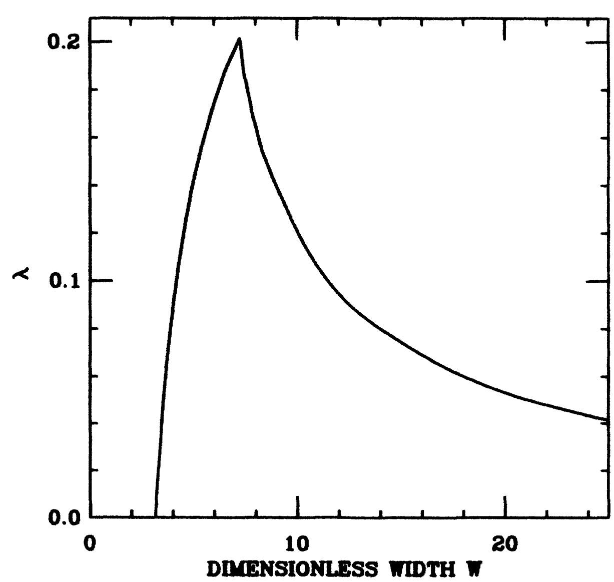

FIG. 3. B-phase transverse resonance frequency of the lowest spin mode as a function of dimensionless width, $\omega^2 = \omega_L^2 + \lambda_t \Omega_B^2$.

amplitude of the static order parameter.[5] Once more, the bulk value is achieved at $W \approx 50$ while in the absence of the gradient terms the theory predicts $\lambda_t = \frac{1}{3}$ for the planar state.

IV. CONCLUSION

We have demonstrated that NMR is, in principle, a very sensitive probe of the order parameter of superfluid ^{3}He. Perhaps most significantly, we have shown that the NMR signal for the *B*-like planar phase is very different from that for the *A* phase. Moreover, the longitudinal NMR signal for the *A* phase is strongly dependent on the flow rate. We hope that future experiments on superfluid ^{3}He in confined geometries will attempt to measure these predicted differences.

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